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ABOUT UNIFORMLY CONNECTED AND UNIFORMLY CHAINED SPACES

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Abstract. In modern mathematics, the concept of the limit of the projective spectrum of objects of a certain category plays a very important role. In the theory of uniform spaces there is considered a category, where uniform spaces act as objects, and uniformly continuous mappings act as morphisms. Therefore, it is important task to study what are the limits of projective spectra composed of objects of a certain class. This article examines the question of finding out what properties do the limits of projective spectra composed of uniformly connected and uniformly chained spaces have with respect to certain uniformly continuous mappings. The properties of uniform connectivity and being uniformly chained are tools for studying the structure of uniform space. The property of uniform connectivity is an interesting generalization of the topological property of connectivity, and therefore allows us to consider the corresponding concept for a much wider class of spaces in order to identify new analogies between topological spaces. The property of being uniformly chained is related to the concept of pseudo-compactness, which is important when considering sets of bounded uniformly continuous functions. What is described means the relevance of the topic of this work.

Key-words: *uniformed space, inversed spectrum, projective limit, uniformly connectedness, uniformly chainedness*.

Аннотация. В современной математике понятие проективного спектра, составленного из объектов некоторой категории, играет очень важную роль. В теории равномерных пространств рассматривается категория, объекты которой суть равномерные пространства, а морфизмы которой суть равномерно непрерывные отображения. Поэтому вопрос о том, какие пространства представляют из себя пределы проективных спектров, составленных из пространств определённого класса, является важным. Данная статья рассматривает некоторые свойства проективных пределов обратных спектров, составленных из равномерно связных и равномерно сцепленных пространств с наборами равномерно непрерывных отображений. Свойства равномерной связности и равномерной сцепленности представляют собой инструменты для исследования структуры равномерного пространства. Равномерная связность является интересным свойством, обобщающим свойство связности топологического пространства, и потому оно позволяет рассматривать гораздо более широкий класс пространств с целью установить новые взаимосвязи между топологическими пространствами. Свойство равномерной сцепленности определённым образом соотносится с понятием псевдокомпактности, которое имеет значение при рассмотрении множеств ограниченных равномерно непрерывных функций. Описанное выше свидетельствует в пользу актуальности темы данной работы.

Ключевые слова: равномерное пространство, обратный спектр, проективный предел, равномерная связность, равномерная сцепленность.

Аннотация. Заманбап математикада кандайдыр категориянын объектилерден түзүлгөн проективдүү пределдер жөнүндө түшүнүгү өтө маанилүү ролун ойнойт. Бир калыптуу мейкиндиктердин теориясында бир калыптуу мейкиндиктерди өзүнүн объектилериндей, бир калыптуу үзгүлтүксүз чагылтууларды өзүнүн морфизмдериндей ээлеген категория каралат. Ошол себептен кандайдыр класска кире турган мейкиндиктерден түзүлгөн проективдүү спектрлердин пределдеринин болгондугунун амалга ашырылуусу жөнүндө суроосу сөзсүз дагы маанилүү болот. Бул макала бир калыптуу байланыштырылган жана бир калыптуу бирдиктендирилген мейкиндиктерден түзүлгөн проективдүү спектрлеринин пределдеринин бир нече өзгөчөлүктөрүн карай жүрөт. Бир калыптуу байланыштырылгандыгынын жана бир калыптуу бирдиктендирилгендигинин касиеттери бир калыптуу мейкиндикти изилдөө үчүн колдонулган түшүнүктөр болушат. Бир калыптуу байланыштырылгандык топологиялык мейкиндиктин байланыштырылгандыгынын касиетин жалпыландыруучу кызыктуу касиет болот, ошондуктан бул касиет топологиялык мейкиндиктер арасында жаңы байланыштарды белгилөөнүн максаты менен мейкиндиктердин жакшылап кеңдетилген классын каралууда кармоого жол берет. Бир калыптуу бирдиктендирилгендигинин касиети чектелген бир калыптуу үзгүлүтүксүз функцияларынын көптүктөрүнүн каралуусунда өзгөчө маанисине ээлеген псевдокомпакттуулук деген касиети менен ошолордун бир бирине өзгөчө мамалесине ээ болот. Жогорудагы сүрөттөнгөн ошол макаланын актуалдуулугун далилдеп көрсөтөт.

Ачкыч сөздөр: бир калыптуу мейкиндик, тескери спектр, проективдүү предел, бир калыптуу байланыштырылгандык, бир калыптуу бирдиктендирилгендик.

1. Introduction

The theory of uniformed spaces has appeared as a constructive generalization of the structure of metrical space to the much wider case, that has been able to contain the meanings of all completely regular topological spaces via describing of them.

Definition 1.1 ([1]). *Let (X,U) be a uniformed space. A finite set* $\{A_i : i = 1..n\}$ *of subsets of X is called chained, if for any* $i = 1..n-1$ *the assertion* $A_i \cap A_{i+1} \neq \emptyset$ *holds.*

The concept of uniform connectivity was introduced with the aim of generalizing the connectedness property for topological spaces to the case of uniform spaces and exploring the corresponding generalization, trying to identify analogues of important statements concerning connected spaces from the theory of general topology in relation to the introduced definition in the theory of uniform spaces.

Definition 1.2 ([1]). *The uniformed space (X,U) is called uniformly connected, if any uniformly continuous mapping of (X,U) to the discrete uniformed space (Y,D) is constant.*

Proposition 1.3 ([1]). *Uniformed space (X,U) is uniformly connected space, if and only if for any uniformed cover* $\alpha \in U$ *for any pair of points x and y of the space X there exists a finite subset* $\{A_i : i = 1..n\}$ *of the cover* α*, that is chained sequence, and* $x \in A_1, y \in A_n$.

Proposition 1.4 ([1]). *Uniformed space (X,U) is uniformly connected, if and only if there is no disjunctive uniformed covers of (X,U).*

As indicated above, another tool for studying the structure of uniform space is the concept of uniformly chained space.

Definition 1.5 ([1]). *Uniformed space (X,U) is called uniformly chained, if for any cover* $\alpha \in U$ *there exists such a natural number n, that for any points x and y of the set X there exists a subsystem of* α*, that is a such chained sequence* $\{A_i : i = 1..k\}$*, that it is hold that* $x \in A_1, y \in A_n, k \leq n$.

As it can be seen due to the introduced definition and proposition 1.3, the property of uniform chainedness is stronger than uniform connectedness.

The concept of uniform chainedness was introduced in [1] for the case of uniform spaces. It allows us to draw interesting relationships between the structure of a uniform space and properties associated with uniform connectedness, pre-compactness and pseudo-compactness of uniform spaces.

Thus, uniform chainedness and the concepts introduced above play an interesting role in the study of uniform structures.

Let us be given a projective spectrum of uniform spaces: $\{(X_i, U_i): i \in I, >\}$ with uniformly continuous surjective mappings: $\{\left(f_i^j: (X_i, U_i) \to (X_j, U_j)\right): i > j; i, j \in I\}$, where for any $i, j, k \in I: j > i > k$, there is hold that $f_i^j * f_k^i = f_k^j$.

It is well known that the limit of the corresponding projective spectrum is a set X composed of all possible directions obtained by choosing from the sets $\{(X_i, U_i): i \in I\}$ of such points $x_i \in X_i$, that for any $i, j \in I$ it is hold that $f_i^j(x_i) = x_j$ ([2], [3]). In the category of uniform spaces and uniformly continuous mappings, the uniformity U on X, which is the uniformity of the corresponding limit, represents the supremum of structures $\pi_i^{-1}(U_i)$, obtained by taking systems of inverse images of the uniformities of the spectrum components ([1], [2]). Systems $\pi_i^{-1}(U_i)$ represent pseudo-uniformities due to known reasons. $\pi_i^{-1}(U_i)$ forms the basis of the minimal structure for which the mapping for a specifically considered i, which is π_i, will be uniformly continuous ([1], [2], [3]).

2. **Consideration of questions about uniformly connected and uniformly chained spaces studied in this article.**

2.1.Projective spectrum of uniformly connected spaces.

Let us consider the inverse spectrum composed of uniformly connected spaces. Let's select arbitrary two points on X: x and y. Consider an arbitrary element α' of the family $\pi_i^{-1}(U_i)$. For it there exists $\alpha \in U_i: \pi_i^{-1}(\alpha) = \alpha'$. Let us select their images under the projection $\pi_i: \pi_i(x) = x_i, \pi_i(y) = y_i$. Then, in view of the uniform connectivity of the space $(X_i U_i)$, for an arbitrary covering of the family U_i there is a chained finite sequence A_j of elements of the corresponding covering, where $j = 1..n$ for some natural

n, for which it holds that $A_1 \ni x_i, A_n \ni y_i$, and for all $j = 1..n-1$ it is true that $A_j \cap A_{j+1} \neq \emptyset$. This means that for a covering α there is a finite chained sequence of elements α with the corresponding properties. Let us denote the corresponding sequence by A_j with the natural set of indices indicated above for some *n*. Then we get that $\pi_i(x) \in A_1, \pi_i(y) \in A_n$. So, $x \in \pi_i^{-1}(A_1), y \in \pi_i^{-1}(A_n)$. Moreover, for all *i* from *1* to *n-1* we obtain that $A_i \cap A_{i+1} \neq \emptyset => \pi_i^{-1}(A_i \cap A_{i+1}) \neq \emptyset$ by the surjectivity of π_i, therefore from above what was proved about the preimages of intersection, we obtain that $\pi_i^{-1}(A_i) \cap \pi_i^{-1}(A_{i+1}) \neq \emptyset$. But for any i from 1 to *n* we get that $A_i \in \alpha =>$ $\pi_i^{-1}(A_i) \in \pi_i^{-1}(\alpha) = \alpha'$. We find that there are elements of the covering α' for which the conditions shown above are satisfied. This means that a chained finite sequence has been found for it, the first element of which contains *x*, and the last element contains *y*. This means that in view of the arbitrariness of considering a covering from the family $\pi_i^{-1}(U_i)$ and in view of the arbitrariness of considering points *x* and *y*, we obtain that for any points *x* and *y* for any covering from the family $\pi_i^{-1}(U_i)$ there is corresponding chained sequence. Then we obtain that for any covering from $\pi_i^{-1}(U_i)$ for any pair of points there is a chained sequence corresponding to what was described, and therefore, by proposition 1.3, $\pi_i^{-1}(U_i)$ is the base of a uniformly connected pseudo-uniformity (in meaning similar to the shown meaning). In a view of the consideration for an arbitrary index $i \in I$, we find that for any $i \in I$ the corresponding pseudo-uniformity with the base $\pi_i^{-1}(U_i)$ is uniformly connected, and, moreover, π_i is uniformly continuous.

According to what has been described, the minimal structure for which all canonical projections are uniformly continuous is the supremum of the distinguished pseudo-uniformities, and is *U*. Consider an arbitrary element of the base for *U*, obtained of all possible intersections of element-wise coverings of the families $\pi_i^{-1}(U_i) = U_i'$. Let us select the corresponding family $\{(X_i, U_i): i = 1..n\}$, where *n* is a natural number, without limiting the generality of consideration when redesignating the indices of a set of $i \in I$, when selecting a finite set $\{(X_i, U_i): i = 1..n\}$, one covering of which we select in this case. Due to the fact that $\{(X_i, U_i): i = 1..n\}$ forms the inverse spectrum, there exists $j \in I: (\forall i = 1..n: j > i)$. Then f_j^i are uniformly continuous for *i=1..n*. Then for an arbitrary covering α_i' from the considered set of distinguished coverings from the families U_i' it is true that there exists $\alpha_i \in U_i: \alpha_i' = \pi_i^{-1}(\alpha_i)$. However, for any such $\alpha_i \in U_i$ it is true that $(f_j^i)^{-1}(\alpha_i) \in U_j$, since all f_j^i are uniformly continuous. So, we get from that, what has been shown higher:

$$(f_j^i)^{-1}(\alpha_i) \in U_j => U_j' \ni \pi_j^{-1}\left((f_j^i)^{-1}(\alpha_i)\right) = \left(\left(f_j^i\right)^{-1} * \pi_j^{-1}\right)(\alpha_i)$$

$$= \left(\pi_j * f_j^i\right)^{-1}(\alpha_i),$$

but $\pi_j * f_j^i = \pi_i$, that is why: $\pi_i^{-1}(\alpha_i) \in U_j'$. Id est, $\alpha_i' \in U_j'$. It means that for all $\alpha_i' \in U_i'$ it is hold that if $i = 1..n$, then $\alpha_i' \in U_j'$. So we get, that $\{\alpha_i': i = 1..n\} \subset U_j'$, that means that because of what has been proved by us higher and because of the definition of the base we have the following interrelation: $\wedge\{\alpha_i': i = 1..n\} \in U_j'$. But because of what we have proven: $i \in I$ means the uniformly connectedness of (X, U_i'). So, the U_j' is uniformly connected, and that is why, using the proposition 2, we get that U_j' does not contain disjunctive coverings, so the covering $\wedge\{\alpha_i': i = 1..n\} \in U_j'$ is not disjunctive too. Then because of arbitrarity of the considering of the covering of the base of *U* in the form $\wedge\{\alpha_i': i = 1..n\}$ we get that *U* does not contain disjunctive coverings. It is possible to show that the absence of disjunctive coverings in the base of *U* implies the absence of disjunctive coverings in the system *U* as in itself. That is why because of the proposition 1.4 and because of structural paricularability of every component of the considered spectrum we get that *U* is uniformly connected uniformity.

It is corollary of that the projective limit, as it is known ([1], [3]), of the spectrum, containing only particularable spaces, is particularable space in the sense of that axiom, that differs uniformed space from the pseudouniformed space.

Theorem 2.1. *Projective limit of the spectrum, consisted of uniformly connected spaces and of uniformly continuous mappings, is uniformly connected uniformed.*

2.2. Projective limit of uniformly chained spaces.

Let us consider the case, when the projective spectrum is consisted of uniformly chained spaces. Then for every uniformly chained space (X_i, U_i), where $i \in I$, let us represent the pseudouniformed structure, generated by the system $\pi_i^{-1}(U_i)$ on a set X, analogically to that how it was done higher. Similar to considering the spectrum of uniformly connected spaces for an arbitrary index $i \in I$, we find that for any $i \in I$ the corresponding pseudo-uniformity with the base $\pi_i^{-1}(U_i)$ is uniformly chained and ensures uniform continuity of π_i. Moreover, according to the properties of projective spectra ([1]), the minimal structure of interest, ensuring uniform continuity of all canonical projections, is the supremum of the distinguished pseudo-uniformities $\pi_i^{-1}(U_i)$, and coincides with U. Let us consider an arbitrary element of the base for U, obtained as all possible intersections of element-wise coverings of the families $\pi_i^{-1}(U_i) = U_i{}'$. The intersection of finite coverings from the same $U_i{}'$ is an element of $U_i{}'$ as presented above. Therefore, this consideration is similar to that carried out above for the case of uniformly connected spaces. Therefore, from the above we obtain that:

$$(f_j^i)^{-1}(\alpha_i) \in U_j => U_j' \ni \pi_j^{-1}\left((f_j^i)^{-1}(\alpha_i)\right) = \left(\left(f_j^i\right)^{-1} * \pi_j^{-1}\right)(\alpha_i)$$

$$= \left(\pi_j * f_j^i\right)^{-1}(\alpha_i),$$

but $\pi_j * f_j^i = \pi_i$, so $\pi_i^{-1}(\alpha_i) \in U_j'$. Id est, $\alpha_i' \in U_j'$. It means that for all $\alpha_i' \in U_i'$ it is hold that if $i = 1..n$, than $\alpha_i' \in U_j'$. So we get, that $\{\alpha_i' : i = 1..n\} \subset U_j'$, that means that because of that, what has been proved by us higher, and because of the definition of the base we have the following interrelation: $\wedge \{\alpha_i' : i = 1..n\} \in U_j'$. But because of what we have proven: $i \in I$ means the uniformly chainedness of (X, U_i'). So, the U_j' is uniformly chained, and that is why every covering of it corresponds to the such condition, corresponding to which of any covering of some space means the uniformly chainedness of this space. So, the covering $\wedge \{\alpha_i' : i = 1..n\}$ corresponds to this condition too. Because of arbitrarity of the considering of the element of the base of the system U in the form of $\wedge\{\alpha_i'\colon i = 1..n\}$, we get that all elements of the considered base corresponds to the condition, corresponding to which of any covering of some space means the uniformly chainedness of this space. But it is possible to show that the corresponding of all elements of the considered base to this condition means the corresponding of all elements of the considered system *U* to this condition. Being hold of last means the uniformly chainedness of the system *U*.

So, due to being of uniformities of the structures of all components of the considered spectrum we get by the similar to the shown higher for the case of considering of the spectrum of uniformly connected spaces way that the system *U* is uniformly chained uniformity.

Theorem 2.2. *The projective limit of the spectrum, consisted of uniformly chained spaces and uniformly continuous mappings, is a uniformly chained space.*

2.3. A dense subspace of a uniformly connected space

Let a uniform space *(X,U)* be given, which is a dense subspace of a uniformly connected space *(Y,V)*. Then consider an arbitrary open uniform covering *β* of the space *X* and arbitrary two points: *x* and *y*. Then there exists $\gamma \in V\colon \beta = \{A \cap X\colon A \in \gamma\}$. *X⊂Y*, which means $x, y \in Y$. However, according to the properties of uniform space, uniformity *V* has a base of open coverings. This means there is an open $\alpha \in V$ that is inscribed in *γ*. Then note that $\{A \cap X\colon A \in \alpha\}$ will be inscribed in $\beta = \{A \cap X\colon A \in \gamma\}$. According to proposition 1, there is a finite chained sequence of elements α connecting the points *x* and *y*. Then there exists $A_i\colon i = 1..n,$ where for $i = 1..n-1$ it is true that $A_i \cap A_{i+1} \neq \emptyset, x \in A_1, y \in A_n$. But from the openness of α it follows that $A_i\colon i = 1..n$ is a system of open sets, and from the chainedness of this system it follows that each of its sets is non-empty. This means $A_i\colon i = 1..n$ and $A_i \cap A_{i+1} \neq \emptyset\colon i = 1..n-1$ are neighborhoods of some points in the space *Y*. This means that in each set of each of these systems there is at least one point in the space *X* due to the density of the space *X* in the space *Y*. This means that the intersection of each of the sets of each of these systems with *X* will be non-empty: $A_i \cap X \neq \emptyset\colon i = 1..n$ and $A_i \cap A_{i+1} \cap X \neq \emptyset\colon i =$

$1..n-1$. However, $A_i \cap A_{i+1} \cap X = (A_i \cap X) \cap (A_{i+1} \cap X)$, therefore, we obtain that the system of sets $A_i \cap X: i = 1..n$ is a system of non-empty sets, where $x, y \in X, x \in A_1, y \in A_n$ means that $x \in A_1 \cap X, y \in A_n \cap X$, and the intersection of the set $A_i \cap X$ with $A_{i+1} \cap X$ is non-empty for all $i = 1..n-1$. This means that the system $A_i \cap X: i = 1..n$ is chained, and the uniformity of the covering α means that there is some uniform covering $\alpha' = \{A \cap X: A \in \alpha\}$ for *U*, for which the system $A_i \cap X: i = 1..n$ is a subsystem. However, from what was shown above: $\alpha' = \{A \cap X: A \in \alpha\}$ is inscribed in $\beta = \{A \cap X: A \in \gamma\}$. This means, by the definition of inscribedness, for each element of the system $A_i \cap X: i = 1..n$ there is an element $B \in \beta$, into which the first one is included. This means that we can select a subsystem $B_i: i = 1..n$ of the system *β* such that for any $i = 1..n$ it will be true that $A_i \cap X \subset B_i$. But then, according to what was shown above, we obtain that $x \in B_1, y \in B_n$, and $B_i \cap B_{i+1} \supset (A_i \cap X) \cap (A_{i+1} \cap X) \neq \emptyset$. Then we find that for the system $B_i: i = 1..n$, it is true that $B_i \cap B_{i+1} \neq \emptyset$, and $x \in B_1, y \in B_n$, while from the above it follows that the system $B_i: i = 1..n$ is a subsystem of the system *β*. Due to the arbitrariness of considering a uniform covering of the space *(X,U)* and a pair of points of the set *X*: *x* and *y*, we obtain that for any points *x* and *y* from *X* and any uniform covering of the space *(X,U)* there is a finite chained subsystem $B_i: i = 1..n$ covering *β* connecting points *x* and *y*, according to what was proven above. Therefore, by the proposition 1 we obtain that the space *(X,U)* is uniformly connected. Due to the arbitrariness of considering an everywhere dense subspace *(X,U)* of some uniformly connected space *(Y,V)*, we obtain that a dense subspace of any uniformly connected space is uniformly connected, according to what was proven above. So, the following proposition holds.

Proposition 2.3. *Any dense subspace of an arbitrary uniformly connected space is uniformly connected.*

Let now be given a uniform space *(X,U)* that is dense in a uniform chained space *(Y,V)*. Then consider an arbitrary open $\beta \in U$ and select for it, similarly to the case considered above $\gamma \in V: \beta = \{A \cap X: A \in \gamma\}$. For such $\gamma \in V$, we select an open $\alpha \in V$, which is inscribed in γ. Then $\alpha' = \{A \cap X : A \in \alpha\}$ will be inscribed in $\beta = \{A \cap X : A \in \gamma\}$. By the definition of uniformly chainedness, there is a number N_α such that for any points *x* and *y* in the space *Y* there is a finite chained sequence of elements α connecting points *x* and *y* of length $n < N_\alpha$. Consider arbitrary $x, y \in X$. For a given pair, we denote the corresponding chained system as $A_i : i = 1..n$. It is true that $x \in A_1, y \in A_n$, for $i = 1..n-1$: $A_i \cap A_{i+1} \neq \emptyset$. From the openness of α it follows that $A_i : i = 1..n$ is a system of open sets, and the chainedness of the latter means that each $A_i : i = 1..n$ is non-empty. This means that $A_i : i = 1, n$ and $A_i \cap A_{i+1} \neq \emptyset : i = 1..n-1$ are neighborhoods of some points in the space *Y*. This means that it is similar to the proof above with uniformly connectedness of everywhere dense subspaces of uniformly connected spaces : $A_i \cap X \neq \emptyset : i = 1..n$ and $(A_i \cap X) \cap (A_{i+1} \cap X) \neq \emptyset : i = 1..n-1$. Therefore, we obtain that the system $A_i \cap X : i = 1, n$ is a system of non-empty sets, and $x, y \in X, x \in A_1, y \in A_n$ means that $x \in A_1 \cap X, y \in A_n \cap X$, with this $(A_i \cap X) \cap (A_{i+1} \cap X) \neq \emptyset : i = 1..n-1$. This means that the system $A_i \cap X : i = 1..n$ is chained, and $\alpha \in V$ means that $U \ni \alpha' = \{A \cap X : A \in \alpha\}$, for which $A_i \cap X : i = 1..n$ is a subsystem. But from what was shown above, $\alpha' = \{A \cap X : A \in \alpha\}$ is inscribed in $\beta = \{A \cap X : A \in \gamma\}$. This means, similarly to the proof of the previous proposition, we can select a subsystem $B_i : i = 1, n$ of the system β such that $B_i \cap B_{i+1} \neq \emptyset$, and $x \in B_1, y \in B_n$, where $n < N_\alpha$. The arbitrariness of *x* and *y* means that for any pairs *x* and *y* from *X* there is a chained sequence of covering elements β connecting *x* and *y*, with a smaller N_α. The arbitrariness of $\beta \in U$ implies that for any $\beta \in U$ there is a number *N* such that for any *x* and *y* from *X* there is a chained sequence of sets from β connecting *x* and *y*, having length less than *N*. That is, by definition we obtain that the space *(X,U)* is uniformly chained. From the arbitrariness of considering *(X,U)*, everywhere dense in some uniformly chained *(Y,V)*, we obtain that any dense subspace of any uniformly chained space is uniformly chained by what was proved above. This means that we have just proved the following statement.

Proposition 2.4. *Any dense subspace of an arbitrary uniformly chained space is uniformly chained.*

2.4. Some mappings that preserve the properties of uniform connectedness and uniform chainedness.

Theorem 2.5. *If f:(X,U)→(Y,V) is uniformly quotient mapping, and the inverse image of each point at f is a uniformly connected space, then if the space (Y,V) is uniformly connected, then the space (X,U) is uniformly connected.*

Proof. Let *f:(X,U)→(Y,V)* be uniformly factorial, and inverse image at *f* of any point of the space *(Y,V)* is uniformly connected space. Then we put the space *(Y,V)* uniformly connected. Let in the space *(X,U)* there exist a disjoint uniform covering *α*. This assumption we will assign symbolically as (*). Let us consider a certain point near the space *Y* and its inverse image. However, on $f^{-1}(y)$ the uniformity of *V* induces the uniformity of the subspace. Therefore $\alpha' = \{A \cap f^{-1}(y): A \in \alpha\}$ is a uniform covering of $f^{-1}(y)$. However, for arbitrary $A', B' \in \alpha'$ it is true that there exist $A, B \in \alpha$ for which $A \cap f^{-1}(y) = A', B \cap f^{-1}(y) = B'$. If *A* and *B* are different, it is true that $A \cap B = \emptyset$, due to the disjointness of the covering *α*. So, $A' \cap B' \subset A \cap B = \emptyset$. That is, $A' \cap B' = \emptyset$. That is, for arbitrary $A', B' \in \alpha'$ it is true that $A' \cap B' = \emptyset$, which means that the covering α' is disjoint. And the space $f^{-1}(y)$ with the uniformity induced on it does not contain disjoint coverings as it is uniformly connected by condition in accordance with the proposition.2. This means that either α' contains only one set, or more than one, but among its elements only one is a non-empty set. If α' contains only one set, then for arbitrary $A', B' \in \alpha'$ it is true that $A \cap f^{-1}(y) = A' = B \cap f^{-1}(y) = B'$, which means that $\emptyset \neq A' \cap B' \subset A \cap B$, that is, that $\emptyset \neq A \cap B$, which in turn implies *A*=*B* by disjunction *α*, which contradicts the difference between *A* and *B*. This means that among the elements *α* only one is a non-empty set. Then among the elements $\alpha' = \{A \cap f^{-1}(y): A \in \alpha\}$ only one is not empty as a set. This means $f^{-1}(y)$ intersects non-emptyly only with one of the sets from *α*. And, since *α* is a covering, that is, for any *x* of the set $f^{-1}(y)$ there is an element of *α* containing *x*, then $f^{-1}(y)$ lies in some set from

α. Since there is only one set of α with non-empty intersection with $f^{-1}(y)$, $f^{-1}(y)$ lies in one and only one set of α. Due to the arbitrariness of *y*, for any *y* the inverse image $f^{-1}(y)$ lies in one of the sets of the family α. Let us assign this fact conditionally as combination of symbols: (1). There are no points $f^{-1}(y)$ in other sets from α, since α is disjoint.

Consider $f(\alpha)$ and assume that for some *A* and *B* from α, $f(A) \cap f(B) \neq \emptyset$ is true. Then we get that there are $y \in f(A), f(B)$. That is, there are $a \in A$ and $b \in B$ for which $f(a) = f(b) = y$. This means there is a *y* for which $a \in A$ and $b \in B$ are such that $a, b \in f^{-1}(y)$. That is, $f^{-1}(y) \cap A \neq \emptyset$, $f^{-1}(y) \cap B \neq \emptyset$. But from (1) for any point *y* its inverse image $f^{-1}(y)$ lies entirely in one of the elements of the disjoint α. Therefore, $f^{-1}(y)$ intersects only with one of the elements of this covering. This means $f^{-1}(y) \cap A \neq \emptyset$, $f^{-1}(y) \cap B \neq \emptyset$ shows us that *A*=*B*. Thus, for any *A* and *B* from α it is true that $f(A) \cap f(B) \neq \emptyset$ implies *A*=*B*, meaning *f(A)=f(B)*. Let us assign this fact of that *f(A)=f(B)* conditionally as combination of symbols: (2). Moreover, for any sets A' and B' from *f(α)*, there exist sets *A* and *B* from *α* such that $f(A) = A', f(B) = B'$. So for the sets A' and B' of the covering *f(α)*, the existence of such *A* and *B* coverings *α* such that $f(A) = A', f(B) = B'$, and proved above in (2) mean that $A'=B'$ follows from $A' \cap B' \neq \emptyset$. That is, a non-empty intersection of the covering sets *f(A)* means their equality. Thus, from the difference between the two sets in *f(A)* it follows that if their intersection were non-empty, they would be equal, which contradicts their difference. Therefore, different sets from *f(A)* do not intersect. This means *f(α)* is disjoint by definition. However, due to the uniform factor nature of the mapping *f*, the covering *f(α)* is uniform in the image space. Moreover, the image space is uniformly connected, and therefore, according to proposition 1.4, does not contain disjoint uniform coverings. We obtain a contradiction from assumption (*) about the existence of a disjoint $\alpha \in U$. This means that there are no disjoint coverings in *U*, therefore proposition 1.4 means that the space *(X,U)* is uniformly connected. Q.E.D.

Proposition 2.6. *Let we be given a uniformly continuous surjective mapping* $f: (X, U) \to (Y, V)$, *then if* (X, U) *is the uniformly connected space, then* (Y, V) *is the uniformly connected space.*

Proof. This proposition can be easily proven from the definition of uniform connectedness. When considering an arbitrary h – uniformly continuous mapping *(Y,V)* into a discrete uniform space, we obtain that *fh* is a uniformly continuous mapping *(X,U)* into the corresponding discrete space. Due to the uniform connection of *(X,U)*, the mapping *fh* turns out to be a constant mapping. The constancy of *fh* entails that at each point of its domain of definition, *h* also takes on the same value. Therefore, an arbitrary uniformly continuous mapping *h* of the space *(Y,V)* into a discrete one turns out to be constant. This becomes a reason to assert that *(Y,V)* itself is uniformly connected in the case of uniform connection *(X,U)*. This proves proposition 2.6.

Proposition 2.7. *Let we be given a uniformly continuous surjective mapping* $f: (X, U) \to (Y, V)$, *then if* (X, U) *is the uniformly chained space, then* (Y, V) *is the uniformly chained space.*

Proof. Let us consider an arbitrary uniform covering β of the space *Y*. We can select the corresponding uniform covering $\alpha = f^{-1}(\beta)$. For it, due to the uniform chainedness of the space *X*, there exists a corresponding natural number n. Let us be given two points of image space: *a*, *b*. According to surjectivity, they have corresponding points in their prototypes: *c*, *d*. Then for points *c*, *d* there is a finite chained sequence of length less than *n* connecting them. Then we take this system for considering by us in the following form: $A_i: i = 1..k;\ k < n$. Because of surjectivity of *f* we take, that $f(A_i): i = 1..k;\ k < n$ is subsystem of β. Case of it is that for surjective mapping *f* the operator ff^{-1} makes any subset of *Y* be returning to itself. And $A_i \cap A_{i+1} \neq \emptyset: i = 1..k - 1$ means that $f(A_i) \cap f(A_{i+1}) \supset f(A_i \cap A_{i+1}) \neq \emptyset$, so *f* inducts a finite chained sequence of sets of the family β on the space *(Y,V)*. Additionally to this, such a sequence of subsets of the space *Y* connects the pair of points: $f(c) = a$ and

$f(d) = b$. And more, this sequence has been indexed with the order of indexing, which is the same with the order of indexing of sequence, consisted of A_i: $i = 1..k$, that is why it formally has the number of its elements, that is $k < n$; with having of it in our mind some of sets of got by us secondarily sequence are able to be the same. It means, that for an arbitrary uniformed cover β of the space Y has been found such a number n, that for an arbitrary pair of elements of Y there has been found as existing in this sense a chained sequence of sets of system β, which connects with each other the corresponding elements of Y and has the length, less then n. So, (Y, V) is uniformly chained, and, because of arbitrariness of considering of uniformly continuous surjective mapping of a uniformly chained space, it means by itself, that any uniformly continuous surjective mapping with its acting on uniformly chained space means uniformly chainedness of the space of image.

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INDEX OF COMPLETENESS OF UNIFORM SPACES

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The completeness index of uniform spaces was introduced and studied by A.A. Borubaev. It is remarkable in that it guarantees the decomposition of a uniform space into good inverse spectra. In this article, special types of the completeness index of complete uniform spaces are introduced, namely, the index of τ-completeness of complete uniform spaces, which characterizes the "degree of completeness" of complete uniform spaces. The behavior of the τ-completeness index of complete of uniform spaces under basic operations on uniform spaces is studied. In particular, a solution is given to A.A. Borubaev's problem concerning classes of uniform spaces that allow a perfect mapping onto a complete τ-metrizable (in particular, separably metrizable) space.

Key words. completeness index, τ-uniformly complete Čech space, Cauchy filter, τ-metrizable space.

Бир калыптуу мейкиндиктердин индекстин толуктугу А.А. Бөрүбаев тарабынан киргизилген жана изилденген, ал маанилүү, анткени ал бир калыптуу мейкиндиктердин тескери спектрлерге ажырашына кепилдик берет. Бул макалада толук бир калыптуу мейкиндиктердин индекстин толуктугунун атайын түрлөрү киргизилет, тактап айтканда толук бир калыптуу мейкиндиктердин индекс τ-толуктугу, ал бир калыптуу толук мейкиндиктердин "толуктук даражасын" мүнөздөйт. Бир калыптуу мейкиндиктердеги негизги амалдарына τ-толук бир калыптуу индекстин толуктугу абалы изилденет. Тактап айтканда, А.А. Бөрүбаевдин толук метризацияланган мейкиндикке жеткилең чагылган бир калыптуу мейкиндиктердин класстары жөнүндөгү көйгөйү чечилет.

Урунттуу сөздөр. индекстин толуктугу, Чех боюнча бир калыптуу τ-толук мейкиндик, Коши филтри, τ-метризацияланган мейкиндик.

Индекс полноты равномерных пространств был введен и изучен А.А. Борубаевым, он замечателен, тем, что он гарантирует разложимость равномерного пространства в хорошие обратные спектры. В настоящей статье вводятся специальные виды индекса полноты полных равномерных пространств, а именно, τ-индекса полноты полных равномерных пространств, которое характеризует «степень полноты» полных равномерных пространств. Изучается поведение τ-индекса полноты полных равномерных при основных операциях над равномерными пространствами. В частности, дается решение проблемы А.А. Борубаева о

классах равномерных пространств, допускающих совершенное отображение на полное τ-метризуемое (в частности сепарабельно метризуемое) пространство.

Ключевые слова. индекс полноты, τ-равномерно полное по Чеху пространство, фильтр Коши, τ-метризуемое пространство.

Let (X, U) be a uniform space.

Definition 1. Let$H \subset U$ be a subsystem consisting of covers of cardinality $\leq \tau$. The space (X, U) is called $\tau - H$-complete, and the system H is called complete, if every H is Cauchy filter F has at least one point of adherence, i.e., in $\cap \{[M]: M \in F\} \neq \emptyset$.

Definition 2. Let (X, U) be a complete uniform space. The smallest cardinal number μ is called the indexτ**-completeness** of the uniform space (X, U), if there exists a system $H \subset U$consisting of covers of cardinality $\leq \tau$, such that $|H| = \mu$ and (X, U) are $\tau - H$-complete.

The index of τ-completeness of a complete uniform space (X, U) is denoted by $\tau - ic(U)$.

A uniform space (X, U) is called τ-uniformly locally compact if there exists a uniform cover of cardinality $\leq \tau$. $\aleph_0$-uniformly locally compact spaces are called strongly uniformly locally compact**.**

Proposition 1. For a complete uniform space (X, U), the following conditions are equivalent:

1. $\tau - ic(U) = 1$;
2. (X, U) is τ-uniformly locally compact.

Proof. Let the system H consist of a uniform cover $\alpha \in U$ of cardinality $\leq \tau$, and let the uniform space (X, U) be $\tau - H$-complete. Then, there exists a uniform cover $\gamma \in U$ of cardinality $\leq \tau$, consisting of closed subsets of $\gamma \succ \alpha$. We will prove that $\gamma \in U$ consists of compact sets. Let $G \in \gamma$ be some set, and F_G be some filter in the set G. Then, the system $F = \{M \subset X: \exists M_G \in F_G: M_G \subset M\}$ is an H-Cauchy filter. Due to the $\tau - H$-completeness of the space (X, U), this space has a point of adherence $x \in G$, and thus, the point $x \in G$ is a point of adherence of the filter F_G. Hence, G is compact.

If the uniform cover $\alpha \in U$ of cardinality $\leq \tau$ consists of compact sets, then every H-Cauchy filter such that $H = \{\alpha\}$ has a point of adherence. Therefore, $\tau - ic(U) = 1$.

Corollary 1. For a complete uniform space (X, U), the following conditions are equivalent:

1. $c - ic(U) = 1$;
2. (X, U) is strongly uniformly locally compact.

Proposition 2. Let $f: (X, U) \rightarrow (Y, V)$ be a perfect uniformly continuous map of (X, U) onto a complete space (Y, V). Then (X, U) is also complete and $\tau - ic(U) \leq \tau - ic(V)$-complete.

Proof. Let $P \subset V$ be a system of covers of cardinality $\leq \tau$, such that the space (Y, V) is $\tau - P$-complete and $\tau - ic(V) = |P|$-complete. Let $H = \{f^{-1}\gamma: \gamma \in P\}$ be arbitrary. We will show that the space (X, U) is $\tau - H$-complete. Choose an arbitrary H-Cauchy filter F in (X, U). Then, fF is a P-Cauchy filter in (Y, V), and due to the P-completeness of the space (Y, V), it has a point of adherence $y \in Y$. Since the map$f$ is perfect, the filter F has a point of adherence $x \in f^{-1}y$. Therefore, (X, U) is H-complete. Since $|H| \leq |P|$, we conclude that $\tau - ic(U) \leq \tau - ic(V)$.

Lemma 1. Let (X, U) be a complete uniform space and $\tau - ic(U) \leq \mu$. Then, there exists a pseudo-uniformity $B \subset U$, consisting of covers of cardinality $\leq \tau$, such that $w(B) \leq \tau$and the system B are $\tau - B$-complete.

Proof. Let $H \subset U$ be a system of covers of cardinality $\leq \tau$, such that $\tau - ic(U) = |H|$. For each cover $\gamma \in U$, we choose a normal sequence of uniform covers of cardinality $\leq \tau$ $\phi_\gamma = \{\gamma_n\}$, such that $\gamma_1 = \gamma$. Let $H' = \cup \{\phi_\gamma: \gamma \in H\}$ be defined as follows. Denote by H'' the set of all possible finite internal intersections of covers from H', and B is the family consisting of covers into which a cover from H'' can be inserted. It is clear that the family B is a complete pseudo-uniformity, and $w(B) \leq \tau$.

A uniform space (X, U) is called τ**-uniformly** complete Čech space if it is complete and $\tau - ic(U) \leq \aleph_0$. A uniform space (X, U) is called τ**-metrizable** if it has a countable base consisting of covers of cardinality $\leq \tau$. Uniformly $\aleph_0$-metrizable spaces are called separably metrizable spaces**.**

The following theorem is the solution to the problem posed by A.A. Borubaev.

Theorem 1. For a uniform space (X, U), the following conditions are equivalent:

1. The uniform space (X, U) is τ-uniformly complete Čech space;
2. The uniform space is (X, U) mapped onto some complete τ-metrizable uniform space (Y, V) by a perfect uniformly continuous map;
3. The uniform space (X, U) is the limit of the inverse system $\{(X_a, U_a), h_a^b, M\}$ consisting of complete τ-metrizable uniform spaces (X_a, U_a) and perfect uniformly continuous maps "onto" h_a^b.

Proof.1 ⇒ 2 Let the space (X, U) be a τ-uniformly complete Čech space, i.e., it is complete and satisfies $\tau - ic(U) \leq \aleph_0$. By Lemma 1, there exists a pseudo-uniformity $B \subset U$, consisting of covers of cardinality $\leq \tau$, such that $w(B) \leq \aleph_0$ and the system B are strongly B-complete. Then, by Lemma 1.1.13 [1], for the pseudo-uniform space (X, B), there exist a uniform space (Y, V) and a uniformly continuous map $f: (X, B) \to (Y, V)$, such that the family $\{f^{-1}\gamma: \gamma \in V\}$ forms a base of the pseudo-uniformity $B$and satisfies $w(B) \leq \aleph_0$.It remains to check that F_Y has a point of adherence in (Y, V), after which the completeness of (Y, V) follows. Let F_Y be an arbitrary Cauchy filter in (Y, V), consisting of covers of cardinality $\leq \tau$, such that $F = \{M \subset X: M \supset f^{-1}L\}$ corresponds to some $L \in F_Y\}$. Since the family $\{f^{-1}\gamma: \gamma \in V\}$ forms a base of a B-complete pseudo-uniformity, F is a B-Cauchy filter in (X, U). By the strong B-completeness of the space (X, U), the filter F has a point of adherence in (X, U). From the uniform continuity of the map f and $fF = F_Y$, it follows that F_Y has a point of adherence in (Y, V). Therefore, (Y, V) is complete. We will prove that the map $f: (X, B) \to (Y, V)$ is perfect. Let $F_{f^{-1}y}$ be some filter in $f^{-1}y$, $y \in Y$. We define $F_X = \{M \subset X: M \supset M_{f^{-1}y}\ \forall M_{f^{-1}y} \in F_{f^{-1}y}\}$. Since $f^{-1}y =$

$\cap \{\alpha(f^{-1}y): \alpha \in B\}$, F_X is a B-Cauchy filter in (X, B). By the B-completeness of the space (X, U) the filter F_X has a point of adherence $x \in f^{-1}y$. It is clear that $x \in f^{-1}y$is a point of adherence of the filter $F_{f^{-1}y}$, i.e., $f^{-1}y$ is compact. Let $y \in Y$ be an arbitrary point, and let O be an open set in (X, U) such that $O \supset f^{-1}y$. We will show that there exists a set $\alpha \in B$ such that $\alpha(f^{-1}y) \subset O$. Suppose $L_\alpha = \alpha(f^{-1}y) \cap (X \backslash O) \neq \emptyset$ holds for all $\alpha \in B$ and $y \in Y$. It is easy to see that $\eta = \{L_\alpha: \alpha \in B\}$ is a centered system. Define $F = \{C \subset X: C \supset L_\alpha \forall L_\alpha \in \eta\}$. It is clear that F is a filter in X. Let $\alpha \in B$be an arbitrary uniform cover. Then, there exists a uniform cover$\beta \in B$ such that $\beta *\succ \alpha$, i.e., there exist $A \in \alpha$ and $x \in f^{-1}y$ such that $\beta(x) \subset A$. But $\beta(x) \subset \beta(f^{-1}y)$ implies $A \in F$, i.e., F is a B-Cauchy filter in (X, U). Since (X, U) is B-complete, we conclude that $\cap \{[M]: M \in F\} \neq \emptyset$. However, $\cap \{[M]: M \in F\} \subset \cap \{|\alpha(f^{-1}y)| \cap (X \backslash O): \alpha \in B\} = \cap \{\alpha(f^{-1}y) \cap (X \backslash O): \alpha \in B\} = \emptyset$. The obtained contradiction proves the closedness of the map f.

2 $\Rightarrow$ 1. it follows from Proposition 2.

1 $\Rightarrow$ 3 Let $\{(X_a, B_a): a \in M\}$ be the family of all complete τ-metrizable pseudo-uniform spaces (X_a, U_a) such that $B_a \subset U$ holds for all $a \in M$, and each B_a contains a cover of cardinality $\leq \tau$. According to Lemma 1, $\{B_a: a \in M\} \neq \emptyset$. The order relation on the set M is introduced in the standard way: $a < b$ if and only if there exist $B_a \subset B_b$ and $B_a \neq B_b$. Note that if $a, b \in M$, then $B_c = sup\{B_a, B_b\} \subset U$ is a complete τ-metrizable pseudo-uniformity, so M is a directed set.According to Lemma 1.1.13 [1], for each pseudo-uniformityB_a, we can construct a complete τ-metrizable uniform space (X_a, U_a) and a perfect uniformly continuous map $f_a: (X, U) \to (X_a, U_a)$. Let $X_a = \{[x]_a: x \in X\}$, $[x]_a = \cap \{\alpha(x): \alpha \in B_a\}$, $f_a x = [x]_a$, $x \in X$ and$U_a = \{f_a^{\#}\alpha: \alpha \in B_a\}$,$f_a^{\#}\alpha = \{f_a^{\#}A: A \in \alpha\}$, $f_a^{\#}A = X_a \backslash f_a(X \backslash A)$. If $a < b$, then the map $f_a^b: X_b \to X_a$ is defined by the formula:$f_a^b([x]_b) = [x]_a$,$x \in X$. It is easy to see that the map$f_a^b: (X_b, U_b) \to (X_a, U_a)$ is uniformly continuous. Now, we will prove the perfection of the map f_a^b. Since $f_a = f_a^b \circ f_b$ and $f_a$are perfect maps, f_a^b is also a perfect map. It is not difficult to verify that if $a < b < c$, then $f_a^c = f_a^b \circ f_b^c$. Thus, we have obtained an inverse spectrum $S = \{(X_a, U_a), f_a^b, M\}$ consisting of

complete τ-metrizable uniform spaces and perfect uniformly continuous maps "onto" f_a^b. Let$(\hat{X}, \hat{U}) = \underline{lim}$. We will prove that there exists a uniform homeomorphism $f: (X, U) \to (\hat{X}, \hat{U})$. Let f be the diagonal product of the family of maps. Then, f_a, i.e., $fx = \{f_a x\}$, $x \in X$. Therefore, $f: (X, U) \to \prod\{(X_a, U_a): a \in M\}$ is a uniformly continuous map "onto". From $f_a x = f_a^b(f_b x)$, it follows that $\{f_a x\}_{a \in M}$ is a thread of the spectrum S and $fx \in \hat{X}$, i.e., the map $f: (X, U) \to (\hat{X}, \hat{U})$ is uniformly continuous. Next, the bijectivity of the map$f$ follows from the separability of the space (X, U). Clearly, $\cup\{B_a: a \in M\}$ forms a base of uniformity U. It is easy to prove that the map f is a uniform homeomorphism "in". Therefore, $f: (X, U) \to (\hat{X}, \hat{U})$ is a uniform homeomorphism.

3 $\Rightarrow$ 1. Let $(X, U) = \underline{lim_{a_{a_a}}}^{b}$, (X_a, U_a) be a complete τ-metrizable space for each $a \in M$, and let f_a^b be a perfect uniformly continuous map “onto”. Then the projection of $f_a: (X, U) \to (X_a, U_a)$ is uniformly continuous. Since f_a^b is a perfect uniformly continuous map “onto”, it follows that$f_a$is also a perfect uniformly continuous map “onto”.

From Proposition 1, we conclude that (X, U) is a τ-uniformly complete Čech space.

Corollary 2. For a uniform space (X, U), the following conditions are equivalent:

1. The uniform space (X, U) is a uniformly complete Čech space;
2. The uniform space (X, U) mapped onto some complete separably metrizable uniform space (Y, V) by a perfect uniformly continuous map;
3. The uniform space (X, U) is the limit of the inverse system $\{(X_a, U_a), h_a^b, M\}$ consisting of complete separably metrizable uniform spaces (X_a, U_a) and perfect uniformly continuous maps "onto" h_a^b.

Proposition 3. Every τ-uniformly complete Čech space is uniformly B-paracompact.

Proof. Let (X, U) be a τ-uniformly complete Čech space. Then, there exists a complete τ-metrizable space (Y, V) and a perfect uniformly continuous map “onto”

$f:(X,U)\to(Y,V)$. Let $\{\beta_n\}$ be a base for the τ-metrizable space (Y,V), consisting of covers of cardinality $\leq\tau$. Let$\alpha_n=f^{-1}\beta_n$. We will show that a sequence of $\{\alpha_n\}$ uniform coverings of cardinality $\leq\tau$, for any finite additive coverλ of the space (X,U), satisfies condition (BP) [4]. Choose an arbitrary point $x\in X$. Since $f^{-1}y$ is compact, there exists a finite number of elements L_i from the cover of λ such that λ. Clearly, $\overset{n}{\underset{i=1}{\cup}} L_i\in\lambda$. Let $L=\overset{n}{\underset{i=1}{\cup}} L_i$. Due to the closedness of the map f, the set $O=Y\setminus f(X\setminus L)$ is a neighborhood of the point $y\in Y$. Thus, there exists an $n\in N$ -element cover$\beta_n(y)\subset O$ such that $\alpha_n(x)\subset L$covers $x\in f^{-1}y$. Therefore, (X,U) is uniformly B-paracompact.

Theorem 2. For a uniform space (X,U), the following conditions are equivalent:

1. (X,U) is a τ-uniformly complete Čech space;
2. $(exp_n X, exp_n U)$ is a τ-uniformly complete Čech space.

Proof. Let (X,U) be a τ-uniformly complete Čech space. Then, by Theorem 1, the uniform space (X,U) perfectly maps onto some complete τ-metrizable space (Y,V). It is known that the map $exp_n f:(exp_n X, exp_n U)\to(exp_n Y, exp_n V)$, $exp_n(f)(F)=f(F)$, $F\in exp_n X$, is a perfect uniformly continuous map "onto". By Theorem 0.6.5 (see [1]), the uniform space $(exp_n Y, exp_n V)$ is complete, and by Corollary 0.6.2 (see [1]) and Conclusion 0.6.4 (see [1]), the space $(exp_n Y, exp_n V)$ is τ-metrizable. From Theorem 1, it follows that $(exp_n X, exp_n U)$ is aτ-uniformly complete Čech space.

Corollary 3. For a uniform space (X,U), the following conditions are equivalent:

1. (X,U) is a uniformly complete Čech space;
2. $(exp_n X, exp_n U)$ is a uniformly complete Čech space.

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UNIFORMLY μ-u-PLUME SPACES

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The topological properties of plumeness and paracompactness are independent. However, in the class of uniform spaces, the property of uniform plumeness implies the property of uniform paracompactness. Uniformly plumeness spaces were introduced and studied by A.A. Borubaev. Uniformly plumeness spaces are sufficiently broad, containing all compact, uniformly locally compact, uniformly Čech - complete spaces, and metrizable spaces. In the class of uniform spaces, the general solution to the problem belongs to A.A. Borubaev: uniformly plume spaces, and only they are the images of metrizable uniform spaces under perfect mappings. From this theorem, the theorem of A.V. Arkhangelskii follows, characterizing plume paracompacts as perfect preimages of metric spaces. Naturally, the following question arises: which uniform spaces can be perfectly mapped onto μ-metrizable spaces? Therefore, it is very important to attempt to find a class of uniform spaces that could be perfectly mapped onto μ-metrizable spaces. The present work studies μ-u-uniformly plume spaces.

Keywords: uniformly plume spaces, uniformly Čech-complete, uniform covering, perfect mapping.

Канатчалуу жана паракомпакттуу топологиялык касиеттер көз карандысыз. Бирок, бир калыптуу мейкиндиктер классында бир калыптуу канатчалуу касиеттен бир калыптуу паракомпакттуулук касиет келип чыгат. Бир калыптуу канатчалуу мейкиндик түшүнүгүн А.А. Бөрүбаев киргизген жана изилдеген. Бир калыптуу канатчалуу мейкиндиктер бардык компактуу, бир калыптуу локалдуу компактуу, бир калыптуу Чех боюнча толук мейкиндик жана метрикалык мейкиндиктерди камтыйт. Бир калыптуу мейкиндиктер классында метризацияланган мейкиндиктредин жеткилең элеси жөнүндөгү маселенин жалпы чечилишин А.А. Бөрүбаевге таандык: бир калыптуу канатчаланган мейкиндиктер жана алар гана метризацияланган бир калыптуу мейкиндиктердин жеткилең элеси болуп саналат. Бул теоремадан А.В. Архангельскийдин канатчаланган паракомпакттар метрикалык мейкиндиктердин жеткилең кайра элеси катары мүнөздөлөт деген теоремасы келип чыгат. Төмөнкү суроо келип чыгат: кандай бир калыптуу мейкиндиктер μ-метризацияланган мейкиндикке жеткилең чагылдырылышы мүмкүн? Ошондуктан, бул μ-метрика мейкиндиктерге картада боло турган бир калыптуу мейкиндиктердин классын табууга аракет кылуу абдан маанилүү. Бул макалада μ-u-метрика бир калыптуу канатчалуу мейкиндиктери изилденилет.

Негизги сөздөр. бир калыптуу канатчаланган мейкиндиктер, Чех боюнча бир калыптуу толук мейкиндиктер, бир калыптуу жабдуу, жеткилең чагылдыруу.

Топологические свойства перистости и паракомпактности являются независимыми. Однако, в классе равномерных пространств свойство равномерной перистости влечет свойство равномерной паракомпактности. Равномерно перистые пространства были введены и изучены А.А. Борубаевым. Равномерно перистые пространства достаточно широки, они содержат все компактные, равномерно локально компактные, равномерно полные в смысле Чеха пространства и метризуемые пространства. В классе равномерных пространств общее решение проблемы о совершенных образах метризуемых пространств принадлежит А.А. Борубаеву: равномерные перистые пространства и только они являются образами метризуемых равномерных пространств при совершенных отображениях. Из этой теоремы следует теорема А.В. Архангельского о характеристике перистых паракомпактов как совершенных прообразов метрических пространств. Естественно вытекает следующая задача: какие равномерные пространства могут быть совершенно отображены на μ-метризуемые пространства? Поэтому весьма важна попытаться найти класс равномерных пространств, которые могли бы совершенно отображены на μ-метризуемые пространства. В настоящей работе изучаются μ-u-равномерно перистые пространства.

Ключевые слова. равномерно перистые пространства, равномерно полные по Чеху пространства, равномерное покрытие, совершенное отображение.

Let (X, U) be a uniform space and $H \subset U$ be a subsystem consisting of countable covers. A uniform space (X, U) is called $c - H$-complete, and the system H is called complete, if every H -Cauchy filter F has at least one point of contact, i.e., $\cap \{[M]: M \in F\} \neq \emptyset$.

Let (X, U) be a complete uniform space. The smallest cardinal number τ is called the c -index of the completeness of the uniform space (X, U), if there exists a system $H \subset U$ of countable covers such that $|H| = \tau$ and (X, U) are $c - H$-complete.

The c -index of the completeness of the uniform space (X, U) is denoted by $c - ic(U)$.

A uniform space (X, U) is called u-uniformly Čech–complete if it is complete and $c - ic(U) \leq \aleph_0$.

A uniform space (X, U) is called uniformly plume if there exists a pseudouniformity $P \subset U$ satisfying the following conditions:

1. $w(U) \leq \aleph_0$;
2. $\cap \{\alpha(x): \alpha \in P\} = K_x$ is compact for any $x \in X$;
3. The system $\{\alpha(K): \alpha \in P\}$ is a fundamental system of neighborhoods K_x in (X, τ_U) for each $x \in X$ [1].

Definition 1. A uniform space (X, U) is called $\mu - u$-uniformly plume if there exists a pseudouniformity $P \subset U$ consisting of covers of cardinality $\leq \mu$, satisfying the following conditions:

1. $w(U) \leq \aleph_0$;
2. $\cap \{\alpha(x): \alpha \in P\} = K_x$ is compact for any $x \in X$;
3. The system $\{\alpha(K): \alpha \in P\}$ is a fundamental system of neighborhoods K_x in (X, τ_U) for each $x \in X$.

$\aleph_0 - u$-uniformly plume spaces are called u-uniformly plume spaces.

Theorem 1. Every compact space (X, U) is $\mu - u$-uniformly plume. Proof follows from Definition 1.

Corollary 1. Every compact space (X, U) is u-uniformly plume.

A uniform space (X, U) is called $\mu - u$-metrizable if it has a countable basis consisting of covers of cardinality $\leq \mu$. Uniformly $\aleph_0$-metrizable spaces are called separably metrizable spaces.

Theorem 2. Every $\mu - u$-metrizable space (X, U) is $\mu - u$-uniformly plume. Proof follows from Definition 1.

Corollary 2. Every separably metrizable space (X, U) is u-uniformly plume.

Theorem 3. If a uniform space (X, U) is $\mu - u$-uniformly Čech–complete then (X, U) is $\mu - u$-uniformly plume.

Proof. Let (X, U) be a $\mu - u$-uniformly Čech–complete. Then, by Lemma 1.2.11 ([1]), there exists a pseudouniformity $P \subset U$ consisting of covers of cardinality $\leq \mu$, such that $w(U) \leq \aleph_0$ and the family P are complete. For each $x \in X$, let $K_x = \cap \{\alpha(x): \alpha \in P\}$ be defined. We aim to show that K_x is compact in (X, U). Let F_x be an arbitrary filter in K_x. Let $F = \{E \subset X : E \supset B_x$ be defined for some $B_x \in F_x\}$. Then F is a B -Cauchy filter in (X, U). Since P is a complete system, it follows that $\cap \{[E] : E \in F\} \neq \emptyset$. It is easy to see that K_x is closed in (X, U) and $\cap \{[E] : E \in F\} \subset \{[B_x] : B_x \in F_x\}$. Hence, F_x has a point of contact in K_x, meaning $K_x$is compact. Let G be an open set in (X, U) such that$G \supset K_x$. Let $S_\alpha = \alpha(K_x) \cap (X \backslash O)$ be defined. Suppose that $S_\alpha \neq \emptyset$for some $\alpha \in P$. Then the system $\lambda = \{S_\alpha : \alpha \in P\}$ is centered. Let $F = \{M \subset X : M \supset S_\alpha$ be defined for some $S_\alpha \in \lambda\}$. Then the filter F is a B -Cauchy filter in (X, U), and because P is complete, we have $\cap \{[M] : M \in F\} \neq \emptyset$. If $\alpha \ast\succ \beta$, then $[S_\alpha] \subset S_\beta$ so $\cap \{[M] : M \in F\} \subset \cap \{[S_\alpha] : \alpha \in P\} = \emptyset$, for some $B_x \in F_x\}$, leading to a contradiction. Therefore, $\{\alpha(x) : \alpha \in P\}$ is a fundamental system of neighborhoods for K_x in (X, τ_U) for each $x \in X$.

Corollary 3. If a uniform space (X, U) is u-uniformly Čech–complete, then (X, U) is u-uniformly plume.

Theorem 4. If for a uniform space (X, U) every pseudouniformity $P \subset U$ is μ -bounded, then the following conditions are equivalent:

1. The uniform space (X, U) is $\mu - u$-uniformly plume;
2. The uniform space (X, U) is uniformly plume.

Proof. 1. $\Rightarrow$ 2. Obvious.

Sufficiency. Let (X, U) be uniformly plume space. Then there exists a pseudouniformity $P \subset U$ satisfying conditions 1–3 of Definition 1.4.1 (see [1]). Due to the μ-boundedness of P, there exists a basis $B \subset P$ consisting of covers of cardinality $\leq \mu$. It is then easy to see that conditions 1–3 of Definition 1 are satisfied.

Corollary 4. If for a uniform space (X, U) every pseudouniformity $P \subset U$ is $\aleph_0$-bounded, then the following conditions are equivalent:

1. The uniform space (X, U) is u-uniformly plume;
2. The uniform space (X, U) is uniformly plume.

Theorem 5. For a uniform space (X, U) the following conditions are equivalent:

1. The uniform space (X, U) is u-uniformly plume;

2. The uniform space (X, U) can be mapped onto some separably metrizable uniform space (Y, V) by a perfect uniformly continuous mapping.

Proof. $1. \Rightarrow 2.$ Let the space (X, U) be $\mu - u$-uniformly plume. By Lemma 1.1.13 (see [1]), there exists a pseudouniformity $B \subset U$ consisting of covers of cardinality $\leq \mu$ such that $w(B) \leq \aleph_0$ and the system B is strongly B-complete. Moreover, for the pseudouniform space (X, B), there exist a uniform space (Y, V) and a uniformly continuous mapping "onto" $f: (X, B) \to (Y, V)$ such that the family $\{f^{-1}\gamma: \gamma \in V\}$ forms a basis for the pseudouniformity $B$and $w(B) \leq \aleph_0$.We will show that the mapping $f: (X, B) \to (Y, V)$ is perfect. Let $F_{f^{-1}y}$ be a filter in $f^{-1}y$and $y \in Y$. Since $f^{-1}y = \cap \{\alpha(f^{-1}y): \alpha \in B\}$ is uniformly continuous, F_X is a B -Cauchy filter in (X, B). By the B-completeness of(X, U), the filter F_X has at one cluster point $x \in f^{-1}y$. Clearly, $x \in f^{-1}y$ is also a cluster point of $F_{f^{-1}y}$, meaning that $f^{-1}y$ is compact. Now, let $y \in Y$ be an arbitrary point and let Obe an open set in (X, U) such that $O \supset f^{-1}y$. We will show that there exists a uniform cover $\alpha \in B$ such that $\alpha(f^{-1}y) \subset O$. Suppose, that $L_\alpha = \alpha(f^{-1}y) \cap (X \backslash O) \neq \emptyset$ for every $\alpha \in B$ and $y \in Y$. It is easy to see that $\eta = \{L_\alpha: \alpha \in B\}$ is a centered system. Put $F = \{C \subset X: C \supset L_\alpha, L_\alpha \in \eta\}$. Let F be the filter X. Let $\alpha \in B$ be an arbitrary uniform cover. Then there exists a uniform cover $\beta \in B$ such that $\beta *\succ \alpha$ i.e. that there exist $A \in \alpha$ and $x \in f^{-1}y$ such that $\beta(x) \subset A$. But $\beta(x) \subset \beta(f^{-1}y)$, hence $A \in F$, i.e., F is a B -Cauchy filter in (X, U). Since (X, U) is B-complete, it follows that $\cap \{[M]: M \in F\} \neq \emptyset$. But $\cap [[M]: M \in F\} \subset \cap \{[\alpha(f^{-1}y)] \cap (X \backslash O): \alpha \in B\} = \cap \{\alpha(f^{-1}y) \cap (X \backslash o): \alpha \in B\} = \emptyset$. This contradiction proves that $f$is a closed map. Since f is both compact and closed, it is a perfect map.

$2. \Rightarrow 1.$ It follows from Definition 1 and Lemma 1.1.13 (see [1]).

Corollary 5. For a uniform space (X, U), the following conditions are equivalent:

1. The uniform space (X, U) is u-uniformly plume;

2. The uniform space (X, U) can be mapped onto some separably metrizable uniform space (Y, V) by a perfect uniformly continuous map.

Theorem 6. For a uniform space (X, U), the following conditions are equivalent:

1. (X, U) is $\mu - u$-uniformly plume;
2. $(exp_c X, exp_c U)$ is $\mu - u$-uniformly plume.

Proof. Let $(exp_c Y, exp_c V)$ be a $\mu - u$-uniformly plume. Then it is easy to see that the space (X, U) is $\mu - u$-uniformly plume. Conversely, suppose the space (X, U) is $\mu - u$-uniformly plume. Then the space (X, U) is perfectly mapped onto some μ-metrizable space (Y, V). It is known that the map $exp_c f : (exp_c X, exp_c U) \to (exp_c Y, exp_c V)$, $exp_c(f)(F) = f(F), F \in exp_c X$, is a perfect uniformly continuous map "onto". Thus $(exp_c X, exp_c U)$ is $\mu - u$-uniformly plume.

Corollary 6. For a uniform space (X, U), the following conditions are equivalent:

1. (X, U) is u-uniformly plume.
2. $(exp_c X, exp_c U)$ is u-uniformly plume.

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ABOUT UNIFORMLY LINDELÖF SPACES

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Lindelöf spaces occupy a central place in General Topology. There have been some variants in defining uniformly Lindelöfness of uniform spaces. For example, uniform B-Lindelöfness in the sense of A.A. Borubaev, uniform A-Lindelöfness in the sense of L.V. Aparina, uniform I-Lindelöfness in the sense of J. Isbell. L.V.Aparina showed, that all uniformly A-Lindelöf spaces are complete. From this follows that non complete separable metrizable uniform space are not uniformly A-Lindelöf. It is worthwhile mentioning that all uniformly A-Lindelöf spaces are uniformly B-Lindelöf and, consequently, the properties (B) of all uniformly B-Lindelöf spaces is wider than the properties (A) of all uniformly A-Lindelöf spaces. In this article we show a new approach to the definition of a uniform analogue of Lindelöfness. We introduce uniform Lindelöf space and investigates their connections with other uniform properties.

Keywords. finitely-additive open cover, star finite uniform cover, $\aleph_0$ -boundedness, Lindelöfness, uniformly Lindelöfness.

Линделёфтук мейкиндиктер жалпы топологияда борбордук орунду ээлешет. Бир калыптуу мейкиндиктердин бир калыптуу Линделёфтуулугунун айрым түрлөрү бар. Мисалы, А.А. Бөрүбаевдин маанисиндеги бир калыптуу B-линделёфтуулук, Л.В. Апаринанын маанисиндеги бир калыптуу A-линделёфтуулук, Дж. Исбеллдин маанисиндеги I-линделёфтуулук. Л.В.Апарина бардык бир калыптуу A-линделёфтук мейкиндиктер толук экенин далилдеген. Мындан толук эмес сепарабелдүү метризацияланган мейкиндиктер бир калыптуу A -линделёфтуу эмес экени келип чыгат. Бардык бир калыптуу A-линделёфтук мейкиндиктер бир калыптуу B -линделёфтуу болгондуктан бардык бир калыптуу B -линделёфтук мейкиндиктердин (B) классы бардык бир калыптуу A -линделёфтук мейкиндиктердин (A) классынан кенен болот. Бул макалада Линделёфтуулуктун аныктамасынын жаңы түрү сунушталат. Бир калыптуу Линделёфтук мейкиндик киргизилет жана анын башка бир калыптуу касиеттер менен болгон байланыштары изилденент.

Негизги сөздөр.чектүү-аддитивдүү ачык жабдуу, жылдыздуу чектүү бир калыптуу жабдуу, $\aleph_0$-чектелгендик, Линделёфтуулук, бир калыптуу Линделёфтуулук.

Линделёфовы пространства занимают центральное место в общей топологии. Существуют некоторые варианты определения равномерной Линделёфовости равномерных пространств. Например, равномерно B-линделёфовость в смысле А.А. Борубаева, равномерно

A-линделёфовость в смысле Л.В. Апариной, равномерная I-линделёфовость в смысле Дж. Исбелла. Л.В. Апарина доказала, что все равномерно A-линделёфовы пространства полны. Отсюда следует, что не полные сепарабельно метризуемые пространства не являются равномерно A-линделёфовыми. Следует отметить, что все равномерно A-линделёфовы пространства равномерно B-линделёфовы и, следовательно, класс (B) всех равномерно B-линделёфовых пространств шире, чем класс (A) всех равномерно A -линделёфовых пространств. В данной статье мы предлагаем новый подход к определению равномерного аналога Линделёфовости. Вводятся равномерно Линделёфовы пространства и изучается их связь с другими равномерными свойствами.

Ключевые слова. конечно-аддитивное открытое покрытие, звездно конечное равномерное покрытие, $\aleph_0$-ограниченность, Линделёфовость, равномерная Линделёфовость.

Throughout this work all uniform spaces are assumed to be Hausdorff and mappings are uniformly continuous.

For coverings α and β of the set X, the symbol $\alpha \succ \beta$ means that the covering α is a refinement of the covering β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, for coverings α and β of a set X, we have: $\alpha \wedge \beta = \{A \cap B: A \in \alpha, B \in \beta\}$. The covering α is finitely-additive if $\alpha^{\angle} = \alpha$, $\alpha^{\angle} = \{\cup\, \alpha_0: \alpha_0 \subset \alpha \text{ is finite}\}$. $\alpha(x) = \cup\, St(\alpha, x)$, $St(\alpha, x) = \{A \in \alpha: A \ni x\}$, $x \in X$, $\alpha(M) = \cup\, St(\alpha, M)$, $St(\alpha, M) = \{A \in \alpha: A \cap M \neq \emptyset\}$, $M \subset X$.

A uniform space (X, U) is called uniformly A-Lindelöf, if for each open covering α exist a countable uniformly covering $\beta = \{B_n: n \in N\}$ and $\gamma \in U$ such that $\beta \succ \alpha^{\angle}$ and $\gamma(\bar{B}_n) \subset B_{n+1}$ for any $n \in N$ [2]; a uniform space (X, U) is called uniformly B-Lindelöf, if it is both uniformly B-paracompact and $\aleph_0$-bounded [1]; a uniform space (X, U) called uniformly$I$-Lindelöf, if every uniform cover has a locally finite uniform refinement [3]; a uniform space (X, U) called uniformly$A$-paracompact, if every open covering has a locally finite uniform refinement [1]; a uniform space (X, U) called strongly uniformly$A$-paracompact, if every open covering has a star finite uniform refinement [4]; a uniform space (X, U) called uniformly B-paracompact, if for each finitely-additive open covering λ of (X, U) there exists such sequence uniform covering $\{\alpha_i: i \in N\} \subset U$, that following condition is realized: for each point $x \in X$

there exist such number $i \in N$and $L \in \lambda$that $\alpha_i(x) \subset L$ (U) [1]; a uniform space (X, U) is called $\aleph_0$-bounded if the uniformity U has a base consisting of countable coverings [1]; a uniform space (X, U) is called strongly uniformly locally compact if the uniformity of U contains a locally finite uniform covering whose closure of each element is compact [2]; a uniform space $(X, U)$is called uniformly locally Lindelöf if the uniformity of U contains a uniform covering whose closure of each element is Lindelöf [5]; a uniformly continuous mapping $f: (X, U) \to (Y, V)$ of uniform space (X, U) onto a uniform space (Y, V) is called a precompact, if for each $\alpha \in U$ there exist a uniform covering $\beta \in V$ and finite uniform covering $\gamma \in U$, such that $f^{-1}\beta \wedge \gamma \succ \alpha$[1]; a uniformly continuous mapping $f: (X, U) \to (Y, V)$ of uniform space (X, U) onto a uniform space (Y, V) is called a uniformly perfect, if it is both precompact and perfect [1]; a uniformly continuous mapping $f: (X, U) \to (Y, V)$ of uniform space (X, U) onto a uniform space (Y, V) is called a uniformly open, if f maps each open uniform covering $\alpha \in U$ to an open uniform covering $f\alpha \in V$ [1]; for the uniformity U by τ_U we understand the topology generated by this uniformity; for the Tychonoff space X by $U_X$we understand a universal uniformity.

Let (X, U)be a uniform space.

Definition 1. A uniform space (X, U) is said to be uniformly Lindelöf, if it is both strongly uniformly A-paracompact and $\aleph_0$-bounded.

Theorem 1. If (X, U) is a uniformly Lindelöf space then its topological space (X, τ_U) is uniformly Lindelöf. Conversely, if (X, τ) is Lindelöf topological space, then the uniform space (X, U_X) is uniformly Lindelöf.

Proof. Let α be an arbitrary finitely-additive open cover of (X, τ_U) . Then for covering α exists a star finite uniform covering β such that $\beta \succ \alpha$. Since the space (X, U) is $\aleph_0$-bounded the covering β contain countable uniformly subcovering β_0. Then for any $i \in N$ we have $B_i \subset \overset{j=1}{\underset{k}{\cup}} A_j$. The system $\{B_i \cap A_j\}$, $i = 1,2, \dots, n$, $j = 1,2, \dots, k$ forms a countable open covering which is refinement in covering α. Consequently, the space (X, τ_U) is Lindelöf.

Conversely, if (X, τ) is Lindelöf, then the system of all open coverings forms a base of universal uniformity $U_X$of the space (X, τ). It follows from this that $(X, U_X)$is uniformly Lindelöf.

Theorem 2. Any compact space is uniformly Lindelöf.

Proof. Since any compact uniform space is precompact and strongly uniformly A-paracompact. Consequently, the uniform space (X, U) is uniformly Lindelöf.

Theorem 3. Any uniformly Lindelöf space is strongly uniformly$A$-paracompact.

Proof. Implies from the definitions of uniformly Lindelöfness.

Corollary 1. Any uniformly Lindelöf space is uniformly A-paracompact.

Theorem 4. Any closed subspace of a uniformly Lindelöf space is uniformly Lindelöf.

Proof. Let (M, U_M) be a subspace of (X, U). Note that space (M, U_M) is $\aleph_0$-bounded. Let α_M be an arbitrary finitely-additive open cover of the subspace (M, U_M). Then there exits finitely-additive open family $\alpha$of (X, U), such that $\alpha \wedge \{M\} = \alpha_M$. Denote $\beta = \{\alpha, X \backslash M\}$. Obviously the covering β is finitely-additive open covering of (X, U).Then exist star finite uniform covering $\lambda \in U$ such that $\lambda \succ \beta$. It is implies$\lambda_M \in U_M$, hence $\lambda_M \succ \alpha_M$. Then it is easy to see that λ_M is a star finite uniform cover of (M, U_M). Thus, (M, U_M) is uniformly Lindelöf.

Corollary 2. Any uniformly Lindelöf space is uniformly locally Lindelöf.

Corollary 3. The real space R with natural uniformity is uniformly Lindelöf.

Proof. It is clear that R is $\aleph_0$-bounded. Let α be a finitely-additive open cover of R. Put $\beta = \{(n-1, n+1): n = 0, \pm 1, \pm 2, \ldots\}$. Then $\beta$is star finite uniform cover of R. Since $[n-1, n+1]$ is compact, there exists a finite family $\{A_1, A_2, \ldots, A_n\}$, $A_i \in \alpha$, $i = 1,2, \ldots, n$ such that $(n-1, n+1) \subset [n-1, n+1] \subset \overset{i=1}{\underset{n}{\cup}} A_i$. Therefore, $\beta \succ \alpha^{\angle}$. Hence, R is uniformly Lindelöf.

Proposition 4. Any uniformly Lindelöf space is uniformly B-Lindelöf.

Proof. Let (X, U) be a uniformly Lindelöf space. As is known, every strongly uniformly A-paracompact space is strongly uniformly B-paracompact [4]. Hence, the uniform space is uniformly B-Lindelöf.

Proposition 4. Any uniformly B-Lindelöf space is uniformly I-Lindelöf.

Proof. Let (X, U) be a uniformly B-Lindelöf space. Then it follows from the definition of uniformly B-Lindelöfness [1] that (X, U) is a uniformly I-Lindelöf.

Corollary 4. Any uniformly Lindelöf space is uniformly I-Lindelöf.

Proposition 5. Any uniformly Lindelöf space is complete.

Proof. Let (X, U) be a uniformly Lindelöf space. The completeness of (X, U) follows from the fact that every strongly uniformly A-paracompact space is complete [4].

Non complete separable uniform space is not uniformly Lindelöf. For example, the space $(0,1)$ is a uniformly B-Lindelöf, but not uniformly Lindelöf.

Lemma 1. Let $f: (X, U) \to (Y, V)$ be a precompact mapping of a uniform space (X, U) to a uniform space (Y, V). If a space (Y, V) is $\aleph_0$-bounded, then (X, U) is also $\aleph_0$-bounded.

Proof. Let $f: (X, U) \to (Y, V)$ be a precompact mapping of a uniform space (X, U) to a uniform space (Y, V) and $\alpha \in U$ be an arbitrary uniformly covering. Then by virtue of the precompactness of $f$there exist such a countable covering $\beta \in V$and finite covering $\gamma \in U$ that $f^{-1}\beta \wedge \gamma \succ f^{-1}\alpha$. But the covering $f^{-1}\beta \wedge \gamma$is countable. Therefore the space (X, U) is $\aleph_0$-bounded.

Lemma 2. Let $f: (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) to a uniform space (Y, V). If β is star finite uniform covering of the space (Y, V), then $f^{-1}\beta$is star finite uniform covering of the space (X, U).

Proof. Let f be a uniformly continuous mapping of a uniform space (X, U) to a uniform space (Y, V) and β be a star finite uniform covering of (Y, V). Then, for each element$B \in \beta$, there exist elements $B_i \in \beta$, $i = 1,2, \dots, n$, such that $B \subset \bigcup_{n}^{i=1} B_i$. From the uniform continuity of the mapping f it follows that the covering $f^{-1}\beta$ is uniform, i.e. $f^{-1}\beta \in U$. Consequently,$f^{-1}B \subset \bigcup_{n}^{i=1} f^{-1}B_i, f^{-1}B_i \in f^{-1}\beta$. Thus, $f^{-1}\beta$is star finite uniform covering of (X, U).

Lemma 3. Let $f:(X,U)\to(Y,V)$ be a uniformly continuous mapping of a uniform space (X,U) to a uniform space (Y,V). If a space (X,U) is $\aleph_0$-bounded, then (Y,V) is also $\aleph_0$-bounded.

Proof. Let $f:(X,U)\to(Y,V)$ be a uniformly continuous mapping of a uniform space (X,U) to a uniform space (Y,V) and $\beta\in V$ be an arbitrary uniformly covering. Then $f^{-1}\beta\in U$. According to Proposition 1.1.6. [1, p. 40] the covering $f^{-1}\beta$ has a countable subcover $f^{-1}\beta_0$. Then β_0 is countable subcover of β. Therefore the space (Y,V) is $\aleph_0$-bounded.

Theorem 5. Let $f:(X,U)\to(Y,V)$ be a uniformly perfect mapping of a uniform space (X,U) onto a uniform space (Y,V). Then uniformly Lindelöf space preserved in the preimage direction.

Proof. According to Lemma 1 the space (X,U) is $\aleph_0$-bounded. Let a uniform space (Y,V) be a uniformly A-paracompact and αbe an arbitrary finitely-additive open covering of (X,U). Then $\{f^{-1}y: y\in Y\}\succ\alpha$. Since f is a closed map, it follows that $\lambda=\{f^{\#}A: A\in\alpha\}$ is finitely-additive open cover of (Y,V), $f^{\#}A=Y\backslash f(X\backslash A)$. By virtue of the Lindelöfness of the space (Y,V) there exist such a star finite covering $\beta\in V$ that $\beta\succ\lambda$. Then $f^{-1}\beta\succ\lambda$and according to Lemma 2 the cover $f^{-1}\beta$ is star finite. Consequently, (X,U) is uniformly Lindelöf.

Theorem 6. Let $f:(X,U)\to(Y,V)$ be a uniformly open mapping of a uniform space (X,U) onto a uniform space (Y,V). If (X,U) is uniformly Lindelöf space then uniform space (Y,V) is uniformly Lindelöf.

Proof. Let f be a uniformly open mapping of a uniform space (X,U) onto a uniform space (Y,V) and αbe an arbitrary finitely-additive open covering. Then $f^{-1}\alpha$ is a finitely-additive open covering of the space (X,U) and due to the uniformly A-paracompactness we refined a star finite uniformly open cover $\beta\in U$in the cover $f^{-1}\alpha$. Since f is openness, it follows that $f\beta\succ\alpha$. Hence, (Y,V) is strongly uniformly A-paracompact. Note that (Y,V) is $\aleph_0$-bounded. Consequently, the space (Y,V) is uniformly Lindelöf.

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UNIFORMLY PARACOMPACT AND COUNTABLE UNIFORMLY PARACOMPACT SPACES

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It is well known that paracompactness plays an important role in General Topology. Therefore, the finding of uniform analogues of paracompact and countable paracompact spaces is an important and interesting problem in the theory of uniform spaces. Many mathematicians turned to this problem and as a result, different variants of uniform paracompactness of uniform spaces appeared. For example, the following types of uniform paracompactness are introduced and investigated: uniform R-paracompactness in the sense of M.D. Rice, uniform F-paracompactness in the sense of Z. Frolik, uniform B-paracompactness in the sense of A.A. Borubaev, uniform P-paracompactness in the sense of B.A. Pasynkov, uniform A-paracompactness in the sense of L.V. Aparina, uniform I-paracompactness in the sense of J.R. Isbell. In this work we introduce and study uniformly τ-finally paracompact and countable uniformly τ-finally paracompact spaces. In particular, the characterizations of uniformly τ-finally paracompact spaces by using of Hausdorff compact extensions and mappings are obtained.

Key words: *uniformly τ-finally paracompactness, countable uniformly τ-finally paracompactness, finitely additive open covering, σ-locally finite uniform covering, covering cardinality $\leq \tau$.*

Паракомпактар жалпы топологияда маанилүү ролду ойной тургандыгы жакшы белгилүү. Ошондуктан паракомпактуу жана санактуу паракомпактуу мейкиндиктердин бир калыптуу аналогдорун табуу бир калыптуу мейкиндиктер теориясынын маанилүү жана кызыктуу маселеси болуп саналат. Бир нече математиктер бул маселеге кайрылышып натыйжасында бир калыптуу мейкиндиктердин бир калыптуу паракомпактуулугунун түрдүү варианттары пайда болгон. Мисалы, бир калыптуу паракомпактуулуктун төмөнкүдөй типтери киргизилген жана изилденген: М.Д. Райстын маанисиндеги бир калыптуу R -паракомпактуулук, З. Фроликтин маанисиндеги бир калыптуу F-паракомпактуулук, А.А. Бөрүбаевдин маанисиндеги бир калыптуу B-паракомпактуулук, Б.А. Пасынковдун маанисиндеги бир калыптуу P-паракомпактуулук, Л.В. Апаринанын маанисиндеги бир калыптуу A-паракомпактуулук, Дж.Р. Исбеллдин маанисиндеги бир калыптуу I-паракомпактуулук. Бул макалада бир калыптуу τ-финалдык паракомпактуулук *жана санактуу бир калыптуу τ-финалдуу паракомпактуулуктар киргизилген жана изилденген. Маселен, бир калыптуу τ-финалдуу паракомпактуу мейкиндиктердин Хаусдорфтук кеңейүү жана чагылдыруулар боюнча мүнөздөмөлөрү табылган.*

Негизги сөздөр: *бир калыптуу τ-финалдуу-паракомпактуулук, санактуу бир калыптуу τ-финалдуу-паракомпактуулук, чектүү аддитивдүү ачык жабдуу, σ-локалдуу чектүү бир калыптуу жабдуу, $\leq \tau$ кубаттуулуктагы жабдуу.*

Хорошо известно, что паракомпакты играют важную роль в общей топологии. Поэтому, нахождение равномерных аналогов паракомпактных и счетно паракомпактных пространств является важной и интересной задачей теории равномерных пространств. Многие математики обратились к этой задаче, и в результате получили различные варианты равномерных паракомпактов равномерных пространств. Например, найдены и исследованы следующие типы равномерных паракомпактов: равномерная R -паракомпактность в смысле М.Д. Райса, равномерная F-паракомпактность в смысле З. Фролика, равномерная B-паракомпактность в смысле А.А. Борубаева, равномерная P -паракомпактность в смысле Б.А. Пасынкова, равномерная A-паракомпактность в смысле Л.В. Апариной, равномерная I-паракомпактность в смысле Дж.Р. Исбелла. В данной статье вводятся и исследуются равномерно τ-финально паракомпактные и счетно равномерно τ-финально паракомпактные прострнаства. В частности, при помощи Хаусдорфовых расширений и отображений найдены характеристики равномерно τ-финально паракомпактных пространств.

Keywords: *равномерно τ-финальная паракомпактность, счетно равномерно τ-финальная паракомпактность, конечно аддитивное открытое покрытие, σ-локально конечное равномерное покрытие, покрытие мощности $\leq \tau$.*

For covers α and β of a set X, we have: $\alpha \wedge \beta = \{A \cap B: A \in \alpha, B \in \beta\}.H \subset X$. For covers α and β of the set X, the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$such that $A \subset B$.For a covering α of X we put$\alpha^{\angle} = \{\cup \alpha_0: \alpha_0 \subset \alpha$ is finite$\}$. The covering α is finitely additive if $\alpha^{\angle} = \alpha$.

Definition 1. A uniform space (X, U) is called uniformly τ-finally paracompact if every finitely additive open cover has a σ-locally finite uniform refinement of cardinality $\leq \tau$.

Proposition 1. If (X, U) is uniformlyτ-finally paracompact space, then the topological space (X, τ_U) is τ-finally paracompact. Conversely, if (X, τ) is τ-finally paracompact, then the uniform space (X, U_X), where U_X is the universal uniformity, is uniformly τ-finally paracompact.

Proof. Let αbe an arbitrary open covering of the space (X, τ_U). Then for a finitely additive open covering $\alpha^{\angle}$ of (X, U) there exists a σ-locally finite uniform covering cardinality $\leq \tau\beta \in U$ is a refinement of it. Let $\gamma = \langle\beta\rangle$. It is clear that $\gamma = \{\gamma_i : i \in N\}$ is σ-locally finite open uniform covering of (X, U). For each $\Gamma \in \gamma$ choose $A_\Gamma \in \alpha^{\angle}$ such that $\Gamma \subset A_\Gamma$, where $A_\Gamma = \cup_{i=1}^{n} A_i$,$A_i \in \alpha$,$i = 1,2,\ldots,n$. Let $\alpha_0 = \cup \{\alpha_\Gamma : \Gamma \in \gamma\}$, $\alpha_\Gamma = \{\Gamma \cap A_i : i = 1,2,\ldots,n\}$. Then $\alpha_0$is a σ-locally finite open covering cardinality $\leq \tau$of (X, τ_U). This implies $\alpha_0 \succ \alpha$. Thus,(X, τ_U) is τ-finally paracompact.

Conversely, let (X, τ) be aτ-finally paracompact space. Then set of all open coverings forms the base of universal uniformity U_X of (X, τ). It is easy to see that the uniform space (X, U_X) is uniformly τ-finally paracompact.

The following theorem gives a characteristic of uniformτ-finally paracompactness in the spirit of Tamano.

Theorem 1. Let (X, U) be a uniform space and bX be a certain compact Hausdorff extensions. The uniform space $(X, U)$is uniformly τ-finally paracompact, if and only if for each compact $K \subset bX \backslash X$there exists a σ-locally finite uniform covering cardinality $\leq \tau\alpha \in U$ such that $[A]_{bX} \cap K = \emptyset$for all $A \in \alpha$.

Proof. Necessity. Let (X, U) be a uniformly τ-finally paracompact space and $K \subset bX \backslash X$be an arbitrary compact set. Then for each point $x \in X$ there is an open neighborhood $O_x$inbXsuch that $[O_x]_{bX} \cap K = \emptyset$. It is clear that $\gamma = \{O_x \cap X : x \in X\}$is an open covering of (X, U). We form an open covering $\gamma^{\angle}$of (X, U), taking as elements of γ. Then $\gamma^{\angle}$ is finitely additive open covering of (X, U). According to the condition of the theorem there exist a σ-locally finite uniform covering cardinality $\leq \tau\beta \in U$ such that$\beta \succ \gamma^{\angle}$. Then $[B]_{bX} \subset [\cup_{i=1}^{n} (O_{X_i} \cap X)]_{bX} \subset \cup_{i=1}^{n} [O_{X_i}]_{bX}$. Since $[O_{X_i}]_{bX} \cap K = \emptyset$ for each $i = 1,2,\ldots,n$, then $[B]_{bX} \cap K = \emptyset$ for each $B \in \beta$.

Sufficiency. Let α be an arbitrary finitely additive open covering of the space (X, U). Then there is an open family β in bX such that $\beta \wedge \{X\} = \alpha$. Let $K = bX \backslash \cup \beta$. It follows that K is a compactum. Then, by the condition of the theorem, there exists a

σ-locally finite uniform covering cardinality $\leq \tau\gamma \in U$such that $[\Gamma]_{bX} \cap K = \emptyset$ for any $\Gamma \in \gamma = \{\gamma_i : i \in N\}$. Since $[\Gamma]_{bX}$is a compact in bX there is $B_1, B_2, \ldots, B_n \in \beta$ such that $[\Gamma]_{bX} \subset \overset{n}{\underset{i=1}{\cup}} B_i$. Then $\Gamma \subset \overset{n}{\underset{i=1}{\cup}} A_i$, where $\overset{n}{\underset{i=1}{\cup}} A_i \in \alpha$. Consequently, (X, U) is a uniformly τ-finally paracompact space.

A uniform space (X, U) is called strongly uniformly locally compact, if there exists a σ - locally finite uniform covering cardinality $\leq \tau$ consisting of compact subsets.

Proposition 2. Any strongly uniformly locally compact space is uniformly τ-finally paracompact.

Proof. Let $\boldsymbol{\alpha}$ be an arbitrary finitely additive open covering. Then there exists a σ - locally finite uniform covering β cardinality $\leq \tau$ consisting of compact subsets. It is easy to see that the covering β is a refinement of finitely additive open covering α. Hence, β is σ-locally finite uniform covering cardinality $\leq \tau$.

Proposition 3. Any uniformly τ-finally paracompact space is complete.

Proof. It follows from the fact that every uniformly R -τ-finally paracompact space is complete.

Uniform τ-finally paracompactness is preserved when passing to a closed subspace and any disjoint sum of uniform spaces.

Theorem 2. For uniformly perfect mappings uniformly τ-finally paracompactness is preserved in the direction of the preimage.

Proof. Let α be an arbitrary finitely additive open covering of space (X, U). It is clear that the covering $\{f^{-1}y : y \in Y\}$ is refinement in the covering α. Then $\beta = f^{\#}\alpha = \{f^{\#}A : A \in \alpha\}$, where $f^{\#}A = Y \backslash f(X \backslash A)$, is the open covering of the space (Y, V). Considering all possible finite unions of sets of β, we construct an open covering $\beta^{\angle}$. By the condition of the theorem there exist a σ - locally finite uniform covering $\gamma \in V$ cardinality $\leq \tau$ such that $\gamma \succ \beta^{\angle}$. It is easy to see that the covering $f^{-1}\beta^{\angle}$ is a refinement in the covering α. The covering $f^{-1}\gamma$ is a σ-locally finite uniform covering cardinality $\leq \tau$ of the space (X, U). Obviously,$f^{-1} \succ \alpha$. Hence, (X, U) is uniformly τ-finally paracompact.

The following theorem is a uniform analog of Dowker's theorem [8].

Theorem 3. A uniform space (X, U) is uniformly τ-finally paracompact if and only if for every finitely additive open covering $\omega$of space (X, U) there exists a uniformly continuous ω -mapping $f: (X, U) \to (Y, V)$ of a uniform space (X, U) onto a metrizable uniformly τ-finally paracompact space (Y, V).

Proof. Necessity. Let a (X, U) be a metrizable uniformly τ-finally paracompact space and ω be an arbitrary finitely additive open covering. Then the identity map of a space (X, U) is the required uniformly continuous ω -mapping of a space (X, U) to a metrizable uniformly τ-finally paracompact space.

Sufficiency. Let ω be an arbitrary finitely additive open covering of the space (X, U). Then there exists a uniformly ω -continuous mapping $f: (X, U) \to (Y, V)$ of a uniform space (X, U) onto some metrizable uniformly τ-finally paracompact space (Y, V). For each point $y \in Y$there exists a neighborhood $O_y$whose preimage $f^{-1}O_y$is contained in some element of the covering ω. Let $\beta = \{O_y: y \in Y\}$. We form an open covering $\beta^{\angle}$consisting of all possible finite unions of covering elements of β. There exist aσ-locally finite uniform covering $\gamma \in V$cardinality $\leq \tau$ such that$\gamma \succ \beta^{\angle}$. Then covering $f^{-1}\gamma$is a refinement of the covering ω of a uniform space (X, U). We show that $f^{-1}\gamma$is aσ-locally finite uniform covering cardinality $\leq \tau$. Let $f^{-1}\Gamma \in f^{-1}\gamma$ be an arbitrary element and $\Gamma \in \gamma$meets only with a finite number elements of γ . Then element $f^{-1}\Gamma$of the covering $f^{-1}\gamma$also meets only with a finite number elements of $f^{-1}\gamma$. Thus, (X, U) is uniformly τ-finally paracompact.

Theorem 4. The product of a uniformly τ-finally paracompact uniform space (X, U) onto a compact uniform space (Y, V) is uniformly τ-finally paracompact.

Proof. Let a (X, U) be a uniformly τ-finally paracompact space and (Y, V) be a compact uniform space. It is known [1] that the projection $\pi_x: (X, U) \times (Y, V) \to (X, U)$ is uniformly perfect. Then ω is mapping of the product $(X, U) \times (Y, V)$ onto a uniformly τ-finally paracompact space (Y, V) for any finitely additive open covering ω of $(X, U) \times (Y, V)$. Therefore, according to Theorem 3 the uniform space $(X, U) \times (Y, V)$ is uniformly τ-finally paracompact.

Recall, a uniform space (X, U) is called a strongly uniformly τ-finally paracompact if every finitely additive open cover has a star finite uniform refinement of cardinality $\leq \tau$.

Theorem 5. For a uniform space (X, U) the following are equivalent:

1) (X, U) is strongly uniformly τ-finally paracompact;
2) (X, U) is uniformly τ-finally paracompact and the topological space (X, τ_U) is strongly τ-finally paracompact.

Proof. 1) ⇒2) It is obvious.

2) ⇒1). Let α be an arbitrary finitely additive open covering of (X, U). We refine a star finite open covering β cardinality $\leq \tau$ in it. We form a covering $\beta^{\angle}$ consisting of all possible finite unions of covering elements of β. Then $\beta^{\angle}$ is a finitely additive open covering. It is easy to see that $\beta^{\angle}$ is a star finite covering. The covering $\beta^{\angle}$ has refinement a σ-locally uniform covering $\gamma \in U$cardinality $\leq \tau$. Therefore, the star finite uniform covering $\beta^{\angle}$ is a refinement of the finitely additive open covering α. Thus, a uniform space (X, U) is strongly uniformly τ-finally paracompact.

Proposition 4. Any uniformly τ-finally paracompact space (X, U) whose topological space is locally compact is strongly uniformly τ-finally paracompact.

Proposition 5. A locally compact uniform space (X, U) is uniformly τ-finally paracompact if and only if the uniform space (X, U) is strongly uniformly locally compact.

Proof. Necessity. Let (X, U) be a uniformly τ-finally paracompact and a (X, τ_U) be a locally compact. Then each of its points $x$has an open neighborhood O_x, whose closure $[O_x]$ is compact. It is clear that the family $\alpha = \{O_x : x \in X\}$ forms an open covering of space (X, U). We form a covering $\alpha^{\angle}$consisting of all possible finite unions of the covering α. Obviously, a$\alpha^{\angle}$ is finitely additive open covering. There exist a σ -locally finite uniform covering $\beta \in U$ cardinality $\leq \tau$ such that $\beta \succ \alpha^{\angle}$. Each $B \in \beta = \{\beta_i : i \in N\}$is contained in the set $\cup_{i=1}^{n} O_{x_i}$, due to the monotony of the closure $[B] \subset \cup_{i=1}^{n} O_{x_i}$, it is therefore compact $[B]$, as a closed subset of the compact set

$[\cup_{i=1}^{n} \quad O_{x_i}] = \cup_{i=1}^{n} \quad [O_{x_i}]$. So, $[\beta] = \{[B]: B \in \beta\}$ is a σ-locally finite uniform covering consisting of compact subsets. Hence, (X, U) is uniformly locally compact.

Sufficiency. Let γ be a locally finite uniform covering cardinality $\leq \tau$consisting of compact subsets. Then for any finitely additive open covering αwe have $\gamma \succ \alpha^{\angle} = \alpha$. Consequently, (X, U) is uniformly τ-finally paracompact.

Definition 2. Uniform space (X, U) is called countable uniformly τ-finally paracompact if every finitely additive open countable cover has a σ-locally finite uniform refinement of cardinality $\leq \tau$.

Proposition 6. If (X, U) is countable uniform τ-finally paracompact space, then the topological space (X, τ_U) is countable τ-finally paracompact. Conversely, if (X, τ) is countableτ-finally paracompact, then the uniform space (X, U_X), where U_X is the universal uniformity, is countable uniformly τ-finally paracompact.

Proof. Let α be an arbitrary countable open covering of (X, τ_U). Then for a countable finitely additive open covering $\alpha^{\angle}$ of (X, U) there exists a σ-locally finite uniform covering cardinality $\leq \tau\beta \in U$ such that $\beta \succ \alpha^{\angle}$. Let $\gamma = \langle\beta\rangle$. It is clear that $\gamma = \{\gamma_i : i \in N\}$ is σ-locally finite open uniform covering of space (X, U). For each $\Gamma \in \gamma$ choose $A_\Gamma \in \alpha^{\angle}$ such that $\Gamma \subset A_\Gamma$, where $A_\Gamma = \cup_{i=1}^{n} A_i$, $A_i \in \alpha$,$i = 1,2,\ldots,n$. Let $\alpha_0 = \cup \{\alpha_\Gamma : \Gamma \in \gamma\}$, $\alpha_\Gamma = \{\Gamma \cap A_i : i = 1,2,\ldots,n\}$. Then α_0 is a σ-locally finite open covering cardinality $\leq \tau$ of the space (X, τ_U). This implies $\alpha_0 \succ \alpha$. Thus, (X, τ_U) is countable τ-finally paracompact.

Conversely, let (X, τ) be countableτ-finally paracompact space. Then set of all open coverings forms the base of universal uniformity U_X of (X, τ). It is easy to see that the uniform space (X, U_X) is countable uniformly τ-finally paracompact.

Proposition 7. Any strongly uniformly locally compact space is countable uniformlyτ-finally paracompact.

Proof. Let $\boldsymbol{\alpha}$ be an arbitrary countable finitely additive open covering. Then there exists aσ - locally finite uniform covering βcardinality $\leq \tau$ consisting of compact subsets. It is easy to see that the covering β is a refinement of finitely additive open covering α. Hence, β isσ-locally finite uniform covering cardinality $\leq \tau$.

Proposition 8. Any uniformly τ-finally paracompact space is countable uniformly τ-finally paracompact.

Uniform τ-finally paracompactness is preserved when passing to a closed subspace and any disjoint sum of uniform spaces.

Theorem 6. For uniformly perfect mappings uniformly τ-finally paracompactness is preserved in the direction of the preimage.

Proof. Let α be an arbitrary countable finitely additive open covering of space (X, U). It is clear that the covering $\{f^{-1}y: y \in Y\}$ is refinement in the covering α. Then $\beta = f^{\#}\alpha = \{f^{\#}A: A \in \alpha\}$, where $f^{\#}A = Y \backslash f(X \backslash A)$, is the countable open covering of the space (Y, V). Considering all possible finite unions of sets of β, we construct an open covering $\beta^{\angle}$. By the condition of the theorem there exist a σ - locally finite uniform covering $\gamma \in V$ cardinality $\leq \tau$ such that $\gamma \succ \beta^{\angle}$. It is easy to see that the covering $f^{-1}\beta^{\angle}$ is a refinement in the covering α.The covering $f^{-1}\gamma$is a σ-locally finite uniform covering cardinality $\leq \tau$ of the space (X, U). Obviously,$f^{-1} \succ \alpha$. Hence, (X, U) is countable uniformly τ-finally paracompact.

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COUNTABLE-STAR UNIFORM SPACES

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One of the central concepts of the theory of uniform spaces and uniformly continuous mappings is a compact uniform space. In this paper we study countable-star uniform spaces, which are compact types of uniform spaces. Countable-star uniform spaces are uniform spaces in each of their uniform coverings can be refined a countable-star uniform covering. In this case a covering is called countable-star if the power of the star of any point relative to this covering is countable. The class of all countable-star spaces is wider than the class of all pre-Lindelöf spaces. There are examples of countable-star uniform spaces that are not pre-Lindelöf uniform spaces. In particular, it is proved that the product of any number of countable-star uniform spaces is countable-star uniform space, any subspace of a countable-star uniform space is a countable-star uniform space, the sum any number of countable-star uniform space is a countable-star uniform space and the precompact preimage of a countable-star uniform space is a countable-star uniform space.

Keywords. Uniform space, uniform covering, countable-star covering, countable-star uniform space, uniformly continuous mapping.

Бир калыптуу мейкиндиктердин жана бир калыптуу үзгүлтүксүз чагылдырууларды теориясындагы борбордук түшүнүктөрдүн бири болуп компактуу бир калыптуу мейкиндик болуп саналат. Бул илимий макалада бир калыптуу мейкиндиктердин компактуу типтерине таандык болгон санактуу-жылдыздуу бир калыптуу мейкиндиктер изилденет. Санактуу-жылдыздуу мейкиндиктер булар ар бир бир калыптуу жабдууга санактуу-жылдыздуу бир калыптуу жабдууну ичтен сызууга мүмкүн болгон бир калыптуу мейкиндиктер болуп саналат. Мында жабдуу санактуу-жылдыздуу деп аталат, эгерде

каалагандай чекиттин бул жабдууга карата жылдызынын кубаттуулугу санактуу болсо. Бардык санактуу-жылдыздуу бир калыптуу мейкиндиктердин классы бардык пред-Линделёфтук мейкиндиктердин классынан кенен. Санактуу-жылдыздуу бир калыптуу мейкиндик болгон, бирок пред-Линделёфтук мейкиндик болбогон бир калыптуу мейкиндиктердин мисалдары белгилүү. Каалагандай сандагы санактуу-жылдыздуу бир калыптуу мейкиндиктердин көбөйтүндүсү санактуу-жылдыздуу, санактуу-жылдыздуу мейкиндиктердин каалагандай камтылган мейкиндиги санактуу-жылдыздуу, каалагандай сандагы санактуу-жылдыздуу мейкиндиктердин суммасы санактуу-жылдыздуу жана санактуу-жылдыздуу бир калыптуу мейкиндиктердин предкомпактуу кайраэлеси санактуу-жылдыздуу экендиктери далилденген.

Негизги сөздөр. Бир калыптуу мейкиндик, бир калыптуу жабдуу, санактуу-жылдыздуу жабдуу, санактуу-жылдыздуу бир калыптуу мейкиндик, бир калыптуу үзгүлтүксүз чагылдыруу.

Одним из центральным понятием теории равномерных пространств и равномерно непрерывных отображений является компактное равномерное пространство. В данной статье исследуются счетно-звездные равномерные пространства, которые относятся к компактным типам равномерных пространств. Счетно-звездные пространства - это такие равномерные пространства, что в каждое их равномерное покрытие можно вписать счетно-звездное равномерное покрытие. При этом покрытие называется счетно-звездным, если мощность звезды любой точки относительно этого покрытия - счетная. Класс всех счетно-звездных пространств шире класса всех пред-Линделёфовых пространств. Существуют примеры счетно-звездных равномерных пространств, не являющиеся пред-Линделёфовыми равномерными пространствами. В частности, доказываются, что произведение любого числа счетно-звездных равномерных пространств счетно-звездное равномерное пространство, любое подпространство счетно-звездного равномерного пространства счетно-звездное равномерное пространство, сумма любого числа счетно-звездных равномерных пространств счетно-звездное равномерное пространство и предкомпактный прообраз счетно-звездного равномерного пространства является счетно-звездным равномерным пространством.

Ключевые слова. Равномерное пространство, равномерное покрытие, счетно-звездное покрытие, счетно-звездное равномерное пространство, равномерно непрерывное отображение.

It is well known that compact-type spaces play an important role in set-theoretical topology. Find and studying uniform analogues of these classes of compact is an important and interesting problem in uniform topology.

This work studies countable-star spaces.

Throughout this work all uniform spaces are assumed to be Hausdorff and mappings are uniformly continuous.

For covers α and β of a set X, we have: $\alpha \wedge \beta = \{A \cap B: A \in \alpha, B \in \beta\}$. $\alpha(x) = \cup St(\alpha, x)$, $St(\alpha, x) = \{A \in \alpha: A \ni x\}$, $x \in X$, $\alpha(H) = \cup St(\alpha, H)$, $St(\alpha, H) = \{A \in \alpha: A \cap H \neq \emptyset\}$, $H \subset X$. For covers α and β of the set X, the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, the symbol $\alpha * \succ \beta$ means that the cover α is a strongly star refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$such that $\alpha(A) \subset B$. A uniform covering α of uniform space (X, U) is called countable-star uniform covering if $|St(\alpha, x)|$ is countable for each $x \in X$, where $St(\alpha, x) = \{A \in \alpha: A \ni x\}$.

A uniform space (X, U) is called:

(i) precompact, if the uniformity U has a base consisting of finite covers [1], [2];

(ii) totally bounded if for any $\alpha \in U$ there exists a finite set $A \subset X$ such that $\alpha(A) = X$ [1], [2];

(iii) pre-Lindelof, if the uniformity U has a base consisting of countable covers [3], [4];

(iv) compact, if it is precompact and complete [1], [2];

(v) Complete if every Cauchy filter in it converges [1], [2].

A filter F in a uniform space (X, U) is said to be a Cauchy filter, if X for all $\alpha \in U$ [1], [2].

Let $f:(X,U)\to(Y,V)$ be a uniformly continuous mapping of a uniform space (X,U) onto a uniform space (Y,V). The mapping f is called:

(1) precompact, if for each $\alpha\in U$ there exist a uniform cover $\beta\in V$ and a finite uniform cover $\gamma\in U$ such that$f^{-1}\beta\wedge\gamma\succ\alpha$ [1], [2];

(2) uniformly perfect, if it is both precompact and perfect [1], [2].

Definition 1. The uniform space (X,U) called is countable-star if uniformity U has base consisting of countable-star covering.

Proposition 1. Any pre-Lindelöf uniform space is countable-star uniform space.

Proof. Let (X,U) be a pre-Lindelöf space and $\alpha\in U$ be an arbitrary uniform covering. Let $\beta\in U$ be uniform covering such that $\beta *\succ \alpha$. Then there exist countable subset $M\subset X$ such that $\beta(M)=X$. Put $\beta_0=\{\beta(x_1),\beta(x_2),\ldots,\beta(x_n),\ldots\}$. It is easy to see that $|St(\beta_0,x)|$ is countable and $\beta_0\succ\alpha$. Thus, the uniform space (X,U) is countable-star space.

Example. Let X be a discrete uniform space of countable capacity. Then the space X induced with a discrete uniformity whose basis is the family $B=\{\alpha\}$, where $\alpha=\big\{\{x\}:x\in X\big\}$ is a countable-star uniform space. However, it is not a pre-Lindelöf uniform space.

Corollary 1. Any totally bounded uniform space is countable-star uniform space.

Corollary 2. Any compact uniform space is countable-star uniform space.

We will say that the uniformity U on the set X is countable-star if the space (X,U) is countable-star.

Proposition 2. Every uniformity U_1 on the set X which is weaker than some countable-star uniformity U_2 on X is itself countable-star.

Proof. Let $\alpha\in U_1$ be n arbitrary uniform covering. Then $\alpha\in U_2$. Since U_2 is accountable-star uniformity, then there exist countable-star uniform cover $\beta\in U_2$ such that $\beta\succ\alpha$. So, U_1 is countable-star uniformity.

Theorem 1. Any subspace of countable-star space is countable-star.

Proof. Let (M, U_M) be a subspace of countable-star uniform space (X, U) and $\alpha_M \in U_M$ be an arbitrary uniform cover. Then there exists uniform covering $\alpha \in U$ of the uniform space (X, U) such that $\alpha \wedge \{M\} = \alpha_M$. For the covering α there exist countable-star uniform covering $\beta \in U$ such that $\beta \succ \alpha$. Since the mapping $i_M: (M, U_M) \to (X, U)$ is a uniform embedding of the subspace (M, U_M) into the space (X, U), then uniform covering $f^{-1}|_M \beta = \beta_M \in U_M$ is countable-star covering. It's clear that $\beta_M \succ \alpha_M$. Thus, (M, U_M) is countable-star subspace.

Theorem 2. The sum of any number of countable-star uniform spaces is a countable-star uniform space.

Proof. Let $\{(X_a, U_a): a \in M\}$ be an arbitrary family of countable-star uniform spaces and α be an arbitrary uniform cover of the space $(\coprod_{a \in M} X_a, \coprod_{a \in M} U_a)$. Then $i_a^{-1}\alpha \wedge \{X_a\} = \alpha_a \in U_a$ for each $a \in M$, where $i_a: X_a \to \coprod\{X_a: a \in M\}$. Since the uniform space (X_a, U_a) for each $a \in M$ is countable-star, then for uniform covering $\alpha_a \in U_a$ there exist a countable-star uniform covering $\beta_a \in U_a$, $a \in M$ such that $\beta_a \succ \alpha_a$. Let $\beta = \underset{a \in M}{\cup} \alpha_a$. Then according to the sum of uniform spaces of the definition of a uniform covering of uniform space the family β forms uniform covering of $(\coprod_{a \in M} X_a, \coprod_{a \in M} U_a)$. We proof that $\beta \succ \alpha$. Each element of the family β as one of the elements of the family β_a for some $a \in M$, will be contained in some element of the covering α_a, i.e. in some set $i_a^{-1} A \cap X_a$ and therefore in $A \in \alpha$. Thus, each element of the covering β is contained in some element of the covering α, i.e. $\beta \succ \alpha$. Now we prove that the covering β is countable-star. Let $x \in X$ be an arbitrary point and $x \in X_a$. Since the covering β_a is countable-star, then $|St(x, \beta_a)|$ is countable. Then, by virtue of the disjointness of spaces X_a we conclude that $|St(x, \beta)|$ is countable. Therefore, β is a countable-star covering. Hence, the uniform space $(\coprod_{a \in M} X_a, \coprod_{a \in M} U_a)$ is countable-star.

Proposition 3. The completion $(\tilde{X}, \widetilde{U})$ of countable-star uniform space (X, U) is countable-star.

Proof. Let $(\tilde{X}, \widetilde{U})$ is completion of the countable-star uniform space (X, U) and $\tilde{\alpha} \in \widetilde{U}$ be an arbitrary uniform covering. Put $\alpha = \{\tilde{A} \cap X: \tilde{A} \in \tilde{\alpha}\}$. Then $\alpha \in U$ and there exist a countable-star uniform covering $\beta \in U$ such that $\beta \succ \alpha$. It is clear that $\tilde{\beta} \succ \tilde{\alpha}$. We show that $\tilde{\beta}$ is countable-star uniform covering of the uniform space $(\tilde{X}, \widetilde{U})$. Let $\tilde{x} \in \tilde{X}$ be an arbitrary point. Then $i^{-1}\tilde{x} = x \in X$, where $i: X \to \tilde{X}$ is uniformly isomorphism. We proof that the family $St(\tilde{x}, \tilde{\beta})$ is countable. By the completion construction, for every $\tilde{B} \in \tilde{\beta}$ $i^{-1}\tilde{B} = B$, i.e., every element B of the uniform covering β extends to an element $\tilde{B} = \tilde{X} \backslash [X \backslash B]_{\tilde{X}}$ of the uniform covering $\tilde{\beta}$. Since β is countable-star uniform covering of (X, U), then $|St(x, \beta)|$ is countable, where $x = i^{-1}\tilde{x}$. Then $|St(\tilde{x}, \tilde{\beta})|$ is countable. So, the completion $(\tilde{X}, \widetilde{U})$ is countable-star space.

Theorem 3. The product of any number of countable-star uniform spaces is countable-star space.

Proof. Let $\{(X_a, U_a): a \in M\}$ be an arbitrary family of countable-star uniform spaces and B_a be base of uniformity U_a consisting of countable-star coverings. By B' we denote the family of all coverings type $\overset{i=1}{\underset{n}{\wedge}} \pi_{a_i}^{-1}\alpha_{a_i}$, where $\alpha_{a_i} \in B_{a_i}$, $\{a_i: i = 1,2,\ldots,n\}$ is an arbitrary finite subset of the set M, and $\pi_a: X \to X_a$ is projection onto the a -th factor, $a \in M$. Then by the definition of the product of uniform space the system B' is the basis of the product $U = \prod_{a \in M} U_a$ of uniformities U_a. Since the projection $\pi_a: X \to X_a$ is uniformly continuous for each $a \in M$, then for each $a \in M$ the uniform cover $\pi_{a_i}^{-1}\alpha_{a_i}$ is countable-star and the uniform covering $\overset{i=1}{\underset{n}{\wedge}} \pi_{a_i}^{-1}\alpha_{a_i}$ is also countable-star. So, the product $(\prod_{a \in M} X_a, \prod_{a \in M} U_a)$ is countable-star.

Theorem 4. The precompact preimage of countable-star uniform space is countable-star.

Proof. Let $f: (X, U) \to (Y, V)$ is a precompact mapping and the uniform space (Y, V) is countable-star. Let $\alpha \in U$ be an arbitrary uniform covering. Then from the precompactness of the mapping f there exists countable-star uniform covering $\beta \in V$ and finite uniform covering $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$. It is easy to see that $f^{-1}\beta \wedge \gamma = \alpha_0$ is uniform covering and $|St(\alpha_0, x)|$ is countable. Thus, (X, U) is countable-star uniform space.

Corollary 3. The uniform perfect preimage of countable-star uniform space is countable-star.

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NECESSARY AND SUFFICIENT CONDITIONS OF EXISTENCE OF QUASI-DOUBLE LINES OF A PAIR OF DISTRIBUTIONS IN THE FIVE-DIMENSIONAL SPACE

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Abstract. In domain $\Omega \subset E_5$ it is considered a set of smooth lines such than through a point $X \in \Omega$ passed one line of a given set. The moving frame $\mathfrak{R}$ is a Frenet frame for the line ω^1 of the given set. Integral lines of the vector fields $\vec{e_i}$ are forming the Frenet net. There exist a point F_3^2 on the tangent of the line ω^3 of the Frenet net. When the point X is shifted in the domain Ω, the point F_3^2 describes it′s domain in E_5. The partial mapping f_3^2 is defined such that $f_3^2(X) = F_3^2$. Necessary and sufficient conditions of existence of a quasi-double lines of a pair of distributions in partial mapping f_3^2 are proved in the case, when a Frenet net is a Frenet cyclical net.

Аннотация. В области Ω евклидова пространства E_5 задано семейство гладких линий так, что через каждую точку $\mathrm{X} \in \Omega$ проходит одна линия заданного семейства. Подвижной репер $\mathfrak{R}$ выбран так, чтобы он являлся репером Френе для линии ω^1 заданного семейства гладких линий. Интегральные линии координатных векторов репера Френе образуют сеть Френе. На касательной к линии ω^3 сети Френе определятся точка F_3^2. Когда точка X движется в области Ω, точка F_3^2 описывает свою область Ω_3^2. Таким образом определяется частичное отображение евклидова пространства, которое переводит точку X в точку F_3^2.
В случае, когда сеть Френе является циклической сетью Френе, доказаны необходимое и достаточные условия существования квазидвойной линии пары распределений в рассматриваемом частичном отображении.

Аннотация. E_5 мейкиндигинин Ω аймагында ушундай жылма сызыктардын көптүгү берилген: ар бир $\mathrm{X} \in \Omega$ чекити аркылуу берилген көптүктүн бир гана сызыгы өтөт. $\mathfrak{R}$ кыймылдуу реперин берилген көптүктүн ω^1 сызыгы үчүн Френенин репери боло тургандай тандап алабыз. e_i вектордук талааларынын

интегралдык сызыктары Френенин торчосун түзүшөт. Бул торчонун ω^3 сызыгынын жанымасында F_3^2 чекити жашайт. X чекити Ω аймагында кыймылга келгенде F_3^2 чекити E_5 мейкиндигинде өзүнүн аймагын сызып чыгат. Натыйжада X чекити F_3^2 чекитине өтө тургандай чагылтуусу аныкталат. Френенин торчосу циклдик торчо болгон учурда f_3^2 бөлүктөп чагылтуусунда бөлүштүрүүлөрдүн түгөйүнүн квазикошмок сызыктарынын жашашынын зарыл жана жетиштүү шарттары далилденген.

Key words: partial mapping, Frenet cyclic net, Frenet frame, pseudofocus, pair of distributions, quasi-double line, Euclidean space

Ключевые слова: частичное отображение, циклическая сеть Френе, репер Френе, псевдофокус, пара распределений, квазидвойная линия пары, евклидово пространство.

Урунттуу сөздөр: Бөлүктөп чагылтуу, Френенин циклдик торчосу, Френенин торчосу, псевдофокус, түгөй бөлүштүрүүлөр, квазикошмок сызыктар, Евклиддик мейкиндик.

1. Introduction

In domain $\Omega \subset E_5$ it is considered a set of smooth lines such that through a point $X \in \Omega$ passed one line of given set. The moving frame $\Re = (X, \vec{e_i})(i, j, k = \overline{1,5})$ is a Frenet frame of for the line ω^1 of the given set of smooth lines. Derivation formula of the frame $\Re$ have a form:

$$d\vec{X} = \omega^i \vec{e}_i, \qquad d\vec{e}_i = \omega_i^k \vec{e}_k \tag{1}$$

$$D\omega^i = \omega^k \wedge \omega^i{}_k, \quad D\omega_i^k = \omega_i^j \wedge \omega_j^k, \quad \omega_i^j + \omega_j^i = 0 \tag{2}$$

Integral lines of vector fields $\vec{e_i}$ are forming the Frenet net Σ_5 for the line ω^1 of the given set of lines. Since frame $\Re$ is constructed on tangent of lines of the net Σ_5, the forms ω_i^k are principal forms [3]; in orther words:

$$\omega_i^k = \Lambda_{ij}^k \omega^j \tag{3}$$

Taking in to account (3) from (2) we have:

$$\Lambda_{ij}^k = -\Lambda_{kj}^i \tag{4}$$

If we differentiate externally (3) then we have:

$$D\omega_i^k = d\Lambda_{ij}^k \wedge \omega^j + \Lambda_{ij}^k D\omega^j$$

By using formula (2) from here we have got:

$$\omega_i^j \wedge \omega_j^k = d\Lambda_{ij}^k \wedge \omega^j + \Lambda_{ij}^k \wedge \omega^\ell \wedge \omega_\ell^j$$

Taking in to account (3) from last formula is follows that:

$$\omega_i^j \wedge \Lambda^k_{j\ell}\omega^\ell = d\Lambda_{ij}^k \wedge \omega^j - \Lambda_{ij}^k \omega_\ell^j \wedge \omega^\ell$$

or $\Lambda^k_{j\ell}\omega_i^j \wedge \omega^\ell = d\Lambda_{ij}^k \wedge \omega^j - \Lambda_{ij}^k \wedge \omega_\ell^j \wedge \omega^\ell$

From here we found: $d\Lambda_{ij}^k \wedge \omega^j - \Lambda_{i\ell}^k \omega_j^\ell \wedge \omega^j - \Lambda^k_{j\ell}\omega_i^j \wedge \omega^\ell = 0$

or $(d\Lambda_{ij}^k - \Lambda_{i\ell}^k \omega_j^\ell - \Lambda_{\ell j}^k \omega_i^\ell) \wedge \omega^j = 0$

By using lemma of Cartan from here we have: $d\Lambda_{ij}^k - \Lambda_{i\ell}^k \omega_j^\ell - \Lambda_{\ell j}^k \omega_i^\ell = \Lambda_{ijm}^k \omega^m$

or

$$d\Lambda_{ij}^k = (\Lambda_{ijm}^k + \Lambda_{il}\Lambda^l{}_{jm} + \Lambda_{lj}^k \Lambda^l{}_{im})\omega^m \quad \textbf{(5)}$$

The system of variable $\{\Lambda_{ij}^k, \Lambda_{ijm}^k\}$ is formed geometrical object of second order. The Frenet formulas for the line ω^1 of given set have a form

$$d_1\vec{e}_1 = \Lambda_{11}^2 \vec{e}_2,\ d_1\vec{e}_2 = \Lambda_{21}^1 \vec{e}_1 + \Lambda_{21}^3 \vec{e}_3,\ d_1\vec{e}_3 = \Lambda_{31}^2 \vec{e}_2 + \Lambda_{31}^4 \vec{e}_4,$$

$$d_1\vec{e}_4 = \Lambda_{41}^3 \vec{e}_3 + \Lambda_{41}^5 \vec{e}_5,\ d_1\vec{e}_5 = \Lambda_{51}^3 \vec{e}_4$$

and $\Lambda_{11}^3 = -\Lambda_{31}^1 = 0, \quad \Lambda_{11}^4 = -\Lambda_{41}^1 = 0 \quad \Lambda_{11}^5 = -\Lambda_{51}^5 = 0$ **(6)**

$$\Lambda_{21}^5 = -\Lambda_{51}^2 = 0, \quad \Lambda_{21}^4 = -\Lambda_{41}^2 = 0, \quad \Lambda_{31}^5 = -\Lambda_{51}^3 = 0 \quad \textbf{(7)}$$

There $k_1^1 = \Lambda_{11}^2,\ k_2^1 = \Lambda_{21}^3, k_3^1 = \Lambda_{31}^4, k_4^1 = \Lambda_{41}^5 = -\Lambda_{51}^4$ are a first, a second, a third and a fourth curvature of line ω^1 respectively (there d_1 is the symbol of differentiation along line ω^1).

A pseudofocus [4] $F_i^j (i \neq j)$ of tangent of the line ω^i of the Σ_5 is defined by radius-vector:

$$\vec{F}_i{}^j = \vec{X} - \frac{1}{\Lambda_{ij}^j}\vec{e}_i = \vec{X} + \frac{1}{\Lambda_{jj}^i}\vec{e}_i \quad \textbf{(8)}$$

There are exist four pseudofocuses on each tangent $(X,\overrightarrow{e_i})$: on the straight line $(X,\overrightarrow{e_1})\_F_1^2,F_1^3,F_1^4,F_1^5$, on $(X,\overrightarrow{e_2})\_F_2^1,F_2^3,F_2^4,F_2^5$, $(X,\overrightarrow{e_3})\_F_3^1,F_3^2,F_3^4,F_3^5$, on $(X,\overrightarrow{e_4})\_F_4^1,F_4^2,F_4^3,F_4^5$, $(X,\overrightarrow{e_5})\_F_5^1,F_5^2,F_5^3,F_5^4$.

The net Σ_5 in $\Omega\subset E_5$ is called a Frenet cycle net [5] if the frames

$\Re_1=(\mathrm{X},\vec{e}_1,\vec{e}_2,\vec{e}_3,\vec{e}_4,\vec{e}_5)$, $\Re_2=(\mathrm{X},\vec{e}_2,\vec{e}_3,\vec{e}_4,\vec{e}_5,\vec{e}_1)$, $\Re_3=(\mathrm{X},\vec{e}_3,\vec{e}_4,\vec{e}_5,\vec{e}_1,\vec{e}_2)$, $\Re_4=(\mathrm{X},\vec{e}_4,\vec{e}_5,\vec{e}_1,\vec{e}_2,\vec{e}_3)$, $\Re_5=(\mathrm{X},\vec{e}_5,\vec{e}_1,\vec{e}_2,\vec{e}_3,\vec{e}_4)$ are Frenet frames for lines $\omega^1,\omega^2,\omega^3,\omega^4,\omega^5$ respectively of net Σ_5 simultaneously.

Let the net Σ_5 be Frenet cyclical net [4]. We will denote it by $\widetilde{\Sigma_5}$. Pseudofocus $F_3^2\in(X,\overrightarrow{e_3})$ are defined by radius-vector:

$$F_3^2=\vec{X}-\frac{1}{\Lambda_{32}^2}\vec{e}_3=\vec{X}+\frac{1}{\Lambda_{32}^2}\vec{e}_3 \qquad \textbf{(9)}$$

When the point X is moving in the domain $\Omega\subset E_5$, pseudofocus F_3^2 describes it's domain Ω_3^2. Thus defined the partial mapping $f_3^2\colon\Omega\to\Omega_3^2$ such that $f_3^2(X)=F_3^2$. We will join to $\Omega_3^2\subset E_5$ the moving frame $\Re'=(F_3^2,\overrightarrow{c_i})$, where vectors $\overrightarrow{c_i}$ have a form:

$$\vec{c}_1=\vec{e}_1+\frac{C_{321}^2}{(\Lambda_{32}^2)^2}\vec{e}_3-\frac{\Lambda_{31}^i}{\Lambda_{32}^2}\vec{e}_i;\ \vec{c}_2=\vec{e}_2+\frac{C_{322}^2}{(\Lambda_{32}^2)^2}\vec{e}_3-\frac{\Lambda_{32}^i}{\Lambda_{32}^2}\vec{e}_i;$$

$$\vec{c}_3=\vec{e}_3+\frac{C_{323}^2}{(\Lambda_{32}^2)^2}\vec{e}_3-\frac{\Lambda_{33}^i}{\Lambda_{32}^2}\vec{e}_i;\ \vec{c}_4=\vec{e}_4+\frac{C_{324}^2}{(\Lambda_{32}^2)^2}\vec{e}_3-\frac{\Lambda_{34}^i}{\Lambda_{32}^2}\vec{e}_i;$$

$$\vec{c}_5=\vec{e}_5+\frac{C_{325}^2}{(\Lambda_{32}^2)^2}\vec{e}_3-\frac{\Lambda_{35}^i}{\Lambda_{32}^2}\vec{e}_i;$$

Taking into account that the Frenet net is a cyclic Frenet net we have:

$$\vec{c}_1=\vec{e}_1-\frac{\Lambda_{31}^2}{\Lambda_{32}^2}\vec{e}_2+\frac{C_{321}^2}{(\Lambda_{32}^2)^2}\vec{e}_3-\frac{\Lambda_{31}^4}{\Lambda_{32}^2}\vec{e}_4;\ \vec{c}_2=\vec{e}_2+\frac{C_{322}^2}{(\Lambda_{32}^2)^2}\vec{e}_3-\frac{\Lambda_{32}^4}{\Lambda_{32}^2}\vec{e}_4;$$

$$\vec{c}_3=\left[1+\frac{C_{323}^2}{(\Lambda_{32}^2)^2}\right]\vec{e}_3-\frac{\Lambda_{33}^4}{\Lambda_{32}^2}\vec{e}_4; \qquad \textbf{(10)}$$

$$\vec{c}_4=-\frac{\Lambda_{34}^2}{\Lambda_{32}^2}\vec{e}_2+\frac{C_{324}^2}{(\Lambda_{32}^2)^2}\vec{e}_3+\vec{e}_4;\ \vec{c}_5=-\frac{\Lambda_{35}^2}{\Lambda_{32}^2}\vec{e}_2+\frac{C_{325}^2}{(\Lambda_{32}^2)^2}\vec{e}_3-\frac{\Lambda_{35}^4}{\Lambda_{32}^2}\vec{e}_4+\vec{e}_5.$$

It is considered a pair of 3-dimensional distributions $\Delta_{(145)}=\left(X,\overrightarrow{e_1},\overrightarrow{e_4},\overrightarrow{e_5}\right)$ $f_4^3\left(\Delta_{(145)}\right)=\Delta'_{(145)}=(F_3^2,\overrightarrow{c_1},\overrightarrow{c_4},\overrightarrow{c_5})$. Let the line α belong to the distribution $\Delta_{(145)}$.

Tangent vector of line α has the form: $\vec{\alpha} = \alpha^1\vec{e}_1 + \alpha^4\vec{e}_4 + \alpha^5\vec{e}_5$. We find the tangent vector $\vec{\bar{a}}$ of the line $\vec{\bar{a}} = f_4^3(\alpha)$: $\vec{\bar{a}} = \alpha^1\vec{c}_1 + \alpha^4\vec{c}_4 + \alpha^5\vec{c}_5$

Taking into account formulas (10) from here we have:

$$\vec{\bar{a}} = \alpha^1 e_1 + (\alpha^1 c_1^2 + \alpha^4 c_4^2 + \alpha^5 c_5^2)\vec{e}_2 + (\alpha^1 c_1^3 + \alpha^4 c_4^3 + \alpha^5 c_5^3)\vec{e}_3 + \\ +(\alpha^1 c_1^4 + \alpha^4 + \alpha^5 c_5^4)\vec{e}_4 + \alpha^5\vec{e}_5 \qquad \textbf{(11)}$$

Where c_i^j is $j-th$ coordinate of the vector $\vec{c}_i$.

Definition. If the tangent of the line α at the point X and tangent of the line $f_4^2(\alpha) = \bar{\alpha}$ at the point F_4^2 belong to the same 3-dimensional space $\left(X, \overrightarrow{e_1}, \overrightarrow{e_4}, \overrightarrow{e_5}\right)$, then lines α, $\bar{\alpha}$ are called quasi-double lines of a paire of distributions $(\Delta_{(145)}, \Delta'_{(145)})$ in the partial mapping $f_3^2\colon \Omega \to \Omega_3^2$.

From the condition $\alpha, \bar{\alpha} \in \left(X, \overrightarrow{e_1}, \overrightarrow{e_4}, \overrightarrow{e_5}\right)$ we have:

$$\begin{cases} \alpha^1 c_1^2 + \alpha^4 c_4^2 + \alpha^5 c_5^2 = 0, \\ \alpha^1 c_1^3 + \alpha^4 c_4^3 + \alpha^5 c_5^3 = 0. \end{cases}$$

From here, taking into account formulas (10), we find:

$$\begin{cases} \Lambda_{31}^2\alpha^1 + \Lambda_{34}^2\alpha^4 + \Lambda_{35}^2\alpha^5 = 0, \\ \mathrm{C}_{321}^2\alpha^1 + \mathrm{C}_{324}^2\alpha^4 + \mathrm{C}_{325}^2\alpha^5 = 0. \end{cases}$$

From here we have:

$$\alpha^1 = \begin{vmatrix} \Lambda_{34}^2 & \Lambda_{35}^2 \\ \mathrm{C}_{324}^2 & \mathrm{C}_{325}^2 \end{vmatrix};\ \alpha^4 = \begin{vmatrix} \Lambda_{35}^2 & \Lambda_{31}^2 \\ \mathrm{C}_{325}^2 & \mathrm{C}_{321}^2 \end{vmatrix};\ \alpha^5 = \begin{vmatrix} \Lambda_{31}^2 & \Lambda_{34}^2 \\ \mathrm{C}_{321}^2 & \mathrm{C}_{324}^2 \end{vmatrix} \qquad \textbf{(12)}$$

If the coordinates of the tangent vector $\vec{\alpha}$ satisfy the conditions (12), then the lins $\alpha, \bar{\alpha}$ are quasi-double lines of a pair of distributions $(\Delta_{(145)}, \Delta'_{(145)})$ in the partial mapping f_3^2.

Thus the following theorem is proven.

Theorem: lines α and $\bar{\alpha} = f_3^2(\alpha)$ are quasi-double lines of a pair of distributions $(\Delta_{(145)}, \Delta'_{(145)})$ in the partial mapping f_3^2 if and only if when the coordinates of the tangent vector $\vec{\alpha}$ of the line α are satisfy the conditions (12).

Here $\Lambda_{31}^2, \Lambda_{34}^2, \Lambda_{35}^2$ are the second, fourth, third curvatures of lines $\omega^1, \omega^4, \omega^5$ of Frenet cyclic net of *t* respectively.

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MSC 68T10

NECESSARY EMERGING OF PARTIAL PATTERNS AND METHODIC OF ARTIFICIAL INTELLIGENCE

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Surveyed the hypothesis of arising the Universe as an order (cosmos) from chaos and ideas on this topic including the law of the transformation of quantity into quality when quantity exceeds the limit. Arising of partial patterns under random exposures in discrete media (irgöö) and continuous media (synergetic) is mentioned; examples of necessary emerging of partial patterns including Kolmogorov-Arnold theorem with artificial intelligence are given.

Keywords: cosmos, chaos, quantity, quality, pattern, artificial intelligence

Ааламдын тартип (космос) катары башаламандыктан келип чыгышы жөнүн-дөгү гипотезаны изилдеген жана чегинен ашып кеткенде сандын сапатка айлануу мыйзамын камтыган бул темадагы идеялар каралат. Дискреттик чөйрөдө (иргөө) жана үзгүлтүксүз чөйрөдө (синергетика) кокустуктар астында жарым-жартылай схемалардын пайда болушу айтылат; жасалма интеллект менен Колмогоров-Арнольд теоремасы, анын ичинде жарым-жартылай моделдердин зарыл пайда болушунун мисалдары келтирилген.

Урунттуу сөздөр: Аалам (космос), башаламандык, сан, сапат, сайма, жасалма интеллект.

Сделан обзор гипотезы возникновения Вселенной как порядка (космоса) из хаоса и идей на эту тему, в том числе закон перехода количества в качество, когда количество превышает предел. Упоминается возникновение частичных закономерностей при случайном воздействии в дискретных средах (иргөө) и сплошных средах (синергетика); приведены примеры необходимого возникновения частичных закономерностей, включая теорему Колмогорова-Арнольда, с использованием искусственного интеллекта.

Ключевые слова: космос, хаос, закономерность, искусственный интеллект.

1. Introduction

We consider the hypothesis of arising the Universe as an order (cosmos) from chaos. Platon (4th century B.C.) explained arising patterns in nature as tending to non-material ideas.

G.F.Hegel (1800s) declared the law of the transformation of quantity into quality; the change in quantity will transform into the change in quality when it exceeds the limit.

Remark. Probably, G.F.Hegel meant addition of matter to the system. Further investigations demonstrated that input and dissipation of energy are necessary. We do not consider patterns arising with addition of matter (fractals). Also, the system must be compact.

We distinguish two items: (*) arising of partial patterns under random exposures; (**) necessary emerging of partial patterns.

(*) Basing on ancient notion of phenomenon of irgöö (random vibration of balls of different sizes of same material in a wide symmetrical vessel yields migration of the biggest one to the center of their surface) in Kyrgyz language and results of numerical experiment (2012) we formulated the effect of irgöö, the effect of numerosity and the notion of constant of numerosity. If there are sufficiently many (greater than the constant of numerosity) discrete objects within a compact domain and their arbitrary arrangement has a non-zero probability then a partial pattern under random exposures may arise.

The spontaneous formation of hexagonal structures (solidified lava, honeycombs, the latest example at the pole of Saturn) is known. Basing on the phenomenon of Rayleigh-Bénard cells (1900) notion of synergetic (in narrow sense) appeared. If there is a dissipative object of homogeneous continuous media then a partial pattern under random exposures may arise.

Remark. The natural appearance of definite waves on water under the influence of wind, on sand under the influence of wind or water (at the bottom) can be related to synergetic too (the phase of wave is random). Crystallization is not related to synergetic because it is defined by quantum (discrete) properties of atoms and molecules.

(**) F.P.Ramsey [1], Theorem 2. For any natural numbers m,n,r there is a natural k such that if X has at least k elements, and we partition the sets of subsets of X of size n into r parts, then there is a subset Y of X with at least m elements such that every subset of Y of size n is in the same part of the partition. The value of k is denoted as the *Ramsey Number* $R_n(m,r)$.

Particular case for two-colored-edge-complete graph. For any natural numbers c_1,c_2, there is a natural $k= R(c_1,c_2)$ such a graph with k vertices contains a 1st-colored-subgraph(clique) of c_1 vertices and/or a 2nd-colored-subgraph of c_2 vertices.

These results are interpreted as follows. There cannot be complete disorder in a (stationary discrete) object. Given a large enough number of elements, some of them are bound to form a pattern.

2. Examples of necessary emergent partial ordering

Example 1. Take two identical objects and other two identical objects and arrange them randomly in a row. It turns out that symmetry or order must arise: 0II0 0I0I 00II.

Example 2. Compose a 2×3-rectangle of three white squares and three black squares. Then there will arise an anti-symmetrical 2×2- and/or 2×3-rectangle.

These examples continue results [1].

Example 3. We fix a string (of any length d) at the ends and hit it in an arbitrary way. It turns out that the resulting sound (any periodic function with a zero at the beginning) can be expanded into a Fourier series:

$f(x)= b_1 \sin (x/d\cdot\pi)+ b_2 \sin (2x/d\cdot\pi)+ b_3 \sin (3x/d\cdot\pi)+\ldots$

And a set of several numbers - Fourier coefficients: b_1, b_2, b_3 … gives a sufficient idea of the sound. For example, a person is recognized by it.

Example 4. Let us take any (bounded) object. It can be (approximately) represented by a graph of a (continuous) multivariate function f defined on a bounded domain (on a coordinate parallelepiped). The Kolmogorov–Arnold representation theorem (or “superposition theorem”) [2-4] states that f can be written as a finite composition of continuous functions of a single variable and the binary operation of addition.

f(x[1],…x[n])=Sum{ U[q] (Sum{V[q][p](x[p]):p=1..n}):q=0..2n},

G. Lorentz [5] proved that all functions *U[q]* may be same. Particularly, *f(x,y)=Sum{ U (V[q](x)+ W[q](y)):q=0..4}.*

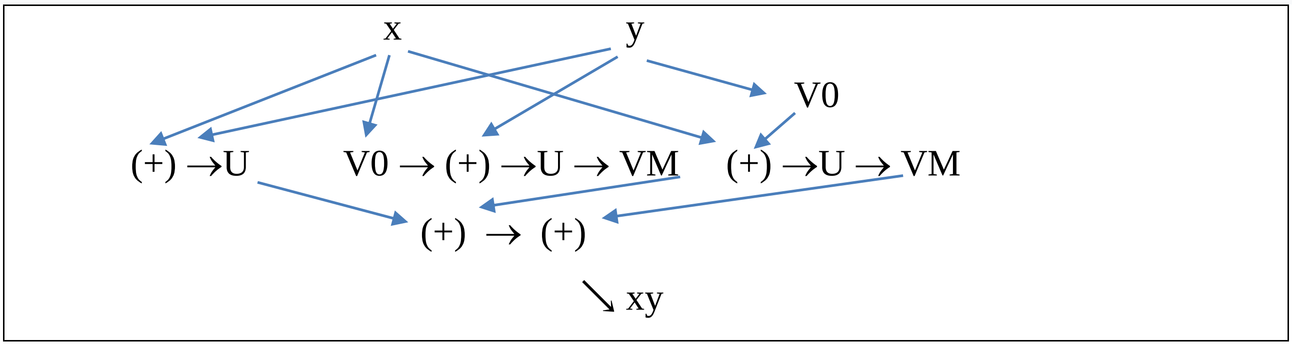
Remark. P. A. Ostrand [6] generalized this theorem to real-valued functions on compact metric spaces.

The simplest example. We need to calculate the product *xy* if we only have devices that can do everything with one number, and a device to add two numbers.

The Kolmogorov-Arnold theorem states that this is possible.

We transform: $xy = (x+y)^2/2 + (-(x+0\cdot y)^2/2) + (-(0\cdot x+y)^2/2)$.

We need two pieces *V0* of multiplying a number by 0, three pieces of the function $U = z^2/2$, two pieces of the *VM* "*unary minus*" function, five pieces of the function *(+)* of two numbers. We are building a neural network (slightly different from G. Lorenz's type).



Let's use this example to show how AI works. Let's assume that we do not know algebra, we only know how to multiply small numbers, and we need a device for multiplying large numbers (for simplicity, we will consider only non-negative integers).

We calculate 2·3=6, 5·5=25, 7·9=63. We give the task to create a neural network that gives such results.

AI sorts through millions of options, also intelligently, and builds such a neural network. It may contain thousands of elements and thus be incomprehensible to the human mind, but it will work.

At the same time, it will probably give errors for 2·0, 1·17, 9·7 - this is our omission - it was necessary to insert such examples, because the above examples do not give any clue, and it may be wrong for some large numbers, as noted in the literature.

5. Conclusion

The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic.

We hope that facts, techniques and methods listed above would help students and research workers to grope and obtain new results, at the same time to avoid mistakes and hallucinations of AI.

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MSC 80A05

SPECIFICATION OF LAW OF ENTROPY INCREMENT IN AFFECTABLE (ALMOST CLOSED) MULTI-POINT SYSTEMS

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The authors propose general laws and derive estimations from below on entropy increment while transformations with dissipation of energy between definite stationary states (examples of two points) on depending on time in almost isolated, permanently unstable (affectable) systems. For short times it is inversely proportional to the square of the transition time. Such estimations are generalized in this paper, the laws are proven as the addition to the second law of thermodynamic.

Keywords: entropy, thermodynamic, second law, control, differential equation, affectable system, heat.

Авторлор дээрлик обочолонгон, таасир этилүүчү системаларда убакытка жараша белгилүү стационардык абалдардын (эки чекит үчүн мисалы) орто-сунда энергияны диссипациялоо менен трансформациялар жүрүп жатканда энтропиянын өсүүсүнө ылдыйдан баа берип, жалпы мыйзамдарды сунушташат. Кыска убакытта ал өткөөл убакыттын квадратына тескери пропорционалдуу. Мындай баа берүүлөр бул эмгекте жалпыланган, мыйзамдар термодинамика-лык экинчи мыйзамга кошумча катары далилденген.

Урунттуу сөздөр: энтропия, термодинамика, башкаруу, дифференциалдык теңдеме, таасир эттилүүчү система, сүрүлүү, жылуулук.

Авторы предлагают общие закономерности и получают оценки снизу прира-щения энтропии при преобразованиях с диссипацией энергии между опре-деленными стационарными состояниями (примеры для двух точек) в зави-симости от времени в практически изолированных, перманентно неустойчивых системах. На малых временах оно обратно пропорционально квадрату времени перехода. В данной статье такие оценки обобщены, законы доказаны как дополнение ко второму началу термодинамики.

Ключевые слова: энтропия, термодинамика, управление, дифференциальное уравнение, воздействуемая система, трение, теплота.

1. Introduction

The second law of thermodynamics for isolated systems stated that entropy increment is positive always but does not give quantitative estimations (see [1] for discussion). We found two attempts in publications:

Example 1 [2]: the minimal energy (entropy increment) to treat one bit of information is the Shannon-von Neumann-Landauer boundary in entropy form: $\Delta H_{bit} \geq k_B \cdot ln\ 2$, k_B is the Boltzmann constant.

Example 2 [3]: “In any mechanical system the energy that must be expended to work against friction is equal to the product of the frictional force and the distance through which the system travels. Hence the faster a swimmer travels between two points, the more energy he or she will expend, although the distance traveled is the same whether the swimmer is fast or slow.”

Basing on the notion of economic (cruising) speed, taking into account ideas of Ecological Rallies and Shell Eco-marathon for cars we specified the second law of thermodynamic for controlled “almost isolated”, “permanently unstable”, “affectable” systems: the entropy increment is asymptotically inversely proportional to the square of the transition time between two stationary states. We proved some estimations in mathematical models describing such concrete systems with one point.

In this paper we consider systems with two points.

Remark. All actual processes are not ideal. Thus, all definitions and terms below are meant to be “almost”.

2. Known definitions and formula for entropy

There are many definitions of general notion of entropy. We will use the following: entropy is the measure of a system’s thermal energy per unit temperature that is unavailable for doing useful work. Because work is obtained from ordered molecular motion, the amount of entropy is also a measure of the molecular disorder, or randomness, of a system. In processes considered below the entropy increment is defined by the formula $\Delta H=\Sigma(\Delta Q/\Theta)$ where ΔQ is a quantity of transferred heat energy or energy that was converted to heat irreversibly and Θ is an absolute temperature.

3. Addition to the second law

Let there is an almost isolated affectable physical system. Let it is in any stationary state *A* now and there it can pass to any other stationary state *B*.

Definition 1. Denote the minimal entropy increment ΔH of the system for any transition from the state *A* to the state *B* as economic entropy increment ΔH_0 and the minimal time for such transition as economic time T_0.

If such transition is then T_0 may be called adiabatic time too.

Law 1. There exists such positive constant C_0 that for any transition from the state *A* to the state *B* during the time T: $\Delta H \ge max\{\Delta H_0, C_0/T^2\}$ $(T< T_0)$

the dimension of C_0 is *mass* × *length* × *length* / *temperature*.

Proof. It is obvious (and was noted in [3]) that ΔH must increase as T decreases. the dimension of ΔH is one of entropy: *mass* × *length* × *length* /(*time* × *time* × *temperature*). Only *time* is a variable. Hence, $\Delta H \sim$ *mass* × *length* × *length* / *temperature*/($T \cdot T$).

Consider a more concrete situation. Let, in a homogeneous media, any point of mass m does not move in any inertial coordinate system at the moment t_1 and it is at the distance d from its initial state and does not move at the moment $t_2 = t_1 + T$.

Law 2. There exists such positive constant C_1 that for any transition from the state *A* to the state *B* during the time T inequality $\Delta H \ge G_1 \cdot m \cdot d^2/T^2$ for any transition from the state *A* to the state *B* during $T<T_0$ (the dimension of G_1 is 1 / *temperature*).

Proof. To move the point has to acquire velocity ~ d/T, i.e. the kinematical energy $E_{kin} \sim m \cdot d^2/T^2$. As the point is motionless in the final state, this energy has to transform into other kinds of energy. But possibilities of kinematical energy to pass into potential (recuperation), chemical etc. ones during bounded time are bounded. Hence a greater part of kinematical energy has to pass into heat, i.e. the entropy increment must be $\Delta H \sim m \cdot d^2/T^2/\Theta$.

4. Denotations, initial examples

Denote $t \in [0,T]$; $x(t)$ for horizontal movement; $v(t)$ for velocity; $u(t)$ (force) for control function. They meet the boundary conditions

$x(0)=A$; $x(T)=B$; $v(0)=x'(0)=0$; $v(T)=x'(T)=0$.

If $sgn(x''(t))=-sgn(x'(t))$ then there is dissipation of energy (entropy increment).

Example 3. Let $AB=[0, d]$ be an ideal horizontal road of length d. There is the load of mass m being standing at many compressed short almost ideal springs at the point A and the massive brake with the absolute temperature Θ at the point B. It is necessary to deliver the load from the point A to the point B during the time T.

By the Newton's second law, the controlled equation

$$x''(t)=u(t)/m;\ u(t)=const\cdot\delta(t)-const\cdot\delta(T-t), \tag{1}$$

$T_0\sim\infty$; $\Delta H_0\sim 0$.

We release some springs to push the load with velocity $v=d/T$, $x'(t)=const/m$, $x(t)=const\cdot t/m$, $(0<t\le T)$; $x(T)=const\cdot T/m=d$, $const=d/T\cdot m$.

Its kinematical energy will be $m\cdot d^2/T^2/2$ until the point B. All this energy transforms to heat energy at the point B with the absolute temperature Θ,

$$\Delta H=1/2\cdot m\cdot d^2/T^2/\Theta. \tag{2}$$

Remark. Let there be compressed springs (possibility of pushing) on all the way ($u(t)$ may be arbitrary) in this example. If we engage some springs within the segment then the load will pass some part of the road with a less velocity than $v=d/T$ and the time will be greater than T. Hence, the result (2) is optimal for this more general case too.

Example 4. Let $AB=[0, d]$ be a horizontal road of length d with the friction coefficient $\mu>0$. There is the load of mass m being standing at many compressed short almost ideal springs at the point A and the massive brake with the absolute temperature Θ at the point B. It is necessary to deliver the load from the point A to the point B during the time T.

By the Newton's second law, the controlled equation

$$x''(t)=u(t)/m-g\mu;\ u(t)=const_1\cdot\delta(t)-const_2\cdot\delta(T-t). \tag{3}$$

We release some springs to push the load with initial velocity v_1, $v(t)=v_1-g\mu t$, $x(t)=v_1t-g\mu t^2/2$ $(0<t\le T)$. It must be $x(T)=d$, $v_1T-g\mu T^2/2=d$, $v_1=d/T+g\mu T/2$,

The initial kinematical energy to be transformed to heat: $E(T)=m(g\mu T + 2d/T)^2/8$. Minimizing: $T_0=\sqrt{2d/(g\mu)}$; $\Delta H_0=4\cdot 2dg\mu m/8/\Theta = mdg\mu/\Theta$.

If $T<T_0$ then

$$\Delta H= m(g\mu T + 2d/T)^2/8/\Theta \sim md^2T^2/2/\Theta \ (T\to 0). \qquad (4)$$

5. Examples with two points

Example 5. Let there be two points as in Example 3, they move consequently. We have $\Delta H =1/2\cdot(m_1\cdot d_1^2/T_1^2 + m_2\cdot d_2^2/T_2^2)/\Theta$, $T_1 + T_2 = T$.

As well as in Example 3, $T_0\sim\infty$; $\Delta H_0\sim 0$.

For $T<\infty$: $m_1\cdot d_1^2/T_1^2 + m_2\cdot d_2^2/(T-T_1)^2 \downarrow min$.

Differentiating: $-2m_1\cdot d_1^2/T_1^3 + 2m_2\cdot d_2^2/(T-T_1)^3=0$;

$((T-T_1)/T_1)^3=m_2\cdot d_2^2/(m_1\cdot d_1^2)$.

Optimal values $T_1= T/(1+(m_2\cdot d_2^2/(m_1\cdot d_1^2)^{1/3})$; $T_2= T/(1+(m_1\cdot d_1^2/(m_2\cdot d_2^2)^{1/3})$.

Denote $\mu_j= m_j\cdot d_j^2$. We have the minimal entropy increment

$\Delta H =1/2\cdot(\mu_1 (1+(\mu_2/\mu_1)^{1/3})^2/T^2 + \mu_2 (1+(\mu_1/\mu_2)^{1/3})^2/T^2)/\Theta =$

$=1/2\cdot(\mu_1 (1+(\mu_2/\mu_1)^{1/3})^2 + \mu_2 (1+(\mu_1/\mu_2)^{1/3})^2)/T^2/\Theta$.

Example 6. Let there be two points as in Example 4, they move consequently.

We have $\Delta H =((m_1g\mu T_1 + 2d/T_1)^2+(m_2g\mu T_2 + 2d/T_2)^2)/8/\Theta$, $T_1 + T_2 = T$.

Substituting $T_2=T-T_1$ and differentiating yields an equation of 6th power for a critical point. Hence, a computer program is necessary.

6. Conclusion

The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic. We substituted declared Laws 1, 2 by some examples and we hope that these laws would be substituted by investigation of other processes. We see: the less is the time of transition (more generally, of execution of any action), the more is spending of energy for braking and the more is the (unavoidable) entropy increment being approximately proportional to pollution of environment. Probably, it explains some processes of modern civilization.

May be, legal restrictions on cars' speed, for examples [7-9], are also due to this fact.

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MSC 34A45

DEPENDENCE OF SOLUTION DELAY ON ZEROS OF EIGENVALUES

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Abstract. In this work, using an example of a system of autonomous singularly perturbed equations, the dependence of the solution delay interval near an unstable equilibrium position on the zeros of the eigenvalues of the first approximation matrix is proven. This dependence has not been previously investigated.

Аннотация. Бул жумушта автономдук сингулярдык козголгон теңдемелер системасынын мисалын колдонуп, туруксуз тең салмактуулук абалынын жанында чечимдин кармалуу мезгилинин биринчи жакындоо матрицасынын өздүк маанилеринин нөлдөрүнө көз карандылыгы далилденген. Бул көз карандылык мурда изилденген эмес.

Аннотация. В данной работе на примере одной системы автономных сингулярно возмущенных уравнений, доказана зависимость отрезка задержки решения вблизи неустойчивого положения равновесия от нулей собственных значений матрицы первого приближения. Ранее такая зависимость не была исследована.

Keywords: singular perturbation, equilibrium position, stability, solution delay, domain, successive approximations.

Ачкыч сөздөр: сингулярдык козголуу, тең салмактуулук абал, туруктуулук, чыгырылыштын кармалуусу, аймак, удаалаш жакындоо.

Ключевые слова: сингулярное возмущение, положение равновесия, устойчивость, задержка решения, область, последовательные приближения.

Introduction

The concept of solution delay (SD) in the theory of singularly perturbed equations was first discovered in [1]. The essence of this phenomenon is as follows. In [1], the system

$$\varepsilon x'(t,\varepsilon) = A(y)\tilde{x}(t,\varepsilon) + \gamma V(\tilde{x}(t,\varepsilon))\tilde{x}(t,\varepsilon), \tag{1}$$

$$y' = 1 \tag{2}$$

is considered, where $0 < \varepsilon$ is a small real parameter;

$$x(t,\varepsilon)=colon(x_1(t,\varepsilon),x_2(t,\varepsilon)),\tilde{x}(t,\varepsilon)=colon(x_1(t,\varepsilon)-y,x_2(t,\varepsilon)),$$

$$\gamma-const,\ V(\tilde{x}(t,\varepsilon)=(x_1(t,\varepsilon)-y)^2+x_2^2(t,\varepsilon)),$$

$$A(y)=\begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}. \tag{2a}$$

System (1) is called the system of fast variables, and system (2) is called the system of slow variables (in the terminology of L. S. Pontryagin) [2]. System (1) has an equilibrium position at the point $(y;0)$, and the matrix $A(y)$ has eigenvalues $\lambda_{1,2}(y)=y\pm i$. Consequently, the equilibrium position is stable for $y<0$ and unstable for $y>0$. From [2], it follows that if the equilibrium position becomes unstable, the solution of the system of fast variables should immediately leave the unstable equilibrium position and move a finite distance away. The solution of (2) was taken in the form $y=t$, and for the solution of (1), the initial problem

$$\|\tilde{x}(-1,\varepsilon)\|=O(\varepsilon) \tag{3}$$

was considered. It was proven that the solution of problem (1), (3) does not immediately leave the unstable equilibrium position but remains near it on the interval [0,1]. In [4], general methods for investigating singularly perturbed equations under the violation of stability conditions were developed.

In [5], it was proven that in the presence of poles in the eigenvalues of the matrix $A(y)$, the SD interval can be extended to +∞ by changing the initial value t_0. The dependence of SD on the zeros of eigenvalues has not been investigated. In this work, we solve this problem.

Problem Statement

Consider system (1)-(2) under the condition (2a), where $0<\alpha\in\mathbb{R}$ - is a set of real numbers. The solution of (2) is taken in the form $y=t$, and the problem

$$\|\tilde{x}(-t_0,\varepsilon)\|=c_1\varepsilon, \tag{4}$$

is considered, where $t_0>0$. Here and further, the letters $c_1,c_2,\dots$ denote positive constants independent of ε.

The solution of problem (1)-(4) is investigated for SD and the dependence of SD on the zeros of the eigenvalues $\lambda_{1,2}(y) = y \pm i\alpha$, i.e., on α.

Preliminary Transformations and Geometric Constructions

In (1), new unknown functions are introduced:

$x_1(t,\varepsilon) - t = u_1(t,\varepsilon),\ \ x_2(t,\varepsilon) = u_2(t,\varepsilon), u(t,\varepsilon) = colon(u_1(t,\varepsilon), u_2(t,\varepsilon))$

and we obtain (arguments of the unknown function will be omitted):

$$\varepsilon u' = A(t)u - e\varepsilon + \gamma V((u,u))u, \tag{5}$$

$$\|u(-t_0,\varepsilon)\| \le c_1\varepsilon, \tag{6}$$

where $e = colon(1,0),\ (u,u) = u_1^2 + u_2^2$.

In system (5), the second equation is multiplied by i, and then the first equation is added and subtracted from the second, resulting in the system

$$\varepsilon z_k{}' = \lambda_k(t)z_k\ -\ \varepsilon\ + \gamma(z_1,z_2)z_k,\quad k = 1,2 \tag{7}$$

with the condition

$$|z_k(-t_0,\varepsilon)| = \left|z_k^0\right| \le c_1\varepsilon. \tag{8}$$

Problem (6)-(7) is replaced by the integral equation

$$z_k = z_k^0 exp\frac{F_k(t)}{\varepsilon} + \frac{1}{\varepsilon}\int_{-t_0}^{t}[-\varepsilon + \gamma(z_1,z_2)z_k]exp\frac{F_k(t)-F_n(\tau)}{\varepsilon}d\tau \tag{9}$$

where $F_k(t) = \int_{-t_0}^{t}\lambda_k(\tau)d\tau = \int_{-t_0}^{t}(\tau + (-1)^{k-1}i\alpha)d\tau = \frac{1}{2}[(t + (-1)^{k-1}i\alpha) -$

$-(-t_0 + (-1)^{k-1}i\alpha))$

To (9), the method of successive approximations is applied, defined as

$$z_{km} = z_k^0 exp\frac{F_k(t)}{\varepsilon} + \frac{1}{\varepsilon}\int_{-t_0}^{t}[-\varepsilon + \gamma(z_{1m-1},z_{2m-1})z_{km-1}]exp\frac{F_k(t)-F_n(\tau)}{\varepsilon}d\tau, \tag{10}$$

$z_{k0} \equiv 0, m = 1,2,...$

Now, the main task is to estimate and prove the uniform convergence of (10). (10) will be considered in the domain

$$\mathcal{D} = \{t \in C - complex\ domain, |t| < r_0, r_0 > t_0\}.$$

Some geometric constructions are carried out using the functions $ReF_k(t)$.. The functions $ReF_1(t)$ *and* $ReF_2(t)$ have double zeros at the points $t = -i\alpha,\ t = i\alpha$, respectively, and the level lines passing through these points branch out.

Further, it is assumed that $t_0 > \alpha$, and a straight line passing through the points $(-t_0; 0)$ *and* $(0; -\alpha)$. $t_2 = kt_1 + b, t_2 = 0, -kt_0 + b = 0, b = kt_0; t_2 = k(t_1 + 1)$; $t_1 = 0, t_2 = -\alpha, \quad -\alpha = kt_0, \quad k = -\frac{\alpha}{t_0}$.

$$t_2 = -\frac{\alpha}{t_0}t_1 - \frac{\alpha}{t_0}t_0 = -\frac{\alpha}{t_0}(t_1 + t_0).\ t_2 = -\frac{\alpha}{t_0}(t_1 + t_0)$$

The part of this straight line connecting the points $(-t_0; 0)$ and $(0; -\alpha)$ is denoted as (k_1). The segment connecting the points $(0; -\alpha)$ and $(\alpha; 0)$ is denoted as (k_2). The segments $(\overline{k_1})$ and $(\overline{k_2})$ symmetric to (k_1) and (k_2) are defined. Further, symmetry is understood with respect to the real axis.

The domain bounded by $(k_1), (k_2)$ and $(\overline{k_1}, \overline{k_2})$, is denoted as $\mathcal{D}_0$ (Figure 1).

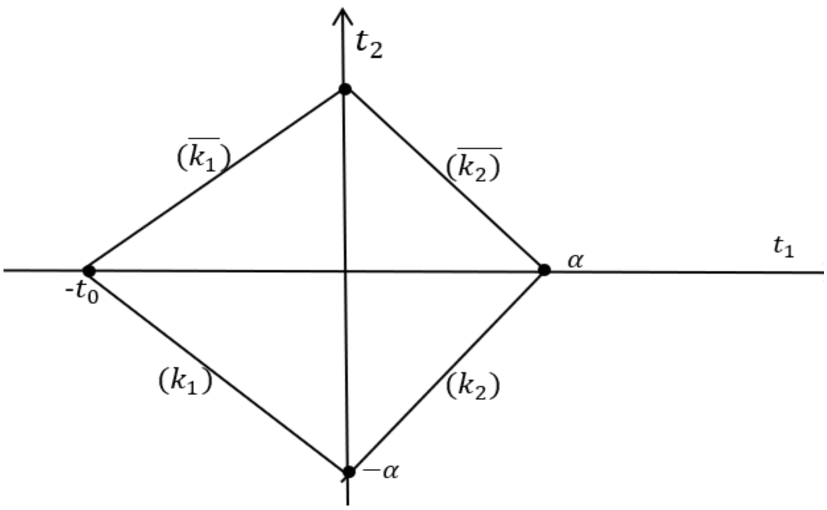

Figure 1. Domain $\mathcal{D}_0$

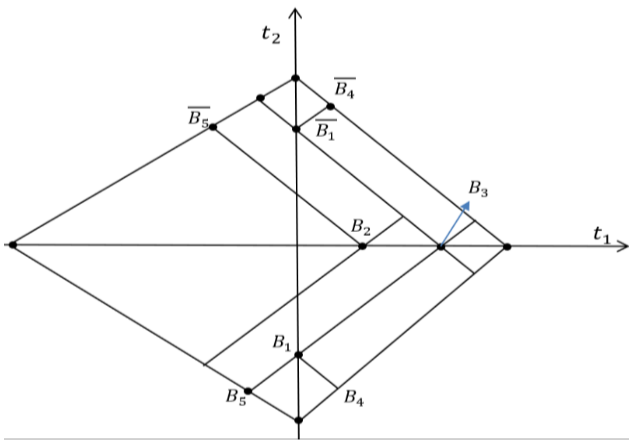

Figure 2. Division of the domain $\mathcal{D}_0$

Segments are drawn through the points $B_1, \overline{B_1}, B_2, B_3$. The segments passing through and $(\overline{B_1}, \overline{B_4})$ are parallel to the straight line $t_1 - t_2 - \alpha$, and the segments passing through the points $(\overline{B_5}, B_2)$, $(\overline{B_1}, \overline{B_4})$, $(B_1 B_4)$ are parallel to the straight line $t_1 + t_2 - \alpha = 0$. The point B_2 has coordinates $(q; 0)$, where $(\alpha - q)$ is independent of ε; B_3 has coordinates $(\alpha - \sqrt{\varepsilon}; 0)$ (Figure 2). After dividing $\mathcal{D}_0$, the domains $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5, \overline{\mathcal{D}_4}, \overline{\mathcal{D}_5}$ are obtained (Figure 3).

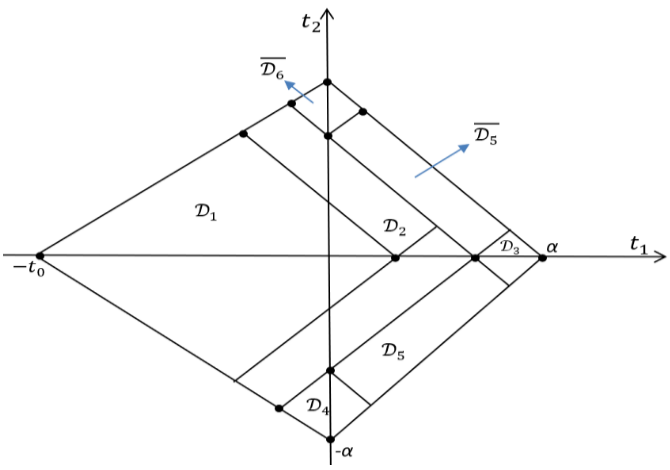

Figure 3. Domains $\mathcal{D}_j (j = 1,2,3,4.5)$ and $\overline{\mathcal{D}_k} (k = 4,5)$

Now, the integration paths for (10) are chosen. For z_{1m}, the path consists of:

If $t \in \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_4 \cup \overline{\mathcal{D}_5}$, then the part of (k_1) connecting the points $(-t_0; 0), (\widetilde{t_1}; t_2)$ and the part of the straight line $\{t_1 - t_2 - \tilde{q} = 0,\ -\alpha \leq \tilde{q} \leq \alpha\}$ connecting the points $(\widetilde{t_1}; \widetilde{t_2}), (t_1; t_2)$; If $t \in \mathcal{D}_3 \cup \mathcal{D}_4$, then the part of (k_1) and the part of (k_2) connecting the points $(-t_0; 0), (\widetilde{t_1}; t_2)$ and the part of the straight line $\{t_1 + t_2 - \tilde{\tilde{q}} = 0,\ -\sqrt{\varepsilon} \leq \tilde{q} \leq \alpha\}$ connecting the points $(\widetilde{t_1}; \widetilde{t_2})$ and $(t_1; t_2)$.

For z_{2m}, the path is chosen symmetrically to the paths of z_{1m}. According to the chosen integration paths, the boundedness and uniform convergence of (9) are proven.

All calculations are carried out as in [6, 7]. As a result, for the solution of (9), we have the estimate

$$\|z\| \leq c_2 \omega(t, \varepsilon), \tag{11}$$

where $0 < c_2$ is independent of ε;

$$\omega(t, \varepsilon) = \begin{cases} \varepsilon, t \in \mathcal{D}_1; \\ \sqrt{\varepsilon}, t \in \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5 \cup \overline{\mathcal{D}_4} \cup \overline{\mathcal{D}_5} \end{cases}.$$

If the estimates are considered for real values of t, then we obtain

$$\|z\| \leq c_2 \omega_0(t, \varepsilon), \tag{12}$$

where $\omega_0(t, \varepsilon) = \begin{cases} \varepsilon, -t_0 \leq t \leq q \\ \sqrt{\varepsilon}, q \leq t \leq \alpha \end{cases}$

From (12), the following theorem follows:

Theorem. Consider problem (1)-(2), (4) under the condition $A(y) = \begin{pmatrix} y & -\alpha \\ \alpha & y \end{pmatrix}$. Then, for the solution of problem (1), (4), SD occurs on the interval $[t_0, \alpha]$, and as α moves away from the boundary of the violation of the stability of the equilibrium position, the SD interval expands, and as it approaches, the SD interval narrows. Note that in [5, 6, 7], cases were considered where the first approximation matrix $A(y)$ as several complex-conjugate eigenvalues, and all eigenvalues influence the stability of the equilibrium position.

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MSC 45A05, 45B05

UNIQUENESS OF SOLUTIONS FOR CERTAIN LINEAR EQUATIONS OF THE THIRD KIND WITH TWO VARIABLES

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В данной работе, с помощью метода неотрицательных квадратичных форм, методам функционального анализа доказывается единственность решений линейных интегральных уравнений третьего рода с с двумя независимыми переменными.

Ключевые слова: линейный, интегральное уравнение, третьего рода, единственность.

In the present article the theorem about uniqueness of the linear integral equations of the third kind two independent variables, with method of nonnegative quadratic forms and functional analysis methods.

Key words: linear, integral equation, third kind, uniqueness.

Бул макалада терс эмес квадраттык формалар усулунун, функционалдык анализдин усулдарынын жардамы менен үчүнчү түрдөгү эки өзгөрмөлүү сызыктуу интегралдык тендемелердин чыгарылыштын жалгыздыгы далилденди.

Негизги сөздөр: сызыктуу, интегралдык тендеме, үчүнчү түрдөгү, жалгыздык.

1. Introduction

Inverse and ill-posed problems are currently attracting great interest. Various problems of the theory of integral equations of the first and third kind were studied in [1-14]. Theories of ill-posed problems, the foundation of which was laid by A.N. Tikhonov [1], M.M. Lavrentiev [2], [3] and V.K. Ivanov [4], is an outstanding achievement of mathematics and allowed a new level in the development of new technologies in various fields of activity. But fundamental results for Fredholm integral equations of the first kind were obtained in [3, 5], where regularizing operators in the sense of M.M. Lavrent'ev were constructed. Results on non-classical Volterra integral equations of the first kind can be found in [6]. In [7,8], problems of regularization, uniqueness and existence of solutions for Volterra integral and operator equations of the first kind are studied. In [8-10], uniqueness theorems were proved and reqularizing operators in the sense of Lavrent'ev were constructed for systems of Volterra and Fredholm integral equations.

In the prezent paper, on the basis of the methods of integral transformation, nonnegative quadratic forms and functional analysis the issues of uniqueness, stability of the solution for a new class of the linear integral equations of the third kind with two independent variables.

Consider the linear integral equations of the third kind with two independent variables

$$Ku \equiv m(t,x)u(t,x) + \int_a^b K(t,x,y)\ u\ (t,y)dy + \int_{t_0}^t H(t,x,s)\ u\ (s,x)ds\ +$$
$$\int_{t_0}^t \int_a^x C(t,x,s,y)\ u(s,y)dyds = f(t,x), (t,x) \in G = \{(t,x) \in R^2: t_0 \le t \le T,$$
$$a \le x \le b\}, \qquad (1)$$

where

$$K(t,x,y) = \begin{cases} A(t,x,y), t_0 \le t \le T, & a \le y \le x \le b, \\ B(t,x,y), t_0 \le t \le T, & a \le x \le y \le b, \end{cases} \qquad (2)$$

$A(t,x,y), B(t,x,y), B(t,x,y), C(t,x,s,y)$ are known functions defined on the domains

$$G_1 = \{(t,x,y), t_0 \le t \le T,\ \ a \le y \le x \le b\},$$

$$G_2 = \{(t,x,y), t_0 \le t \le T,\ \ a \le x \le y \le b\},$$
$$G_3 = \{(t,x,s,y), t_0 \le s \le t \le T,\ \ a \le x \le b\},$$
$$G_4 = \{(t,x,s,y), t_0 \le s \le t \le T,\ \ a \le y \le x \le b\}.$$

$m(t,x)$ and $f(t,x)$ are the known functions, $(t,x) \in G$, $u\{(t,x)$ is an unknown function, $u(t,x) \in L_2(G)$, where $L_2(G)$ is the space of square summable functions in the domain G. Let's denote

$$P(s,y,z) = A(s,y,z) + B(s,z,y), (s,y,z) \in G_1, \qquad (3)$$

Assume that the following conditions are satisfied:

$a)\ m(t,x) \ge 0, for\ \ all\ (t,x) \in G,$

$P(s,b,a) \in C[t_0,T], P(s,b,a) \geq 0$ $for\ \ all\ t \in [t_0,T]$,

$P'_y(s,y,a) \in C(G), P'_y(s,y,a) \leq 0$ $for\ \ all\ (s,y) \in G$,

$P'_z(s,b,z) \in C(G), P'_z(s,b,z) \geq 0$ $for\ \ all\ (s,z) \in G$,

$P''_{zy}(s,y,z) \in C(G_1), P''_{zy}(s,y,z) \leq 0$ $for\ \ all\ (s,y,z) \in G_1$,

$b) H(T,y,t_0) \in C[a,b], H(T,y,t_0) \geq 0$ $for\ \ all\ t \in [a,b]$,

$H'_s(s,y,t_0) \in C(G), H'_s(s,y,t_0) \leq 0$ $for\ \ all\ (s,y) \in G$,

$H'_\tau(T,y,\tau) \in C(G), H'_\tau(T,y,\tau) \geq 0$ $for\ \ all\ (s,y) \in G$,

$H''_{\tau s}(s,y,\tau) \in C(G_3), H''_{\tau s}(s,y,\tau) \leq 0$ $for\ \ all\ (s,y,\tau) \in G_3$,

and for anyone $\nu(t,x) \in L_2(G)$,

$\int_a^x A(t,x,y)\,\nu\,(t,y)dy, \int_x^b B(t,x,y)\ \nu\,(t,y)dy, \int_{t_0}^t H(t,x,s)\,\nu(s,x)ds \in$ G, where $C\,[t_0,T], C(G), C(G_1), C(G_3)$ are the spaces of continuous functions respectively in $[t_0,T], G, G_1, G_3$.

$c)\ C(T,b,t_0,a) \geq 0$

$C'_s(s,b,t_0,a) \in C[t_0,T], C'_s(s,b,t_0,a) \leq 0$ $for\ almost\ all\ \ s \in [t_0,T]$,

$C'_\tau(T,b,\tau,a) \in C[t_0,T], C'_\tau(T,b,\tau,a) \geq 0$ $for\ almost\ all\ \ \tau \in [t_0,T]$,

$C'_y(T,y,t_0,a) \in C[a,b], C'_y(T,y,t_0,a) \leq 0$ $for\ almost\ all\ \ y \in [a,b]$,

$C'_z(T,b,t_0,z) \in C[a,b], C'_z(T,b,t_0,z) \geq 0$ $for\ almost\ all\ \ z \in [a,b]$,

$C''_{sy}(s,y,t_0,a) \in C(G), C''_{sy}(s,y,t_0,a) \geq 0$ $for\ almost\ all\ \ (s,y) \in G$,

$C''_{\tau y}(T,y,\tau,t_0) \in C(G), C''_{\tau y}(T,y,\tau,t_0) \leq 0$ $for\ almost\ all\ \ (y,\tau) \in G$,

$C''_{zs}(s,b,t_0,z) \in C(G), C''_{zs}(s,b,t_0,z) \leq 0$ $for\ almost\ all\ \ (s,z) \in G$,

$C''_{\tau z}(T,b,\tau,z) \in C(G), C''_{\tau z}(T,b,\tau,z) \geq 0$ $for\ almost\ all\ \ (\tau,z) \in G$,

$C'''_{\tau sy}(s,y,\tau,a) \in C(G_3), C'''_{\tau sy}(s,y,\tau,a) \geq 0$ $for\ almost\ all\ \ (s,y,\tau) \in G_3$,

$C'''_{\tau zs}(s,b,\tau,z) \in C(G_3), C'''_{\tau zs}(s,b,\tau,z) \leq 0$ $for\ almost\ all\ \ (s,\tau,z) \in G_3$,

$C'''_{zsy}(s,y,t_0,z) \in C(G_1), C'''_{zsy}(s,y,t_0,z) \geq 0$ $for\ almost\ all\ \ (s,y,z) \in G_1$,

$C'''_{\tau zy}(T,y,\tau,z) \in C(G_1), C'''_{\tau zy}(T,y,\tau,z) \leq 0$ $for\ almost\ all\ \ (\tau,y,z) \in G_1$,

$C''''_{\tau zsy}(s,y,\tau,z) \in C(G_4), C''''_{\tau zsy}(s,y,\tau,z) \geq 0$ $for\ almost\ all\ \ (s,y,\tau,z) \in G_4$,

$C''_{\tau s}(s,b,\tau,a) \in C(G_5), C''_{\tau s}(s,b,\tau,a) \leq 0$ $for\ almost\ all$

$(s,\tau) \in G_5 = \{\ (s,\tau): t_0 \leq \tau \leq s \leq T\}$,

$C''_{zy}(T,y,t_0,z)\in C(G_6), C''_{zy}(T,y,t_0,z)\le 0$ *for almost all*
$(y,z)\in G_6=\{(y,z): a\le z\le y\le b\}$;
$d)$ With almost everyone $(s,\tau,y,z)\in G_4$ and for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in R$:

$$L(s,\tau,y,z,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\frac{1}{(s-t_0)(y-a)}\Big\{\Big(-P'_y(s,y,a)\Big)\alpha_1^2-H'_s(s,y,t_0)\alpha_2^2-2C(s,y,t_0,a)\alpha_1\alpha_2-(s-t_0)[H''_{\tau s}(s,y,\tau)\alpha_3^2+2C'_\tau(s,y,\tau,a)\alpha_3\alpha_1]-(y-a)\left[2C'_z(s,y,t_0,z)\alpha_2\alpha_4+P''_{zy}(s,y,z)\alpha_4^2\right]-2(s-t_0)(y-a)C''_{\tau z}(s,y,\tau,z)\alpha_3\alpha_4\Big\}\ge 0.$$

$e)$ If in front of everyone $(s,y,\tau,z)\in G_4, L(s,\tau,y,z,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=0$, then at least one of the following conditions is met:
$1)\alpha_1=0,\ 2)\alpha_2=0,\ 3)\alpha_3=0,\ 4)\alpha_4=0.$

Theorem 1. Let conditions $a),b),c),d)$ *and* $e)$ be satisfied. Then the solution of the equation (1) is unique in $L_2(G)$.

Proof: In the proof we use integration by parts, and methods of functional analysis, the method of non-negative quadratic forms.
By virtue of (2), we write equation (1) in the form

$$m(t,x)u(t,x)+\int_a^x A(t,x,y)\,u\,(t,y)dy+\int_x^b B(t,x,y)\,u\,(t,y)dy+\int_{t_0}^t H(t,x,s)\,u(s,x)ds+\int_{t_0}^t\int_a^x C(t,x,s,y)\,u(s,y)dyds=f(t,x),(t,x)\in G=\{(t,x)\in R^2: t_0\le t\le T,\ a\le x\le b\}, \tag{4}$$

We multiply both sides of equation (4) by the functions *u(t,x)* and integrate over the domain G:

$$m(t,x)u(t,x)+\int_a^b\int_{t_0}^T\int_a^y A(s,y,z)\,u(s,z)u(s,y)dzdsdy$$
$$+\int_a^b\int_{t_0}^T\int_y^b B(s,y,z)\,u(s,z)u(s,y)dzdsdy$$
$$+\int_a^b\int_{t_0}^T\int_{t_0}^s H(s,y,\tau)\,u(\tau,y)u(s,y)d\tau dsdy$$

$$+\int_a^b \int_{t_0}^T \int_{t_0}^s \int_a^y C(s,y,\tau,z)\, u(\tau,z)\ u(s,y) dz d\tau ds dy$$

$$= \int_a^b \int_{t_0}^T f(s,y) u(s,y) ds dy. \qquad (5)$$

Integrating by parts and using the Dirichlet formula we obtain

$$\int_a^b \int_{t_0}^T \int_a^y A(s,y,z)\, u(s,z) u(s,y) dz ds dy$$

$$+\ \int_a^b \int_{t_0}^T \int_y^b B(s,y,z)\, u(s,z) u(s,y) dy ds dz$$

$$+\int_a^b \int_{t_0}^T \int_{t_0}^s H(s,y,\tau)\, u(\tau,y) u(s,y) d\tau ds dy$$

$$+\int_a^b \int_{t_0}^T \int_{t_0}^s \int_a^y C(s,y,\tau,z)\, u(\tau,z)\ u(s,y) dz d\tau ds dy$$

$$= \int_a^b \int_{t_0}^T f(s,y) u(s,y) ds dy.$$

$$m(t,x)u(t,x) + \int_a^b \int_{t_0}^T \int_a^y [A(s,y,z) + B(s,z,y)]\, u(s,z) u(s,y) dz ds dy$$

$$+\ + \int_a^b \int_{t_0}^T \int_{t_0}^s H(s,y,\tau)\, u(\tau,y) u(s,y) d\tau ds dy$$

$$+\int_a^b \int_{t_0}^T \int_{t_0}^s \int_a^y C(s,y,\tau,z)\, u(\tau,z)\ u(s,y) dz d\tau ds dy$$

$$= \int_a^b \int_{t_0}^T f(s,y) u(s,y) ds dy. \qquad (6)$$

Taking into account notation (3), we obtain

$$m(t,x)u(t,x) + \int_a^b \int_{t_0}^T \int_a^y P(s,y,z) u(s,z) u(s,y) dz ds dy$$

$$+\int_a^b \int_{t_0}^T \int_{t_0}^s H(s,y,\tau)\, u(\tau,y) u(s,y) d\tau ds dy$$

$$+\int_a^b \int_{t_0}^T \int_{t_0}^s \int_a^y C(s,y,\tau,z)\,u(\tau,z)\; u(s,y)dzd\tau dsdy$$

$$= \int_a^b \int_{t_0}^T f(s,y)u(s,y)dsdy. \qquad (7)$$

Let us transform each of the integrals on the left side of equation (7). Integrating by parts twice and applying the Dirichlet formula for the first integral, we obtain

$$\int_a^b \int_{t_0}^T \int_a^y P(s,y,z)u(s,z)u(s,y)dzdsdy$$

$$= -\int_a^b \int_{t_0}^T \int_a^y P(s,y,z)\frac{\partial}{\partial z}(\int_z^y u(s,\nu)d\nu)dz\,u(s,y)dsdy$$

$$= \frac{1}{2}\int_{t_0}^T P(s,b,a)\left(\int_a^b u(s,\nu)d\nu\right)^2 ds$$

$$-\frac{1}{2}\int_{t_0}^T \int_a^b P'_y(s,y,a)\left(\int_a^b u(s,\nu)d\nu\right)^2 dyds$$

$$+\frac{1}{2}\int_{t_0}^T \int_a^b P'_z(s,b,z)\left(\int_z^b u(s,\nu)d\nu\right)^2 dzds$$

$$-\frac{1}{2}\int_{t_0}^T \int_a^b \int_a^y P''_{zy}(s,y,z)\left(\int_z^y u(s,\nu)d\nu\right)^2 dzdsdy, \qquad (8)$$

where $P'_t(t,x,s), P'_y(t,x,s)$ are the partial derivatives.

Integrating by parts twice and applying the Dirichlet formula for the second integral, we have

$$\int_a^b \int_{t_0}^T \int_{t_0}^s H(s,y,\tau)u(\tau,y)u(s,y)d\tau dsdy$$

$$= -\int_a^b \int_{t_0}^T \int_{t_0}^s H(s,y,\tau)\frac{\partial}{\partial \tau}(\int_\tau^s u(\xi,y)d\xi)d\tau\,u(s,y)dsdy$$

$$= \frac{1}{2}\int_a^b H(T,y,t_0)\left(\int_{t_0}^T u(\xi,y)d\xi\right)^2 dy -$$

$$\frac{1}{2}\int_a^b\int_{t_0}^T H_s'(s,y,t_0)\left(\int_{t_0}^s u(\xi,y)d\xi\right)^2 dsdy$$

$$+\frac{1}{2}\int_a^b\int_{t_0}^T H_\tau'(T,y,\tau)\left(\int_\tau^T u(\xi,y)d\xi\right)^2 d\tau dy$$

$$-\frac{1}{2}\int_a^b\int_{t_0}^T\int_{t_0}^s H_{\tau s}''(s,y,\tau)\left(\int_\tau^T u(\xi,y)d\xi\right)^2 dzdsdy. \quad (9)$$

To transform the third integral we use the relation

$C\nu_{\tau z}'' = (C\nu)_{\tau z}'' - (C_\tau'\nu)_z' - (C_z'\nu)_\tau' + C_{\tau z}''\nu.$

Then, integrating by parts, we have

$$\int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y C(s,y,\tau,z)\,u(\tau,z)\;u(s,y)dzd\tau dsdy$$

$$=\int_a^b\int_{t_0}^T C(s,y,t_0,a)(\int_\tau^s\int_a^y u(\xi,\nu)\;\;d\nu d\xi)u(s,y)dyds$$

$$+\int_a^b\int_{t_0}^T\int_{t_0}^s C_\tau'(s,y,\tau,a)\,(\int_\tau^s\int_a^y u(\xi,\nu)\;\;d\nu d\xi)u(s,y)dsd\tau dy$$

$$+\int_a^b\int_{t_0}^T\int_{t_0}^s C_z'(s,y,t_0,z)\,(\int_{t_0}^s\int_z^y u(\xi,\nu)\;\;d\nu d\xi)u(s,y)dydsdz$$

$$+\int_a^b\int_{t_0}^T\int_\tau^T\int_z^b C_{\tau z}''(s,y,\tau,z)\int_\tau^s\int_z^y u(\xi,\nu)\;\;d\nu d\xi)u(s,y)dydsd\tau dz.$$

Further, keeping in mind that

$C\nu\nu_{sy}'' = \frac{1}{2}(C\nu^2)_{sy}'' - \frac{1}{2}(C_s'\nu^2)_y' - \frac{1}{2}\left(C_y'\nu^2\right)_s' + \frac{1}{2}C_{sy}''\nu^2 - C\nu_y'\nu_s',$

and applying the Dirichlet formula we

get$\int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y C(s,y,\tau,z)\,u(\tau,z)u(s,y)dzd\tau dsdy =$

$\frac{1}{2}C(T,b,t_0,a)\left(\int_a^b\int_{t_0}^T u(\xi,\nu)d\xi d\nu\right)^2 -$

$$-\frac{1}{2}\int_{t_0}^T C_s'(s,b,t_0,a)\left(\int_a^b\int_{t_0}^s u(\nu,\xi)d\nu d\xi\right)^2 ds$$

$$-\frac{1}{2}\int_a^b C_y'(T,y,t_0,a)\left(\int_a^y\int_{t_0}^T u(\nu,\xi)d\nu d\xi\right)^2 dy +$$

$$+\frac{1}{2}\int_a^b\int_{t_0}^T C''_{sy}(s,y,t_0,a)\left(\int_{t_0}^s\int_a^y u(\xi,\nu)d\nu d\xi\right)^2 dsdy$$

$$-\int_a^b\int_{t_0}^T C(s,y,t_0,a)\left(\int_{t_0}^s u(\xi,y)d\xi\right)\left(\int_a^y u(s,\nu)d\nu\right)dyds+$$

$$+\frac{1}{2}\int_{t_0}^T C'_\tau(T,b,\tau,a)\left(\int_\tau^T\int_a^b u(\xi,\nu)d\nu d\xi\right)^2 d\tau$$

$$-\frac{1}{2}\int_{t_0}^T\int_\tau^T C''_{\tau s}(s,b,\tau,a)\left(\int_\tau^s\int_a^b u(\xi,\nu)d\nu d\xi\right)^2 dsd\tau-$$

$$-\frac{1}{2}\int_a^b\int_{t_0}^T C''_{\tau y}(T,y,\tau,t_0)\left(\int_\tau^T\int_a^y u(\xi,\nu)d\nu d\xi\right)^2 d\tau dy$$

$$+\frac{1}{2}\int_a^b\int_{t_0}^T\int_\tau^T C'''_{\tau sy}(s,y,\tau,a)\left(\int_\tau^s\int_a^y u(\xi,\nu)d\nu d\xi\right)^2 dsd\tau dy-$$

$$-\int_a^b\int_{t_0}^T\int_\tau^T C'_\tau(s,y,\tau,a)\left(\int_\tau^s u(\xi,y)d\xi\right)\left(\int_a^y u(s,\nu)d\nu\right)dsd\tau dy$$

$$+\frac{1}{2}\int_a^b C'_z(T,b,t_0,z)\left(\int_{t_0}^T\int_z^b u(\xi,\nu)d\nu d\xi\right)^2 dz-$$

$$-\frac{1}{2}\int_a^b\int_{t_0}^T C''_{zs}(s,b,t_0,z)\left(\int_{t_0}^s\int_z^b u(\xi,\nu)d\nu d\xi\right)^2 dsdz$$

$$-\frac{1}{2}\int_a^b\int_z^b C''_{zy}(T,y,t_0,z)\left(\int_{t_0}^T\int_z^y u(\xi,\nu)d\nu d\xi\right)^2 dydz+$$

$$+\frac{1}{2}\int_a^b\int_{t_0}^T\int_z^b C'''_{zsy}(s,y,t_0,z)\left(\int_{t_0}^s\int_z^y u(\xi,\nu)\,d\nu d\xi\right)^2 dydsdz-$$

$$-\int_a^b\int_{t_0}^T\int_z^b C'_z(s,y,t_0,z)\left(\int_{t_0}^s u(\xi,y)d\xi\right)\left(\int_z^y u(s,\nu)d\nu\right)dydsdz+$$

$$+\frac{1}{2}\int_a^b\int_{t_0}^T C''_{\tau z}(T,b,\tau,z)\left(\int_\tau^T\int_z^b u(\xi,\nu)d\nu d\xi\right)^2 d\tau dz-$$

$$-\frac{1}{2}\int_a^b\int_{t_0}^T\int_z^b C'''_{\tau zy}(T,y,\tau,z)\left(\int_\tau^T\int_z^y u(\xi,\nu)d\nu d\xi\right)^2 dyd\tau dz-$$

$$-\frac{1}{2}\int_a^b\int_{t_0}^T\int_\tau^T C'''_{\tau zs}(s,b,\tau,z)\left(\int_\tau^s\int_z^b u(\xi,\nu)\,d\nu d\xi\right)^2 dsd\tau dz+$$

$$+\frac{1}{2}\int_a^b\int_{t_0}^T\int_\tau^T\int_z^b C^{(IV)}_{\tau zsy}(s,y,\tau,z)\left(\int_\tau^s\int_z^y u(\xi,\nu)d\nu d\xi\right)^2 dyds d\tau dz-$$

$$-\int_a^b\int_{t_0}^T\int_\tau^T\int_z^b C''_{\tau z}(s,y,\tau,z)\left(\int_\tau^s u(\xi,y)d\xi\right)\left(\int_z^y u(s,\nu)d\nu\right)dydsd\tau dz.$$

From here, applying the Dirichlet formulas, we obtain

$$\int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y C(s,y,\tau,z)\,u(\tau,z)u(s,y)dzd\tau dsdy=$$

$$=\frac{1}{2}C(T,b,t_0,a)\left(\int_a^b\int_{t_0}^T u(\xi,\nu)d\xi d\nu\right)^2-$$

$$\frac{1}{2}\int_{t_0}^T C'_s(s,b,t_0,a)\left(\int_a^b\int_{t_0}^s u(\nu,\xi)\,d\nu d\xi\right)^2 ds-$$

$$\frac{1}{2}\int_a^b C'_y(T,y,t_0,a)\left(\int_a^y\int_{t_0}^T u(\nu,\xi)\,d\xi d\nu\right)^2 dy+$$

$$+\frac{1}{2}\int_a^b\int_{t_0}^T C''_{sy}(s,y,t_0,a)\left(\int_{t_0}^s\int_a^y u(\xi,\nu)\,d\nu d\xi\right)^2 dsdy-$$

$$\int_a^b\int_{t_0}^T C\,(s,y,t_0,a)\left(\int_{t_0}^s u(\xi,y)\,d\xi\right)\left(\int_a^y u(s,\nu)d\nu\right)dsdy+$$

$$+\frac{1}{2}\int_{t_0}^T C'_\tau(T,b,\tau,a)\left(\int_\tau^T\int_a^b u(\xi,\nu)\,d\xi d\nu\right)^2 d\tau$$

$$-\frac{1}{2}\int_{t_0}^T\int_{t_0}^s C''_{\tau s}(s,b,\tau,a)\left(\int_\tau^s\int_a^b u(\xi,\nu)\,d\nu d\xi\right)^2 d\tau ds-$$

$$-\frac{1}{2}\int_a^b\int_{t_0}^T C''_{\tau y}(T,y,\tau,t_0)\left(\int_\tau^T\int_a^y u(\xi,\nu)d\nu d\xi\right)^2 d\tau dy+$$

$$\frac{1}{2}\int_a^b\int_{t_0}^T\int_{t_0}^s C'''_{\tau sy}(s,y,\tau,a)\left(\int_\tau^s\int_a^y u(\xi,\nu)d\nu d\xi\right)^2 d\tau ds-$$

$$-\int_a^b\int_{t_0}^T\int_{t_0}^s C'_\tau(s,y,\tau,a)\left(\int_\tau^s u(\xi,y)d\xi\right)\left(\int_a^y u(s,\nu)d\nu\right)d\tau ds+$$

$$+\frac{1}{2}\int_a^b C'_z(T,b,t_0,z)\left(\int_{t_0}^T\int_z^b u(\xi,\nu)d\nu d\xi\right)^2 dz-$$

$$\frac{1}{2}\int_a^b\int_{t_0}^T C''_{zs}\,(s,b,t_0,z)\left(\int_{t_0}^s\int_z^b u(\xi,\nu)d\nu d\xi\right)^2 dsdz-$$

$$-\frac{1}{2}\int_a^b \int_a^y C''_{zy}(T,y,t_0,z)\left(\int_{t_0}^T \int_z^y u(\xi,\nu)d\nu d\xi\right)^2 dzdy +$$

$$+\frac{1}{2}\int_a^b \int_{t_0}^T \int_a^y C'''_{zsy}(s,y,t_0,z)\left(\int_{t_0}^s \int_z^y u(\xi,\nu)d\nu d\xi\right)^2 dzdsdy -$$

$$-\int_a^b \int_{t_0}^T \int_a^y C'_z(s,y,t_0,z)\left(\int_{t_0}^s u(\xi,y)d\xi\right)\left(\int_z^y u(s,\nu)d\nu\right) dzdsdy +$$

$$+\frac{1}{2}\int_a^b \int_{t_0}^T C''_{\tau z}(T,b,\tau,z)\left(\int_\tau^T \int_z^b u(\xi,\nu)d\nu d\xi\right)^2 d\tau dz$$

$$-\frac{1}{2}\int_a^b \int_{t_0}^T \int_a^y C'''_{\tau zy}(T,y,\tau,z)\left(\int_\tau^T \int_z^y u(\xi,\nu)d\nu d\xi\right)^2 dzd\tau dy -$$

$$-\frac{1}{2}\int_a^b \int_{t_0}^T \int_{t_0}^s C'''_{\tau zs}(s,b,\tau,z)\left(\int_\tau^s \int_z^b u(\xi,\nu)d\nu d\xi\right)^2 d\tau dsdz +$$

$$+\frac{1}{2}\int_a^b \int_{t_0}^T \int_{t_0}^s \int_a^y C^{(IV)}_{\tau zsy}(s,y,\tau,z)\left(\int_\tau^s \int_z^y u(\xi,\nu)d\nu d\xi\right)^2 dzd\tau dsdy -$$

$$-\int_a^b \int_{t_0}^T \int_{t_0}^s \int_a^y C''_{\tau z}(s,y,\tau,z)\left(\int_\tau^s u(\xi,y)d\xi\right)\left(\int_z^y u(s,\nu)d\nu\right)dzd\tau dsdy. \qquad (10)$$

Substituting expressions (8), (9) and (10) into (7), we obtain$m$(t,x)u(t,x) +

$\frac{1}{2}C(T,b,t_0,a)\left(\int_a^b \int_{t_0}^T u(\xi,\nu)d\xi d\nu\right)^2 + \frac{1}{2}\int_{t_0}^T\{P(s,b,a)\left(\int_a^b u(s,\nu)d\nu\right)^2 -$

$C'_s(s,b,t_0,a)\left(\int_a^b \int_{t_0}^s u(\nu,\xi)d\nu d\xi\right)^2 +$

$+C'_s(T,b,s,a)\left(\int_s^T\int_a^b u(\xi,\nu)d\xi d\nu\right)^2\Bigg\}ds + \frac{1}{2}\int_a^b\{H(T,y,t_0)\left(\int_{t_0}^T u(\xi,y)d\xi\right)^2 -$

$C'_y(T,y,t_0,a)\left(\int_a^y \int_{t_0}^T u(\nu,\xi)d\nu d\xi\right)^2 +$

$$+C'_y(T,b,t_0,y)\left(\int_{t_0}^T \int_y^b u(\xi,\nu)d\nu d\xi\right)^2\Bigg\}dy +$$

$$\frac{1}{2}\int_a^b \int_{t_0}^T \int_{t_0}^s \int_a^y \Bigg\{L\left(s,y,\tau,z,\int_a^y u(s,\nu)d\nu, \int_{t_0}^s u(\xi,y)d\xi, \int_\tau^s u(\xi,y)d\xi, \int_z^y u(s,\nu)d\nu\right)$$

$$+\frac{1}{(s-t_0)(y-a)}\times$$

$$\times\left[P_y'(s,b,y)\left(\int_y^b u(s,\nu)d\nu\right)^2 + H_s'(T,y,s)\left(\int_s^T u(\xi,y)d\xi\right)^2\right.$$

$$+ +C_{sy}''(s,y,t_0,a)\left(\int_{t_0}^s\int_a^y u(\xi,\nu)d\nu d\xi\right)^2$$

$$- C_{sy}''(T,y,s,t_0)\left(\int_s^T\int_a^y u(\xi,\nu)d\nu d\xi\right)^2$$

$$- -C_{ys}''(s,b,t_0,y)\left(\int_{t_0}^s\int_y^b u(\xi,\nu)d\nu d\xi\right)^2$$

$$\left.+ C_{sy}''(T,b,s,y)\left(\int_s^T\int_y^b u(\xi,\nu)d\nu d\xi\right)^2\right]$$

$$+ +\frac{1}{y-a}\left[C_{\tau sy}'''(s,y,\tau,a)\left(\int_\tau^s\int_a^y u(\xi,\nu)d\nu d\xi\right)^2\right.$$

$$\left.- C_{\tau ys}'''(s,b,\tau,y)\left(\int_\tau^s\int_y^b u(\xi,\nu)d\nu d\xi\right)^2\right]$$

$$+ \frac{1}{s-t_0}\left[C_{zsy}'''(s,y,t_0,z)\left(\int_{t_0}^s\int_z^y u\,(\xi,\nu)d\nu d\xi\right)^2\right.$$

$$\left.- C_{szy}'''(T,y,s,z)\left(\int_s^T\int_z^y u\,(\xi,\nu)d\nu d\xi\right)^2\right] +$$

$$\left.+C_{\tau zsy}^{(IV)}(s,y,\tau,z)\left(\int_\tau^s\int_z^y u(\xi,\nu)d\nu d\xi\right)^2\right\} dz d\tau ds dy -$$

$$-\frac{1}{2}\int_{t_0}^T\int_{t_0}^s C_{\tau s}''(s,b,\tau,a)\left(\int_\tau^s\int_a^b u(\xi,\nu)d\nu d\xi\right)^2 d\tau ds -$$

$$-\frac{1}{2}\int_a^b\int_a^y C_{zy}''(T,y,t_0,z)\left(\int_{t_0}^T\int_z^y u(\xi,\nu)d\nu d\xi\right)^2 dz dy = \int_a^b\int_{t_0}^T f(s,y)u(s,y)ds dy.$$

(11)

Substituting expressions (8), (9) and (10) into (7), we obtain $f(t,x) \equiv 0, (t,x) \in G$. Then, taking into account conditions 1), 2), 3) and 4) from (11) we have $\int_{t_0}^s u(\xi,y)d\xi = 0, \forall(s,y) \in G$ or $\int_a^y u(s,\nu)d\nu = 0, \forall(s,y) \in G$.

From here $u(t,x)=0$, in front of everyone $(t,x)\in G$. Theorem 1 has been proven.

Example. Consider the equation (1) with $A(t,x,y)=$ $m_0(t)\beta_0(x)\gamma_0(y), B(t,x,y)=m_0(t)m_1(x)\beta_0(y),$

$$H(t,x,s)=m_2(t)\beta_1(x)\beta_2(s), C(t,x,s,y)=\gamma_1(t)\gamma_2(x)\gamma_3(s)\gamma_4(y),$$

where $m_0(t), m_2(t), m_2'(t)\beta_2(t), \beta_2'(t), \gamma_1(t), \gamma_1'(t), \gamma_3(t), \gamma_3'(t)\in C[t_0,T]$,

$$\beta_0(x), \beta_0'(x), \gamma_0(x), \gamma_0'(x), m_1(x), m_1'(x), \beta_1(x), \gamma_2(x), \gamma_2'(x), \gamma_4(x), \gamma_4'(x) \in C[a,b],$$

$$m_0(t)\geq 0, m_2(t)\geq 0, m_2'(t)\leq 0, \beta_2(t)\geq 0, \beta_2'(t)\geq 0, \gamma_1(t)\geq 0, \gamma_1'(t) \leq 0, \gamma_3(t)\geq 0$$

and $\gamma_3'(t)\geq 0$ for everyone $t\in[t_{0,}T], \beta_0(x)\geq 0, \beta_0'(x)\leq 0, \gamma_0(x)+m_1(x)\geq 0,$ $\gamma_0'(x)+m_1'(x)\geq 0, \beta_1(x)\geq 0, \gamma_2(x)\geq 0, \gamma_2'(x)\leq 0, \gamma_4(x)\geq 0, \gamma_4'(x)\geq 0$ for everyone $x\in[a,b], \gamma_3(t_0)=0, \gamma_4(a)=0, \gamma_0(a)+m_1(a)\neq 0, m_0(t)\beta_0'(x)<0$ for almost everyone $(t,x)\in G, m_0(s)\beta_0'(y)[\gamma_0(z)+m_1(z)]m_2'(s)\beta_1(y)\beta_2'(\tau)-$ $-(s-t_0)(y-a)\left[\gamma_1(s)\gamma_2(y)\gamma_3'(\tau)\gamma_4'(z)\right]^2\geq 0$ for everyone $(s,y,\tau,z)\in G_4$.

In this case

$$P(t,x,y)=m_0(t)\beta_0(x)[\gamma_0(y)+m_1(y)], (t,x,y)\in G_1, L(s,y,\tau,z,\alpha_1,\alpha_2,\alpha_3,\alpha_4)$$

$$=\frac{1}{(s-t_0)(y-a)}\{-m_0(s)\beta_0'(y)[\gamma_0(a)+m_1(a)]\alpha_1^2-$$

$$-m_2'(s)\beta_1(y)\beta_2(t)\alpha_2^2$$

$$-\left[(s-t_0)m_2'(s)\beta_1(y)\beta_2'(\tau)\alpha_3^2+(y-a)m_0(s)\beta_0'(y)[\gamma_0(z)+\right.$$

$$\left.+m_1(z)]\alpha_4^2+2(s-t_0)(y-a)\gamma_1(s)\gamma_2(y)\gamma_3'(\tau)\gamma_4'(z)\alpha_3\alpha_4\right]\}, (s,y,\tau,z)\in G_4.$$

Then conditions a), b), c) and d) are satisfied.

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Boundedness of Solutions of a Class of Linear Integro-differential Equations of the Second Order on the Semiaxis

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Annotation. In this paper based on a formula for one class of second order linear integro-differential equations with variable coefficients sufficient conditions of exponential estimate of solutions for one class of second order linear integro-differential equations on the semi axis were established.

Keywords. Formula, linear integro-differential equations, second order, boundedness, semi axis.

Аннотация. Бул макалада өзгөрүлмө коэффициенттери бар экинчи тартиптеги сызыктуу дифференциалдык теңдемелердин бир классы үчүн формуланын негизинде жарым октогу экинчи тартиптеги сызыктуу интегро-дифференциалдык теңдемелердин бир классынын чыгарылыштарынын чектелишинин жетиштүү шарттары көрсөтүлгөн.

Урунттуу сөздөр. Формула, сызыктуу интегро-дифференциалдык теңдеме, экинчи тартиптеги, чектелгендик, жарым ок.

Аннотация. В данной работе на основе формулы для одного класса линейных дифференциальных уравнений второго порядка с переменными коэффициентами установлены достаточные условия ограниченности решений одного класса линейных интегро-дифференциальных уравнений второго порядка на полуоси.

Ключевые слова. Формула, линейное интегро-дифференциальное уравнение, второго порядка, ограниченность, полуось.

Consider the linear integro-differential equation

$$y'' + p(t)y' + q(t)y + \int_{t_0}^{t} K(t,s)y(s)ds = f(t), t \geq t_0, \tag{1}$$

with initial conditions

$$y(t_0) = m, y'(t_0) = n;\ m,\ n \in R, \tag{2}$$

where $K(t,s), p(t), q(t), f(t)$ are known functions.

The questions of boundedness and asymptotic stability of solutions on the semiaxis for differential and integro-differential equations were studied in [1-6]. In this paper, on the basis of the results of [7] and the method of transformations, we establish sufficient conditions for the boundedness of solutions of the integro-differential equation (1) on the semiaxis.

Assume the following conditions are met:

a) $q(t) = q_0(t) + q_1(t),\ \ t \ge t_0,$ (3)

$q_0(t) = a(t)[p(t) - a(t)] - l^2(t) + a'(t), t \ge t_0,$ (4)

$l(t) = r exp\left\{\int_{t_0}^{t}[2a(s) - p(s)]ds\right\},$ (5)

where $a(t), a'(t), q_1(t), p(t), f(t)$ are known continuous functions on $[t_0, \infty)$, $r \in R, r \neq 0$;

b) $a(t) + l(t) \ge 0$ and $a(t) - l(t) \ge 0$ for all $t \in [t_0, \infty)$,

$$\int_{t_0}^{t}|f(s)|\left\{exp\left[-\int_{t_0}^{s}\big(a(\tau) + l(\tau) - p(\tau)\big)d\tau\right]exp\left[-\int_{t_0}^{t}\big(a(\tau) - l(\tau)\big)d\tau\right] + \right.$$

$$\left. + exp\left[-\int_{t_0}^{s}\big(a(\tau) - l(\tau) - p(\tau)\big)d\tau\right]exp\left[-\int_{t_0}^{t}\big(a(\tau) + l(\tau)\big)d\tau\right]\right\}ds \le c_0,$$

for all $t \in [t_0, \infty)$, where $0 \le c_0$ *is a constant*;

c) $|q_1(s)|\left\{exp\left[-\int_{t_0}^{s}(a(\tau) + l(\tau) - p(\tau))d\tau\right]exp\left[-\int_{t_0}^{t}\big(a(\tau) - l(\tau)\big)d\tau\right] + \right.$

$\left. exp\left[-\int_{t_0}^{s}(a(\tau) - l(\tau) - p(\tau))d\tau\right]exp\left[-\int_{t_0}^{t}\big(a(\tau) + l(\tau)\big)d\tau\right]\right\} \le l_1(s),$

for all $(t,s) \in G = \{(t,s): t_0 \le s \le t < \infty\}$, where $l_1(t) \in L_1[t_0, \infty), l_1(t) \ge 0$ for all $t \in [t_0, \infty)$;

d) $\int_{s}^{t}|K(\tau,s)|\left\{exp\left[-\int_{t_0}^{\tau}\big(a(\tau) + l(\tau) - p(\tau)\big)d\tau\right]exp\left[-\int_{t_0}^{t}\big(a(\tau) - l(\tau)\big)d\tau\right] + \right.$

$\left. + exp\left[-\int_{t_0}^{\tau}\big(a(\tau) - l(\tau) - p(\tau)\big)d\tau\right]exp\left[-\int_{t_0}^{t}\big(a(\tau) + l(\tau)\big)d\tau\right]\right\}d\tau \le l_2(s)$

for all $(t,s) \in G = \{(t,s): t_0 \le s \le t < \infty\}$, $l_2(t) \in L_1[t_0, \infty), l_2(t) \ge 0$ for all $t \in [t_0, \infty)$.

Substituting (3) into (1) we have

$$y'' + p(t)y' + q_0(t)y = f(t) - q_1(t)y - \int_{t_0}^{t} K(t,s)y(s)ds, t \geq t_0. \qquad (6)$$

Taking into account the corollary of Theorem 1 from [8], from (6) we obtain

$$y(t) = \frac{y_1(t)}{2l(t_0)}\left[m\big(l(t_0) - a(t_0)\big) - n\right] + \frac{y_2(t)}{2l(t_0)}[n + m(l(t_0) + a(t_0))] + y_3(t), \quad (7)$$

where

$$y_1(t) = exp\left[-\int_{t_0}^{t}(a(s) + l(s))ds\right], \qquad (8)$$

$$y_2(t) = exp\left[-\int_{t_0}^{t}(a(s) - l(s))ds\right], \qquad (9)$$

$$y_3(t) = \int_{t_0}^{t}\left[f(s) - q_1(s)y(s) - \int_{t_0}^{s} K(s,\tau)y(\tau)d\tau\right] \times$$

$$\times [2l(s)y_1(s)y_2(s)]^{-1}[y_1(s)y_2(t) - y_1(t)y_2(s)]ds, t \geq t_0. \qquad (10)$$

Substituting (8), (9), (10) into (7) and taking into account (5) we get

$$y(t) = \frac{1}{2l(t_0)}\left[m\big(l(t_0) - a(t_0)\big) - n\right]exp\left\{-\int_{t_0}^{t}[a(s) + l(s)]ds\right\} +$$

$$+\frac{1}{2l(t_0)}\left[n + m\big(l(t_0) + a(t_0)\big)\right]exp\left\{-\int_{t_0}^{t}[a(s) - l(s)]ds\right\} +$$

$$+\int_{t_0}^{t}\frac{1}{2r}exp\left[\int_{t_0}^{s}p(\tau)d\tau\right]\left[f(s) - q_1(s)y(s) - \int_{t_0}^{s}K(s,\tau)y(\tau)d\tau\right]\times$$

$$\times\{exp\left[-\int_{t_0}^{s}\big(a(\tau) + l(\tau)\big)d\tau\right]\times$$

$$\times exp\left[-\int_{t_0}^{t}\big(a(\tau) - l(\tau)\big)d\tau\right] - exp\left[-\int_{t_0}^{t}\big(a(\tau) + l(\tau)\big)d\tau\right]\times$$

$$\times exp\left[-\int_{t_0}^{s}\big(a(\tau) - l(\tau)\big)d\tau\right]\}ds, \; t \geq t_0. \qquad (11)$$

In the double integral in (11), applying the Dirichlet formula, we introduce the notation:

$$F(t) = \frac{1}{2r}\left[m\big(r - a(t_0)\big) - n\right]exp\left\{-\int_{t_0}^{t}[a(s) + l(s)]ds\right\} +$$

$$+\frac{1}{2r}\left[n + m\big(r + a(t_0)\big)\right]exp\left\{-\int_{t_0}^{t}[a(s) - l(s)]ds\right\} +$$

$$+\frac{1}{2r}\int_{t_0}^{t}f(s)\{exp\left[-\int_{t_0}^{s}\big(a(\tau) + l(\tau) - p(\tau)\big)d\tau\right]exp\left[-\int_{t_0}^{t}\big(a(\tau) - l(\tau)\big)d\tau\right] -$$

$$-exp\left[-\int_{t_0}^{s}\big(a(\tau) - l(\tau) - p(\tau)\big)d\tau\right]exp\left[-\int_{t_0}^{t}\big(a(\tau) + l(\tau)\big)d\tau\right]\}ds, \qquad (12)$$

$$H(t,s)=-\frac{q_1(s)}{2r}\{exp\left[-\int_{t_0}^{s}\big(a(\tau)+l(\tau)-p(\tau)\big)d\tau\right]exp\left[-\int_{t_0}^{t}\big(a(\tau)-l(\tau)\big)d\tau\right]$$

$$-exp\left[-\int_{t_0}^{s}\big(a(\tau)-l(\tau)-p(\tau)\big)d\tau\right]exp\left[-\int_{t_0}^{t}\big(a(\tau)+l(\tau)\big)d\tau\right]\}-$$

$$-\frac{1}{2r}\int_{s}^{t}K(\tau,s)\left\{exp\left[-\int_{t_0}^{\tau}\big(a(\tau)+l(\tau)-p(\tau)\big)d\tau\right]\times\right.$$

$$\times\, exp\left[-\int_{t_0}^{t}\big(a(\tau)-l(\tau)\big)d\tau\right]-\ exp\left[-\int_{t_0}^{\tau}\big(a(\tau)-l(\tau)-p(\tau)\big)d\tau\right]\times$$

$$\left.\times\, exp\left[-\int_{t_0}^{t}\big(a(\tau)+l(\tau)\big)d\tau\right]\right\}d\tau. \qquad (13)$$

$(t,s)\in G=\{(t,s): t_0\le s\le t<\infty\}$.

Taking into account notations (12) and (13), we write equation (11) in the form

$$y(t)=F(t)+\int_{t_0}^{t}H(t,s)y(s)ds, t\ge t_0, \qquad (14)$$

Theorem. Let conditions a), b), c) and d) be satisfied. Then the solution of the Cauchy problem (1)-(2) satisfies the following estimate:

$$|y(t)|\le c_1,\ t\ge t_0, \qquad (15)$$

where $c_1=c_2 exp\left[\frac{1}{2|r|}\int_{t_0}^{\infty}(l_1(s)+l_2(s))ds\right]$,

$c_2=\frac{1}{2|r|}\left[\big|m\big(r-a(t_0)\big)-n\big|+|n+m(r+a(t_0))|+c_0\right]$.

Proof. Considering condition b), c) and d), from (12) and (13) we have:

$$|F(t)|\le c_2,\quad t\ge t_0, \qquad (16)$$

$$|H(t,s)|\le\frac{l_1(s)+l_2(s)}{2|r|},\qquad (t,s)\in G. \qquad (17)$$

Then, by virtue of (16) and (17) from (14) we obtain:

$$|y(t)|\le c_2+\frac{1}{2|r|}\int_{t_0}^{t}[l_1(s)+l_2(s)]|y(s)|ds,\qquad t\ge t_0. \qquad (18)$$

Further, due to the Granwall-Bellman inequality, from (18) we have estimate (15). Theorem proven.

Example. Consider problems (1) - (2) for $t_0=0, p(t)=2(1+t)$, $q(t)=(1+t)^2+2exp\left[-\frac{1}{2}t^2-4t\right], f(t)=3exp\left[-\frac{1}{2}t^2-3t\right], t\in[0,\infty)$,

$$K(t,s)=e^{-\frac{t^2}{2}-3t-s}, (t,s)\in G=\{(t,s): 0\le s\le t<\infty\}.$$

In this case, conditions *a)* are satisfied for

$$a(t) = 1 + t, r = 1, q_0(t) = (1+t)^2, q_1(t) = 2exp\left[-\frac{1}{2}t^2 - 4t\right],$$
$$l(t) = 1, t \in [0, \infty).$$
In addition, conditions b) and c) are satisfied for
$$c_0 = 6, \quad l_1(t) = 4e^{-2t}, \quad l_2(t) = 2e^{-t}.$$
Indeed $a(t) + \ l(t) = 2 + t \geq 0, a(t) - \ l(t) = t \geq 0, t \in [0, \infty)$,

$|f(t)|\{\exp\left[-\int_0^t \left(a(\tau) + l(\tau) – p(\tau)\right) d\tau\right] + \exp[-\int_0^t (a(\tau) - l(\tau) - p(\tau))d\tau]\} =$

$= 3e^{-(t^2/2+3t)}\left(e^{t^2/2} + e^{t^2/2+2t}\right) \leq 6e^{-t}, \ t \in [0, \infty),$

$|q_1(t)|\left\{exp\left[-\int_0^t (a(\tau) + l(\tau) - p(\tau))d\tau\right] + exp\left[-\int_0^t (a(\tau) - l(\tau) -\right.\right.$

$\left.\left.-p(\tau))d\tau\right]\right\}$ $=2e^{-(t^2/2+4t)}\left[e^{t^2/2} + t^2/2\right] \leq 4e^{-2t} = l_1(t) \in L_1[0, \infty),$

$\int_s^t |K(\tau, s)|\{exp\left[-\int_0^\tau (a(\tau) + l(\tau) - P(\tau))d\tau\right] +$

$+exp\left[-\int_0^\tau (a(\tau) + l(\tau) - P(\tau))d\tau\right]\}d\tau \leq \int_s^t e^{-\tau^2/2-3\tau-s}\left(e^{\tau^2/2} + e^{\tau^2/2+2\tau}\right)d\tau \leq$

$\leq 2e^{-s}\int_s^t e^{-\tau}d\tau \leq 2e^{-s} = l_2(s) \epsilon L_1[0, \infty).$

Thus, all conditions are met. In this case
$$c_2 = \frac{1}{2}[|n| + |n + 2m| + 6] = \frac{1}{2}[|n| + |n + 2m| + 6], \qquad (19)$$
$$c_1 = c_2 exp\left[\frac{1}{2|r|}\int_0^\infty (l_1(s) + l_2(s))ds\right] = c_2 e^3.$$
Therefore, to solve this Cauchy problem (1) - (2), estimate (15) is valid for, where the number c_2 is determined by formula (19).

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SYSTEMS OF LOCAL EQUATIONS AND FUNCTIONAL RELATIONS

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The authors considered local operators on non-continuous functions: coinciding of two functions in any neighbor of a point implies a same value of the operator on such functions in the point. The notion of derivative was extended to all functions in metrical spaces. A local equation for a scalar fractal function was constructed. In the paper these results are extended to vector-functions.

Keywords: functional relation, local operator, local equation, ordinary differential equation, fractal

Авторлор үзгүлтүк функциялар боюнча жергилик операторлор карады: чекиттин каалаган айланасында эки функциянын дал келиши чекиттеги мындай функциялар боюнча оператордун бирдей маанисин билдирет. Туунду түшүнүгү метрикалык мейкиндиктердеги бардык функцияларга жайылтылды. Фрактал функциясы үчүн жергилик теңдеме курулду. Бул макалада мындай жыйынтыктар вектор-функцияларга жалпыланат.

Урунттуу сөздөр: функционалдык байланыш, жергилик оператор, жергилик теңдеме, кадимки дифференциалдык теңдеме, фрактал

Авторы рассмотрели локальные операторы над разрывными: совпадение двух функций в любой окрестности точки влечет за собой одинаковое значение оператора над такими функциями в этой точке. Понятие производной было распространено на все функции в метрических пространствах. Было построено локальное уравнение для фрактальной функции. В данной статье эти результаты распространены на вектор-функции.

Ключевые слова: функциональное соотношение, локальный оператор, локальное уравнение, обыкновенное дифференциальное уравнение, фрактал

1. Introduction

We consider the notion “local” in meaning “infinitesimal”. Considered a general task on relations between equations with global and local operators for a same function. Obviously, a global operator involves values of the function in different points of domain. Previously, definitions of local operators involved any kind of smoothness of a function defined on a domain with locally linear structure.
The first local operator on sufficiently non-smooth functions was introduced in [1]: upper and lower right and left derivative numbers of certain continuous functions. Also, there were publications on differential operators on “almost differentiable” (for instance, convex) functions and “differentiable almost everywhere” functions.

Section 2 contains definitions on local operators in topological and metrical spaces.

Section 3 connects local operators and functional relations.

2. Definitions of local operators in topological and metrical spaces

A satisfactory definition of a local operator was given in [2].

Definition 1. A general local operator Q of the function $u:X\to V$, where X is a topological space, at the point $x\in X$ satisfies the condition: if for some neighborhood U of the point x the restrictions $u_1|_U = u_2|_U$ hold then $Q(u_1, x) = Q(u_2, x)$.

Six months ago we asked artificial intelligence. Gemini and ChatGPT could not give satisfactory definitions.

Now we asked chat.deepseek.com:

Definition 2. Let X be a topological space, and let F and G be sheaves of functions (or sections of some bundles) on X. An operator $T:F\to G$ is called a **local operator** if, for every open set $U\subseteq X$ and every function (or section) $f\in F(U)$, the value of $T(f)$ at any point $x\in U$ depends only on the germ of f at x.

In other words, if two functions f and g agree on some neighborhood of x, then $T(f)(x)=T(g)(x)$.

A proper mention of germ proves that artificial intelligence either did not read definition 1 or sufficiently improved it.

Local operators are common in fields like differential equations, functional analysis, and geometry. For example:

Differential operators are local because the value of a derivative at a point depends only on the behavior of the function in an arbitrarily small neighborhood around that point.

In image processing, local filters (such as blurring or sharpening filters) apply changes to each pixel based on its surrounding pixels, making them local operators.

In contrast, a non-local operator depends on the values of the object over a larger or even the entire domain.

A local operator in mathematics can be strictly defined within the context of functional analysis or differential geometry, where it typically acts on sections of a vector bundle or functions defined on a manifold.

In other words, the action of the operator T on a section over an open set U only depends on the values of the section in small neighborhoods within U.

3. Examples of local operators over non-smooth scalar functions

Definition 3 [1]. The upper right derivative number in a point x_0 is the upper limit

$$\limsup\{(f(x)-f(x_0))/(x-x_0)|\ x\to x_0+0\} \tag{1}$$

(if it exists). Similarly, the lower right, the upper left and lower left derivative number are determined.

Such limits exist not for all functions.

Definition 4. We proposed to change (1) to

$$\limsup, \liminf\{(f(x)-f(x_0))/(x-x_0)/(|f(x)-f(x_0)|/|x-x_0|+1)\ |x\to x_0\pm 0\}. \tag{2}$$

Such four real numbers exist for all functions of real variable. We called them “reduced derivative numbers”.

4. Examples of local operators over non-smooth functions in more general spaces

We generalized (2) to functions from metrical spaces to metrical spaces.

Definition 5. Let f be a map of a Banach space B to R; $x\in B$; $v\in B\backslash\{0\}$. Upper and lower reduced derivative numbers by direction:

$$\limsup, \liminf\{(f(x+tv)-f(x))/(t||v||)/(|f(x+tv)-f(x)|/(t||v||)+1)\ |t\to +0\}. \tag{3}$$

It may be generalized to functions from metrical spaces to metrical spaces.

Definition 6. Two numbers

$$liminf, limsup\{\rho(f(x), f(x_0))/\rho(x, x_0)/(\rho(f(x), f(x_0))/\rho(x, x_0)+1) \mid x\to x_0\} \quad (4)$$

may be considered as “lower” and “upper” estimations of “rate of changing” of the function.

5. Non-differential local equation in R^2

We introduced the general notion of functional relations [3], [4].

We consider the functional relation for a continuous function $u(x,y)\in C(R^2\to R)$ (fractal, concentric circles)

$$u(2x,2y)= ku(x,y),\ (x,y)\in R^2. \quad (5)$$

Theorem 1. Solutions of the equation (5) for $k>1$ fulfil the local equation

$$Q(u,0,0)=a_u\in R, \quad (6)$$

$$Q(u,x,y) := limsup\{measure\ \{|(x-x', y-y')|\le h \mid u(x',y')\ge 0\}/(\pi h^2)| \ h\to +0\}; \quad (7)$$

$$a_u := sup\{\ measure\ \{|(x,y)| \ \in[1-t, 2-t]: u(x,y)\ge 0\}|0\le t\le 1/2\}/(\pi(3\text{-}2t)). \quad (8)$$

Proof. The assumption “$u(0,0)\neq 0$” implies contradiction: we have $u(1,1) =k^n u(2^{-n},2^{-n})$ for all $n\in N$, hence $|u(1,1)|$ is unbounded. Hence, $u(0,0)=0$.

Theorem 2. Solutions of the equation (5) for $k<-1$ fulfil the local equation (6) with

$$a_u := sup\{\ measure\ \{|(x,y)| \ \in[1-t, 2-t]: u(x,y)\ge 0\}|0\le t\le 3/4\}/(\pi(3-2t)).$$

Remark. The functional relation for a continuous function

$$u(2x,2y)= ku(-y, x),\ (x,y)\in R^2,\ k>1 \quad (9)$$

presents a spiral fractal.

4. Conclusion

The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic. We hope that the paper would draw attention to general local equations and such methods would yield new classes of properties of solutions of equations with functional relations.

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