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CONSIDERING OF GENERALIZED COMPRESSING MAPPING

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In this paper, Banach's theorem, known in functional analysis, is generalized to the case of linear spaces with a structure consistent with the uniformity specified on the corresponding carrier. In the category of metrical spaces there is proposition, which is called as Banach's theorem. It proposes existence and uniqueness of the fixed point of the compressing operator, acting on the complete metrical space. Uniformed structure is generalization of the idea about metrical space. It has been shown by Borubaev that, how it is possible to use ideas of Boltiansky and Sarymsakov for transbringing some propositions, having place for metrical spaces, to the objects of the category of uniformed spaces and uniformly continuous mappings. In such investigations the concept about multimetrical spaces is used. Uniformed structure can be associated with multimetrical one. It is the basis for having been made of made and being made of being made in this work generalizations.

Key-words: multimetric space, uniformed structure, topological ring, generalized compressing mapping, complete uniformed space, Cauchy-net, limit, fixed point.

В категории метрических пространств существует теорема Банаха, которая утверждает существование и единственность неподвижной точки для сжимающего оператора, заданного в полном пространстве. Равномерная структура является обобщением понятия о метрике, причём Борубаевым А.А. показано то, как можно использовать идеи Болтянского и Сарымсакова для того, чтобы перенести часть утверждений, имеющих место для метрических пространств, на объекты категории равномерных пространств и равномерно непрерывных отображений. При этом используется понятие мультиметрического пространства. Любое равномерное пространство можно ассоциировать с определённой структурой мультиметрического пространства. Это и является основой для осуществлённости осуществлённых и осуществления осуществляемых в данной работе обобщений. В данной работе теорема Банаха, известная в функциональном анализе, обобщена на случай линейных пространств со структурой, согласованной с равномерностью, заданной на соответствующем носителе.

Ключевые слова: мультиметрическое пространство, равномерность, топологическое кольцо, обобщённые сжимающие отображения, полное равномерное пространство, направленность Коши, предел, неподвижная точка.

Метрикалык мейкиндиктердин категориясында толук мейкиндеги аткарылуучу басылтыруучу оператордун кыймылсыз чектинин бар болгондугун жана жалгыздыгын көрсөтүүчү Банахтын теоремасы бар. Бир калыптуу структура метрика жөнүндө түшүнүгүнүн жалпыланышы болот, ал эми Бөрүбаев метрикалык мейкиндиктер үчүн бар болгон байланыштарынан бир кандай байланыштарын Болтянский менен Сарымсаковдун идеяларын пайдаланышы аркылуу бир калыптуу мейкиндиктердин жана чагылтуулардын категориясына алып киртирүүнүн жолун көрсөткөн. Мультиметрикалык түшүнүгү биякта пайдаланылат. Ар кандай бир калыптуу мейкиндик мультиметрикалык мекиндиктин түрүндө элестениши мүмкүн. Ошол амалга ашырылган жалпылануулардын амалга ашырылгандыгы үчүн жана бул иште амалга ашырылуучу жалпылануулардын амалга ашырыла турганы үчүн негизи болот. Бул макалада функционалдык анализде белгилүү Банахтын теоремасын жалпылантырып, бир калыптуулукту ээлеген сызыктуу мейкиндиктерге ошолорго жооп берген бир калыптуулугу сызыктуу мейкиндиктин структурасы менен келишишүүдө болгондугунда алып амалга ашырылгандыгына киртирүүнүн аракети аткарылат.

Урунттуу сөздөр: мультиметрикалык мейкиндик, бир калыптуулук, топологиялык тоголок, жалпылантырылганча кысуучу чагылтуу, толук бир калыптуу мейкиндик, Кошинин жөндөтүлүп интилгендиги, предел, кыймылбас чекит.

Let a complete uniformed space (X, U), that is generated by the set of pseudometrics $\{(X, \rho_i): i \in I\}$. Let the operator $A: (X, U) \to (X, U)$ is compressing, that is that for any $i \in I$ exists such a positive number $\alpha_i < 1$, that for any x of the space X it's true, that:

$$\rho_i(Ax, Ay) \leq \alpha_i \rho_i(x, y) \tag{1}$$

The space $\mathbb{R}^I$, that is made having the natural uniformity and nature acting the same on every component operations of adding and extracting form the semilfield the club and uniformed space.

As it has been well known ([1]), on $(\mathbb{R}^I, +, \cdot)$ the uniformed structure is interunisistented with its algebraic structure in the meaning, that such ring of scalars has uniformly continuous operations of its signature in the relation to the natural uniformity of multiplication on $\mathbb{R}^I$.

Quotient of compressing from $\mathbb{R}^I$.

Net, that is indexed with natural numbers, id est, as consequence, being represented in the form $(A^n b)_{n\in\mathbb{N}}$ for some $b \in X$, is being considered, and as A it will be thought compressing in the meaning (1) operator. For any index *i* and any natural number *m* because of (1) it is true that:

$$\rho_i(Ab, A^{m+1}b) = \rho_i\big(Ab, A(A^m b)\big) \le \alpha_i \rho_i(b, A^m b).$$

Suppose, that for some natural number *n* it is true, that:

$$\rho_i(A^n b, A^{n+m} b) \le \alpha_i^n \rho_i(b, A^m b) \tag{1'}$$

It means, that:

$$\rho_i(A^{n+1}b, A^{n+1+m}b) = \rho_i\big(\, A(A^n b), A(A^{n+m}b)\big) \le \alpha_i \rho_i(A^n b, A^{n+m}b) \le$$

$$\le \alpha_i \alpha_i^n \rho_i(b, A^m b) = \alpha_i^{n+1} \rho_i(b, A^m b),$$

id est: $\rho_i(A^{n+1}b, A^{n+1+m}b) \le \alpha_i^{n+1} \rho_i(b, A^m b)$.

Because of the principle of mathematical induction from higher proved for any *n*, for which $(1')$ is true, it can have been got, that for any natural numbers *m*, *n* it is true, that:

$$\rho_i(A^n b, A^{n+m} b) \le \alpha_i^n \rho_i(b, A^m b) \tag{2}$$

But from the definition of pseudometrics:

$$\rho_i(b, A^m b) \le \rho_i(b, Ab) + \rho_i(Ab, A^m b) \le$$

$$\le \rho_i(b, Ab) + \rho_i(Ab, A^2 b) + \rho_i(A^2 b, A^m b) \le \cdots \le$$

$$\le \rho_i(b, Ab) + \rho_i(Ab, A^2 b) + \cdots + \rho_i(A^{m-1}b, A^m b).$$

So from (2) it can have been got, that:

$$\rho_i(b, A^m b) \le \rho_i(b, Ab) + \rho_i(Ab, A^2 b) + \cdots + \rho_i(A^{m-1}b, A^m b) \le$$

$$\le \rho_i(b, Ab) + \alpha_i \rho_i(b, Ab) + \cdots + \alpha_i^{m-1} \rho_i(b, Ab) \le \frac{\rho_i(b, Ab)}{1 - \alpha_i}.$$

From (2) and from the last interrelation it has been got, that:

$$\rho_i(A^n b, A^{n+m} b) \le \alpha_i^n \rho_i(b, A^m b) \le \alpha_i^n \frac{\rho_i(b, Ab)}{1 - \alpha_i}. \tag{3}$$

But for given b for any $\varepsilon > 0$ it exists such $N \in \mathbb{N}$, that it is true, that $(n > N) => \alpha_i^n \frac{\rho_i(b,Ab)}{1-\alpha_i} < \varepsilon$. So from (3) for a given $b \in X$ for any $\varepsilon > 0$ exists such $N \in \mathbb{N}$, that $(n > N) => \rho_i(A^n b, A^{n+m} b) < \varepsilon$. Then for any $b \in X$ for any $\varepsilon > 0$ exists such $N \in \mathbb{N}$, that: $(n > N, m \in \mathbb{N}) => \rho_i(A^n b, A^{n+m} b) < \varepsilon$. That is why for a given $b \in X$ it is true that net $(A^n b)_{n\in\mathbb{N}}$ is Cauchy-net in the pseudometrical space (X, ρ_i) by the definition of Cauchy-net. Because of having by us as a considered an abstract index $i \in I$, it has been got, that in the relation to the abstract pseudometrical space (X, ρ_i), where $i \in I$, it is true, that $(A^n b)_{n\in\mathbb{N}}$ is Cauchy-net in this space. Then by the properties of supremum it is true, that that, what is (X, U) is supremum of uniformities of pseudometrics of the set $\{(X, \rho_i): i \in I\}$, means that being of $(A^n b)_{n\in\mathbb{N}}$ the Cauchy-net in the relation to every uniformity of the corresponding set means being of $(A^n b)_{n\in\mathbb{N}}$ the Cauchy-net in the supremum of those, id est, in (X, U). But (X, U) is complete, so $(A^n b)_{n\in\mathbb{N}}$ has its limit in this. Let it have been signified as x. It will be needed some evener.

Let an indefinited cover of a canonical base of the supremum of uniformities of the considered set: $\wedge\{\beta_{i_k}^r : k = 1, \dots, n\}$, where $i_k \in I$, $\beta_{i_k}^r \in U_{i_k}$ for any $k = 1, \dots, n$. Here $\beta_{i_k}^r = \{O_{x,i_k}^r : x \in X\}$, $O_{x,i_k}^r = \{y : \rho_{i_k}(x, y) \le r\}$. Let an being indefinited element O_{x,i_k}^r of an indefinited cover $\beta_{i_k}^r$ from the set be considered here. Let an indefinite $\varepsilon > 0$ have been selected. Then for any x and y, for which it is true, that $\rho_{i_k}(x, y) \le \frac{\varepsilon}{\alpha_i}$, it is true, that $\rho_{i_k}(Ax, Ay) \le \alpha_{i_k} \rho_{i_k}(x, y) \le \varepsilon$. By the definition of a uniformly continuous mapping it means, that A is uniformly continuous. Then from another definition of uniformly continuity for any cover $\beta_{i_k}^r$that is from being considered set of coverings it is true, that $A^{-1}(\beta_{i_k}^r) \in U_{i_k}$. Then by the properties of uniformed coverings $A^{-1}(\wedge\{\beta_{i_k}^r : k = 1, \dots, n\}) = \wedge\{A^{-1}\beta_{i_k}^r : k = 1, \dots, n\}$ is uniformed. Because of that $\wedge\{\beta_{i_k}^r : k = 1, \dots, n\}$ is considered as indefinited covering of a canonical base of uniformity U, by the properties of uniformities and uniformly continuous mappings it is true, that preimage of any uniformed covering in the sense of mapping A is uniformed covering, what means that A is uniformly continuous operator. Then A is continuous in the sense of inducted by uniformity U topology by the properties of uniformly continuous mappings. But because of proved higher there is such point x, that $x = \lim((A^n b)_{n\in\mathbb{N}})$, then because of continuity of A it is true, that:

$$Ax = A(\lim((A^n b)_{n\in\mathbb{N}})) = \lim((A^{n+1} b)_{n\in\mathbb{N}}) = \lim((A^n b)_{n\in\mathbb{N}}) = x, \qquad (4)$$

by the properties of continuous mappings.

So, because of (4) it has been got, that there exists such point x of the space X, that $Ax=x$. Id est, it has been proved the following theorem.

Theorem 1. In the sense of the (1) compressing operator has a fixed point, and it has only one fixed point.

Particularly, if for any $i \in I$ there exists such one positive quotient $\alpha_i = a$, that is less than 1, that for any point x of the space X it is true, that:

$$\rho_i(Ax, Ay) \le a\rho_i(x, y) \qquad (5)$$

then with consequence of interconnected sentences, that is analogical to such, which were constructed higher, for the particular case of that, what is that for any $i \in I$ it is true, that $\alpha_i = a$, it can have been got, that if for any $i \in I$ there exists such one positive number $\alpha_i = a$, that is less than 1, then there exists such point x of the space X, that $Ax=x$, and the point with this property is unique.

So, in the resulting of constructed consequence of interconnected sentences it is starting to have been realized, that the operator A, that is compressing in the sense, that for any x and y and for some $a<1$ would be true the interrelation (5), would have unique fixed point. Id est, the following theorem has just been proved.

Theorem 2. In the sense of the (5) compressing operator has a fixed point, and it has only one fixed point.

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COMPACTNESS AND REMAINDERS OF MAPPINGS

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Recently, many concepts and statements of general topology have been extended from the case of spaces to the case of continuous mappings. In this case, space is understood as the simplest continuous mapping of this space into a single-point space. The studies conducted allowed many basic statements of the General Topology of spaces to be transferred to mappings. The method of transferring results from spaces to mappings is universal and not simple, but it allows many results to be generalized. Therefore, the problem of extending some concepts and statements concerning spaces to mappings has not yet been fully solved. In this paper compact types of continuous mappings are investigated, as well as the perfection of the remainder of continuous mappings.

Keywords: continuous mapping, compactness, open cover, finitely additive cover, *remainder*.

Акыркы мезгилдерде жалпы топологиянын бир топ түшүнүктөрү жана жыйынтыктары мейкиндик учурунан бир калыптуу үзгүлтүксүз чагылдыруу учуруна жайылтылган. Мында топологиялык мейкиндик бул топологиялык мейкиндиктин бир чекиттүү мейкиндикке болгон жөнөкөй үзгүлтүксүз чагылдыруусу катары түшүндүрүлөт. Изилдөөлөр жалпы топологиядагы мейкиндиктин көп негизги жыйынтыктарынын чагылдырууларга жайылтылышына себеп болду. Жыйынтыктарды мейкиндиктерден чагылдырууларга жайылтуу ыкмасы универсалдуу болуп саналат жана жөнөкөй эмес, бирок көп жыйынтыктарды жалпылоого мүмкүндүк берет. Ошондуктан мейкиндиктерге байланышкан айрым түшүнүктөрдү жана жыйынтыктарды чагылдырууларга жайылтуу маселеси азыркыга чейин толук чечиле элек. Бул макалада үзгүлтүксүз чагылдыруулардын компакттуу типтери, ошондой эле үзгүлтүсүз чагылдыруулардын өсүндүлөрүнүн жеткилеңдүүлүгү изилденет.

Урунттуу сөздөр: *үзгүлтүксүз чагылдыруу, компакттуулук, ачык жабуу, чектүү аддитивдүү жабуу, өсүндү.*

В последнее время многие понятия и утверждения общей топологии были распространены со случая пространств на случай непрерывных отображений. При этом пространство понимается как простейшее непрерывное отображение этого *пространства в одноточечное пространство. Проведенные исследования позволили перенести на отображения многие основные утверждения общей топологии пространств. Метод перенесения результатов с пространств на отображения является универсальным, и не простым, но позволяющим обобщить многие результаты. Поэтому задача распространения некоторых понятий и утверждений, касающихся пространств, на отображения до сих пор не решена полностью. В данной статье исследуются компактные типы непрерывных отображений, а также совершенность нароста непрерывных отображений.*

Ключевые слова: *непрерывное отображение, компактность, открытое покрытие, конечно аддитивное открытое покрытие, нарост.*

Throughout this paper all topological spaces are assumed to be Tychonoff and mappings to be continuous.

For covers α and β of a set X, we have: $\alpha \wedge \beta = \{A \cap B: A \in \alpha, B \in \beta\}.H \subset X$. For covers α and β of the set X, the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$.

A topological space (X, τ) is called compact if every arbitrary open cover has an open finite refinement [1]. A topological space (X, τ) is called index compactness $\leq \tau$ if every arbitrary open cover has an open cardinality $\leq \tau$ refinement [4]. A topological space (X, τ) is called τ-compact if every arbitrary open cover cardinality $\leq \tau$ has an open finite refinement [4].

Let$f: X \to Y$ be a mapping between topological spaces X and Y.

The mapping f is called to be

1. compact if preimage $f^{-1}y$ is compact for each $y \in Y$ [4];
2. closed if the image of every closed set is closed [4];
3. perfect if it is both compact and closed [4].

Let cX be a compactification of the topological space X. The complement of a given space to its compactification is called a remainder and is denoted by $cX \backslash X$.

The mapping $cf: \hat{X} \to Y$ is compactification of the mapping $f: X \to Y$, if

1) $X \subset \hat{X}$;
2) $[X] = \hat{X}$;

3) cf is perfect [6].

We will call the mapping $cf|_{\hat{X}\setminus X}: \hat{X}\setminus X \to Y$ is remainder of the mapping f[6]. Let $f: (X,\tau) \to (Y,\sigma)$ be a continuous mapping between (X,τ) and (Y,σ).

Definition 1. A continuous mapping $f: (X,\tau) \to (Y,\sigma)$ of a topological space (X,τ) onto a topological space (Y,σ) is said to be compact, if for each finitely additive open covering α of the space (X,τ) there exist open covering β of the space (Y,σ) and finite open covering γ of the space (X,τ), such that $f^{-1}\beta \wedge \gamma \succ \alpha$.

Proposition 1. Let $f: (X,\tau) \to (Y,\sigma)$ be a continuous mapping between (X,τ) and (Y,σ). If (X,τ) is a compact space, then the mapping f is compact.

Proposition 2. If $f: (X,\tau) \to (Y,\sigma)$ is a continuous mapping between (X,τ) and (Y,σ) , $Y = \{y\}$ is compact, then the topological space (X,τ) is compact.

Lemma 1. If α and β are open coverings cardinality $\leq \tau$ of (X,τ), then the covering $\alpha \wedge \beta$ is open covering cardinality $\leq \tau$ of (X,τ).

Proof. Let α and β be finite open coverings of the space (X,τ). It is clear that $\alpha \wedge \beta$ is open covering. From the definition $\alpha \wedge \beta = \{A \cap B: A \in \alpha, B \in \beta\}$ we have $\alpha \wedge \beta$ is a cardinality $\leq \tau$.

Lemma 2. Let $f: (X,\tau) \to (Y,\sigma)$ be a continuous mapping between (X,τ) and (Y,σ). If β is an open covering cardinality $\leq \tau$ of the space (Y,σ), then $f^{-1}\beta$ is open covering cardinality $\leq \tau$ of the space (X,τ).

Proof. Let f be a continuous mapping and β be an open covering cardinality $\leq \tau$ of the space (Y,σ). From the continuity of the mapping f it follows that $f^{-1}\beta$ is open covering. Let $\alpha = f^{-1}\beta$. Since $\alpha = \{f^{-1}B: B \in \beta\}$ and β is cardinality $\leq \tau$, then α is cardinality $\leq \tau$.

Proposition 3. Let $f: (X,\tau) \to (Y,\sigma)$ be acompact mapping of a topological space (X,τ) onto a topological space (Y,σ) and $g: (Y,\sigma) \to (Z,\delta)$ be a compact mapping of a topological space (Y,σ) onto a topological space (Z,δ). Then the mapping $g \circ f: (X,U) \to (Z,W)$ of a topological space (X,τ) onto a topological space (Z,δ) is compact.

Proof. Let $f: (X, \tau) \to (Y, \sigma)$ be a compact mapping of a topological space (X, τ) onto a topological space (Y, σ),$g: (Y, \sigma) \to (Z, \delta)$ – a compact mapping of a topological space (Y, σ) onto a topological space (Z, δ) and α be an arbitrary finitely additive open covering of (X, τ). Then there exist open covering β of the space (Y, σ) and finite open covering γ of the space (X, τ), such that $f^{-1}\beta \wedge \gamma \succ \alpha$. By virtue of the compactness of the mapping g there exist open covering λ of the space (Z, δ) and finite open covering η of the space (Y, σ), such that $g^{-1}\lambda \wedge \eta \succ \beta^{\angle}$. Then $(g \circ f)^{-1}\lambda \wedge f^{-1}\eta \succ f^{-1}\beta^{\angle} \succ f^{-1}\beta$. $(g \circ f)^{-1}\lambda \wedge f^{-1}\eta \wedge \gamma \succ f^{-1}\beta \wedge \gamma \succ \alpha$. Let $f^{-1}\eta \wedge \gamma = \theta$. It is easy to see that θ is finite. Then $(g \circ f)^{-1}\lambda \wedge \theta \succ \alpha$. Therefore, the mapping $g \circ f$ of a topological space (X, τ) onto a topological space (Z, δ) is compact.

Theorem 1. If f and (Y, δ) is compact, then the topological space $(X, \tau)$is compact.

Proof. Let f and (Y, σ) be compact and α be an arbitrary finitely additive open covering of the space (X, τ). Then there exists open covering β of the space (Y, σ) and finite open covering γ of the space (X, τ), such that $f^{-1}\beta \wedge \gamma \succ \alpha$. By virtue of the compactness of the space (Y, σ) there exists a finite open covering β_0, such that $\beta_0 \succ \beta$. Obviously, $f^{-1}\beta_0 \wedge \gamma \succ f^{-1}\beta \wedge \gamma$. By virtue of Lemma 2 the open covering $f^{-1}\beta_0$is finite. Denote $f^{-1}\beta_0 \wedge \gamma = \lambda$. By virtue of Lemma 1 the uniform covering λ is finite. Hence, the space (X, τ) is compact.

Proposition 4. If $f: (X, \tau) \to (Y, \sigma)$ is acompact mapping of a topological space (X, τ) onto a topological space (Y, δ), then the mapping $f|_M: (M, \tau_M) \to (Y, \sigma)$ of a closed subspace (M, τ_M) onto a topological space (Y, δ) is compact.

Proof. Let $f: (X, \tau) \to (Y, \sigma)$ be acompact mapping of a topological space (X, τ) onto a topological space (Y, σ) and (M, τ_M) be a closed subspace of (X, τ). Let $\alpha_M$be an arbitrary finitely additive open covering of the space (M, τ_M). Then there is a finitely additive open covering α of the space (X, τ), such that $\alpha \wedge \{M\} = \alpha_M$, $\alpha = \{\alpha_M, X \setminus M\}$. Since the mapping $f: (X, \tau) \to (Y, \sigma)$ is compact, there exist open covering β of (Y, σ) and finite open covering $\gamma$of (X, τ), such that $f^{-1}\beta \wedge \gamma \succ \alpha$.

Then it is easy to see that $f^{-1}\beta \wedge \gamma_M \succ \alpha_M$, $\gamma_M = \gamma \wedge \{M\}$ and γ_M is finite. Therefore $f|_M$ is compact.

Definition 2. A mapping $f: (X,\tau) \to (Y,\sigma)$ of a topological space (X,τ) onto a topological space (Y,σ) is said to be index compactness $\leq \tau$, if for each finitely additive open covering α of the space (X,τ) there exist open covering β of the space (Y,σ) and open covering γ cardinality $\leq \tau$ of the space (X,τ), such that $f^{-1}\beta \wedge \gamma \succ \alpha$.

Proposition 5. Let $f: (X,\tau) \to (Y,\sigma)$ be a continuous mapping between (X,τ) and (Y,σ). If (X,τ) is index compactness $\leq \tau$ space, then the mapping f is index compactness $\leq \tau$.

Proof. Let (X,τ) be index compactness $\leq \tau$ space and α be an arbitrary finitely additive open covering. Then there exist open covering cardinality $\leq \tau\gamma$ of the space (X,τ), such that $\gamma \succ \alpha$. For open covering β of the space (Y,σ) we have $f^{-1}\beta \wedge \gamma \succ \alpha$. Consequently, f is index compactness $\leq \tau$.

Proposition 6. If $f: (X,\tau) \to (Y,\sigma)$ is a continuous mapping between (X,τ) and (Y,σ) , $Y = \{y\}$ is index compactness $\leq \tau$, then the topological space (X,τ) is index compactness $\leq \tau$.

Proof. Let f be index compactness $\leq \tau$ mappings and α be an arbitrary finitely additive open covering of the space (X,τ). Then there exists open covering β of (Y,σ) and open covering cardinality $\leq \tau\gamma$ of (X,τ), such that $f^{-1}\beta \wedge \gamma \succ \alpha$. Since $Y = \{y\}$, then $f^{-1}\beta \wedge \gamma = \gamma$. Thus, (X,τ) is index compactness $\leq \tau$.

Proposition 7. Let $f: (X,\tau) \to (Y,\sigma)$ be index compactness $\leq \tau$ mapping of a topological space (X,τ) onto a topological space (Y,σ) and $g: (Y,\sigma) \to (Z,\delta)$ be a Lindelöf mapping of a topological space (Y,σ) onto a topological space (Z,δ). Then the mapping $g \circ f: (X,U) \to (Z,W)$ of a topological space (X,τ) onto a topological space (Z,δ) is index compactness $\leq \tau$.

Proof. Let f and g be index compactness $\leq \tau$ continuous mappings and α be an arbitrary finitely additive open covering of (X,τ). Then exist open covering β of the space (Y,σ) and open covering cardinality $\leq \tau\gamma$ of the space (X,τ), such that

$f^{-1}\beta \wedge \gamma \succ \alpha$. There exists open covering λ of the space (Z, δ) and open covering cardinality $\leq \tau\eta$ of the space (Y, σ), such that $g^{-1}\lambda \wedge \eta \succ \beta^{\angle}$. Then $(g \circ f)^{-1}\lambda \wedge f^{-1}\eta \succ f^{-1}\beta^{\angle} \succ f^{-1}\beta$. $(g \circ f)^{-1}\lambda \wedge f^{-1}\eta \wedge \gamma \succ f^{-1}\beta \wedge \gamma \succ \alpha$. Let $f^{-1}\eta \wedge \gamma = \theta$. By Lemmas 1 and 2 the covering θ is cardinality $\leq \tau$. Then $(g \circ f)^{-1}\lambda \wedge \theta \succ \alpha$. Thus$g \circ f$ is index compactness $\leq \tau$.

Theorem 2. If f and (Y, σ) is index compactness $\leq \tau$, then the topological space(X, τ) is index compactness $\leq \tau$.

Proof. Let f and (Y, σ) be index compactness $\leq \tau$ and α be an arbitrary finitely additive open covering of the space (X, τ). Then there exists open covering β of the space (Y, σ) and open covering cardinality $\leq \tau\gamma$ of the space (X, τ), such that $f^{-1}\beta \wedge \gamma \succ \alpha$. By virtue of the index compactness $\leq \tau$ of the space (Y, δ) there exist an open covering cardinality $\leq \tau$ β_0, such that $\beta_0 \succ \beta$. Obviously, $f^{-1}\beta_0 \wedge \gamma \succ f^{-1}\beta \wedge \gamma$. By virtue of Lemma 2 the open covering $f^{-1}\beta_0$ is cardinality $\leq \tau$. Denote $f^{-1}\beta_0 \wedge \gamma = \lambda$. By virtue of Lemma 1 the uniform covering λ is cardinality $\leq \tau$. Hence, the space $(X, \tau)$is index compactness $\leq \tau$.

Proposition 8. If $f: (X, \tau) \to (Y, \sigma)$ is index compactness $\leq \tau$ of a topological space (X, τ) onto a topological space (Y, σ), then the mapping $f|_M: (M, \tau_M) \to (Y, \sigma)$ of a closed subspace (M, τ_M) onto a topological space (Y, σ) is index compactness $\leq \tau$.

Proof. Let $f: (X, \tau) \to (Y, \sigma)$ be index compactness $\leq \tau$ mapping of a topological space (X, τ) onto a topological space (Y, σ) and (M, τ_M) be a closed subspace of (X, τ). Let $\alpha_M$be an arbitrary finitely additive open covering of the space (M, τ_M). Then there is a finitely additive open covering α of the space (X, τ), such that $\alpha \wedge \{M\} = \alpha_M$, $\alpha = \{\alpha_M, X \backslash M\}$. Since the mapping $f: (X, \tau) \to (Y, \sigma)$ is index compactness $\leq \tau$, there exist open covering β of (Y, σ) and open covering cardinality $\leq \tau\gamma$ of (X, τ), such that $f^{-1}\beta \wedge \gamma \succ \alpha$. Then it is easy to see that $f^{-1}\beta \wedge \gamma_M \succ \alpha_M$, $\gamma_M = \gamma \wedge \{M\}$ and $\gamma_M$is cardinality $\leq \tau$. Therefore $f|_M$ is index compactness $\leq \tau$.

Definition 3. A continuous mapping $f: (X, \tau) \to (Y, \sigma)$ of a topological space (X, τ) onto a topological space (Y, σ) is said to be τ-compact, if for each finitely additive open covering α cardinality $\leq \tau$ of the space (X, τ) there exists open covering β of the space (Y, σ) and finite open covering γ of the space (X, τ), such that $f^{-1}\beta \wedge \gamma \succ \alpha$.

Proposition 9. Let $f: (X, \tau) \to (Y, \sigma)$ be a continuous mapping between (X, τ) and (Y, σ). If (X, τ) is τ-compact space then the mapping f is countable compact.

Proof. Let (X, τ) be τ-compact space and α be an arbitrary finitely additive open covering cardinality $\leq \tau$. Then there exist finite open covering γ of the space (X, τ), such that $\gamma \succ \alpha$. For open covering β of the space (Y, σ) we have $f^{-1}\beta \wedge \gamma \succ \alpha$. Consequently, f is index compactness $\leq \tau$.

Proposition 10. If $f: (X, \tau) \to (Y, \sigma)$ is a continuous mapping between (X, τ) and (Y, σ) , $Y = \{y\}$ is τ-compact, then the topological space (X, τ) is τ-compact.

Proof. Let f be τ-compact mappings and α be an arbitrary finitely additive open covering cardinality $\leq \tau$ of the space (X, τ). Then there exists open covering β of (Y, σ) and finite open covering γ of (X, τ), such that $f^{-1}\beta \wedge \gamma \succ \alpha$. Since, $Y = \{y\}$, then $f^{-1}\beta \wedge \gamma = \gamma$. Thus, (X, τ) is τ-compact.

Theorem 3. If f and (Y, σ) is τ-compact, then the space(X, τ) is τ-compact.

Proof. Let f and (Y, σ) be a τ-compact and α be an arbitrary finitely additive open covering cardinality $\leq \tau$of the space (X, τ). Then there exists open covering β of the space (Y, σ) and finite open covering γ of the space (X, τ), such that $f^{-1}\beta \wedge \gamma \succ \alpha$. By virtue of the τ-compactness of the space (Y, σ) there exist a finite open covering β_0, such that $\beta_0 \succ \beta$. Obviously, $f^{-1}\beta_0 \wedge \gamma \succ f^{-1}\beta \wedge \gamma$. By virtue of Lemma 2 the open covering $f^{-1}\beta_0$ is a finite. Denote $f^{-1}\beta_0 \wedge \gamma = \lambda$. By virtue of Lemma 1 the uniform covering λ is a finite. Hence, the space (X, τ) is τ-compact.

Proposition 11. If $f: (X, \tau) \to (Y, \sigma)$ is a τ-compact mapping of a topological space (X, τ) onto a topological space (Y, σ) , then the continuous mapping $f|_M: (M, \tau_M) \to (Y, \sigma)$ of a closed subspace (M, τ_M) onto a topological space (Y, σ) is τ-compact.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be τ-compact mapping of a topological space (X,τ) onto a topological space (Y,σ) and (M,τ_M) be a closed subspace of (X,τ). Let α_M be an arbitrary finitely additive open covering cardinality $\leq\tau$ of the space (M,τ_M). Then there is a finitely additive open covering cardinality $\leq\tau\alpha$ of the space (X,τ), such that $\alpha\wedge\{M\}=\alpha_M$, $\alpha=\{\alpha_M,X\backslash M\}$. Since the mapping $f:(X,\tau)\to(Y,\sigma)$ is τ-compact map, there exist open covering β of (Y,σ) and finite open covering γ of (X,τ), such that $f^{-1}\beta\wedge\gamma\succ\alpha$. Then it is easy to see that $f^{-1}\beta\wedge\gamma_M\succ\alpha_M$, $\gamma_M=\gamma\wedge\{M\}$ and $\gamma_M$is finite. Therefore $f|_M$ is τ-compact.

Theorem 4. For a continuous mapping $f:(X,\tau)\to(Y,\sigma)$ the following condition are equivalent:

1. f is compact;
2. f is τ-compact and index compactness $\leq\tau$.

Proof. $1.\Rightarrow 2.$ Obviously.

$2.\Rightarrow 1.$ Let f is τ-compactand index compactness $\leq\tau$-compact. Let α be an arbitrary finitely additive open covering of the space (X,τ). Then there exists open covering β of the space (Y,σ) and open covering cardinality $\leq\tau\gamma$ of the space (X,τ), such that $f^{-1}\beta\wedge\gamma\succ\alpha$. By virtue of the τ-compactness of the map f for the open covering cardinality $\leq\tau\gamma$ of (X,τ), there exist an open covering β_1 of the space (Y,σ) and finite open covering γ_1 of (X,τ), such that $f^{-1}\beta_1\wedge\gamma_1\succ\gamma$. It is easily to see that $f^{-1}(\beta_1\wedge\beta)\wedge\gamma_1\succ f^{-1}\beta\wedge\gamma\succ\alpha$. Let $\beta_1\wedge\beta=\beta_2$. β_2 is an open cover of (Y,σ). Then $f^{-1}\beta_2\wedge\gamma_1\succ\alpha$. Thus f is compact.

Theorem 5. The remainder $cf|_{cX\backslash X}:cX\backslash X\to Y$ of the mapping $f:X\to Y$ is perfect if and only if the space X is open subspace of cX.

Proof. Let $cf|_{cX\backslash X}:cX\backslash X\to Y$ be a remainder of f is perfect. Then $cX\backslash X$ is closed in cX. Thus, X is opened in $\hat{X}$.

Conversely, subspace X is opened in cX. Then subspace $\hat{X}\backslash X$ is closed in $\hat{X}$. Thus, the remainder $cf|_{\hat{X}\backslash X}:\hat{X}\backslash X\to Y$ is perfect.

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MSC 80A05

ENLARGED LIST OF EFFECTS IN MATHEMATICS AND ITS APPLICATIONS

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In the Institute of mathematics, the history of mathematics was presented as a successful treatment of phenomena arising both in mathematics and in life; phenomena are consequences of effects. In this paper, improved framework definitions and enlarged list of effects in mathematics and its applications are presented, both well-known and new-identified effects and phenomena are surveyed.

Keywords: entropy, thermodynamic, second law, control, differential equation, affectable system, heat.

Математика институтунда математиканын тарыхы математикада жана турмушта кездешүүчү кубулуштарды ийгиликтүү изилдөө катары таанышты-рылган; кубулуштар ар кандай эффекттердин натыйжасы болуп саналат. Бул макалада математика жана анын колдонмолорунда кездешүүчү эффекттердин тактоолорун жакшыртылган түшүндүрмөсү жана кеңейтилген тизмеси берилет, ошондой эле белгилүү жана жаңы аныкталган эффекттер жана кубулуштар кыскача баяндалат.

Урунттуу сөздөр: математика, тарых, эффект, кубулуш, аныктама.

В Институте математики история математики была представлена как последо-вательное исследование явлений, возникающих как в математике, так и в жиз-ни; явления являются следствиями эффектов. В данной работе представлены уточнённые определения и расширенный список эффектов в математике и её приложениях, дан обзор как известных, так и вновь выявленных эффектов и явлений.

Ключевые слова: математика, история, эффект, явление, определение.

1. Introduction

In the Institute of mathematics, the history of mathematics was presented as a successful treatment of phenomena arising both in mathematics and in life; phenomena are consequences of effects (see, for example, [1], [2]). In this paper, improved framework definitions and enlarged list of effects in mathematics and its applications are presented, both well-known and new-identified effects and phenomena are surveyed.

2. Definitions

Notions "effect" and "phenomenon" were used widely including mathematics. We proposed a framework definition for "phenomenon" and noted that in history of mathematics, phenomena and paradoxes appeared in mathematics itself (or in life) and researchers tried to explain or to create any theory around them. We also noted that various phenomena were not logically connected but had the same origin ("effect").

Consider a theorem in general as an implication of conditions $A \Rightarrow B$, or, more concrete, if there is a general class X of objects x and $A \subset X$, $B \subset X$ then $A \subset B$ or $(x \in A) \Rightarrow (x \in B)$. To prove sufficiency of A for B one is to construct an example without both A and B. An (interesting, nonpresumable, single) way of violating B is said to be a *phenomenon*.

If P is a property (or some properties) of elements $x \in X$ having a property E such that a logical proof $(E \wedge C) \Rightarrow P$ (where C is any additional condition) is too complicated and the property P was discovered not by a logical way but by meeting paradoxes, by experiments in physics and chemistry or by computational experiments in mathematics then E is said to be an *effect*.

Briefly, if a (real or imagined) concept has various unexpected consequences and these consequences develop some part of mathematics or bring to life a new branch of mathematics, then this concept is called *effect*, and these consequences are called *phenomena*.

3. **Geometry**

Careful analysis of Euclid's Elements in the 18th century demonstrated that they were not an axiomatic system of geometry, many theorems were proven by "common sense". A complete axiomatic system was developed at the end of the 19th century but it turned out to be too complicated and is not used actually. Moreover, mathematicians firstly found any phenomenon (as examples, cycloid, projective transformation) and further described it by means of analytic geometry. Hence, geometry and its generalizations are developed not on an axiomatic base.

4. Effect of infinity

Anaxagorus, 5th century BC inserted the notion “homoeomeria” (similar-to-part). The phenomenon "A part of an infinite object can be equivalent to itself" was formulated by Galileo, 16th century. This phenomenon refuted the intuitively evident statement fixed as Euclid's fifth axiom "the whole is greater than the part". The further development of mathematics was significantly connected with attempts to overcome such paradoxes. At the end of the 19th century it turned out that Cantor's theory of sets is controversial. To "complete" this theory, various proposals have been made including attempts to overcome "saying about itself" such as Russell's theory of definite descriptions. At last, P.Cohen (1963) demonstrated that it is impossible.
Hence, infinite notions were and are investigated not on any axiomatic base.

5. Effect of singular perturbations

To explain the physical phenomenon of boundary layer in flow of a slightly viscous fluid near a solid boundary L.Prandtl (1905) discovered the effect of singular perturbations. One of mathematical implementations: the effect of parameter $0 < \varepsilon << 1$ by the highest derivative in a differential equation. Some authors discovered phenomena of interior boundary layer. We have searched consequences of the effect of singular perturbations systematically, and we found the phenomenon of singular (lengthening as ε tends to 0) cycle, the phenomenon of the deepening boundary layer and other ones.

6. Integrity effect

Definition. If a change in one part of a mathematical object changes the entire object, then that object is said to be holistic. For example, absolutely elastic solid, sphere, five regular polyhedra are holistic.

Consequences of such property are called phenomena of integrity.

Analyticity effect is a particular case: some tasks being incorrect for continuous and even infinitely differentiable functions became correct for analytical ones, on the base of following phenomena found in Kyrgyzstan: existence of level lines for singularly perturbed ordinary differential equations; convergence of the grid method for partial differential equations out of the domain covered by characteristics; correctness of the final value problem for the heat equation; correctness of some integral equations of the first kind; correctness of the initial value problem for the Laplace elliptical equation. These results were primarily discovered by numerical experiments.

7. Effect of higher dimensions

It is well-known that passing from 1D (numbers) to 2D (2×2-matrices) yielded many phenomena: non-commutability of multi-plication; zero product of non-zero factors etc. Passing to next dimensions yields possibilities of new combinations of eigenvalues of matrices. Passing from 2D to 3D in the theory of non-linear differential equations yielded an unexpected phenomenon of strange attractors. Passing from 3D to 4D yielded "mechanical strange attractor" as a round bowl with odd number of symmetrically arranged bulbs on the edge and a little ball moving within the bowl under gravitation.

8. Effect of self-organizing

New phenomena in virtual spaces are found and will hopely be found. One of them is an “even-odd” phenomenon in a virtual space mentioned above. It may be considered as a consequence of the effect of higher dimensions too because it appears only for many objects, a “constant of numerosity” arises.

9. Continuity-Smoothness Effect

Consequences: such functions are bounded on a closed bounded domain, have extrema, possess derivatives, their graphs have tangent lines, and equations involving such functions have solutions... On their basis, mathematical analysis, the theory of differential and integral equations, and differential geometry are built.

Note. Such consequences seem natural, which is why they are not called "phenomena." On the contrary, in mathematical analysis, "unnatural" examples are considered as "phenomena" and require the introduction of new concepts.

Note. From the 16th century, mathematical analysis developed successfully without any justification (all functions were assumed to be continuous and piecewise smooth). Only after Weierstrass provided an example of a function that is continuous but "jagged" (non-differentiable) everywhere (1875) mathematicians did attempt to justify mathematical analysis through "actual infinity."

10. Conclusion

The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic. We do not consider logical paradoxes because they do not lead to new mathematical results.

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MSC 35G35

SINGULARLY PERTURBED LINEAR EQUATIONS WITH A LOGARITHMIC POLE

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Abstract. In this work, a linear singularly perturbed equation with a logarithmic pole is considered. The objective is to investigate the asymptotic behavior of the solution to the studied equation and the influence of the logarithmic pole. To address this problem, a conformal transformation is applied, which reduces the equation and the domain in the complex plane to a simpler form. Polar coordinates are introduced to facilitate the asymptotic estimates.

Бул жумушта логарифмалык полюстуу сызыктуу сингулярдуу козголгон теңдеме каралат. Каралып жаткан теңдеменин чечиминин асимптотикалык жүрүмү жана логарифмалык полюстун таасири изилденет. Коюлган маселени чечүү үчүн теңдемени жана комплекстүү тегиздиктеги областты жөнөкөй түргө келтирүүчү конформдук өзгөртүү жана полярдык координаталар колдонулган.

В данной работе рассматривается линейное сингулярно возмущенное уравнение с логарифмическим полюсом. Исследуется асимптотическое поведение решения рассматриваемого уравнения и влияние логарифмического полюса. Для решения поставленной задачи применены конформное преобразование, приводящее уравнение и область в комплексной плоскости, где рассматривается уравнение, к более простому виду, и полярные координаты.

Key words. singular perturbation, logarithmic pole, analyticity, harmonic function, level line, integration path, boundary layer line and region.

Сингулярдык козголуу, логарифмалык полюс, аналитикалуулук, гармоникалык функциялар, деңгээл сызыктары, интегралдоо жолу, чектик катмар сызык жана область.

Сингулярное возмущение, логарифмический полюс, аналитичность, гармонические функции, линия уровня, путь интегрирования, погранслойная линия и область.

Introduction. If a small parameter multiplies the highest-order derivative, such equations are called singularly perturbed. We will consider only ordinary singularly perturbed differential equations. Hereafter, such equations will be referred to as SPEs (singularly perturbed equations). Systematic studies of SPEs began in the 1950s.

They were investigated by L.S. Pontryagin [1], A.N. Tikhonov [2], E.F. Mishchenko [3], A.B. Vasilieva [4], M.I. Imanaliev [5], and many others.

Consider the system of equations

$$\varepsilon x' = f(t,x,y), \quad y' = g(t,x,y), \quad x = (x_1, \dots, x_n), y = (y_1, \dots, y_m) \tag{1}$$

with initial conditions

$$x(t_0,\varepsilon) = x^0, \qquad y(t_0,\varepsilon) = y^0, \tag{2}$$

where $0 < \varepsilon$ is a small real parameter; $t \in [t_0, T]$;

By setting $\varepsilon = 0$ in (1), we obtain the unperturbed (degenerate) system

$$f(t,\xi,h) = 0, \quad g(t,\xi,h) = h'. \tag{3}$$

Suppose that the system (3) has a solution $\xi_0(t)$, $h_0(t)$ defined for any $t \in [t_0, T]$.

Problem. If there is a solution to the problem (1)-(2) that exists for $t \in [t_0, T]$, is the limit transition possible for $t \in [t_0, T]$

$$\lim_{\varepsilon\to 0} x(t,\varepsilon) = \xi_0(t), \lim_{\varepsilon\to 0} y(t,\varepsilon) = h_0(t)? \tag{4}$$

This problem was solved in its most general form by A.N. Tikhonov [2]. It was proven that the first limit in (4) holds over the entire interval $[t_0, T]$ but is not uniform. In the vicinity of the point t_0 a so-called boundary layer appears, where $x(t,\varepsilon)$ and $\xi_0(t)$ significantly differ from each other. In proving the relation (4), the stability of the equilibrium point of the associated system (in A.N. Tikhonov's terminology) was used.

Subsequently, A.B. Vasilieva [4] developed the method of boundary functions and performed an expansion of the solution to the problem (1)–(2) in positive integer powers of the small parameter ε. The expansion consists of two functions: one function is significant only in a sufficiently small (along with ε) neighborhood of the point t_0, while the other is significant outside this neighborhood. The first function decays exponentially outside this neighborhood. The asymptotic stability of the equilibrium point was required to perform the expansion [4]. M.I. Imanaliev developed a simpler and more practical algorithm for computing the terms of the solution expansion and performed the expansion of solutions for integro-differential SPEs.

L.S. Pontryagin [1] proposed a different perspective for studying the asymptotic behavior of solutions to autonomous SPEs of type (1).

$$\varepsilon x' = f(x, y), \qquad y' = g(x, y), \tag{5}$$

L.S. Pontryagin suggested starting the investigation of solutions to systems (5)) by analyzing the system (5) (the system of fast variables) based on its stationary solutions. If the system (5) has only equilibrium positions and some stable equilibrium position loses stability at a certain $y = y_1$ bifurcation value of the parameter, then a transition of the solution of system (5), accompanying the stable equilibrium, to another stable equilibrium may occur. In such transitional processes, the methods proposed in [6] are inapplicable.

L.S. Pontryagin proposed another method and performed an asymptotic expansion of solutions up to (and including) the boundary of stability loss. It was assumed that when the equilibrium position loses stability, the solution of (5) immediately moves a finite distance away from the newly unstable equilibrium. However, it turned out that this is not always the case; specifically, when an equilibrium position becomes unstable, the solution of (5) does not immediately leave the newly unstable equilibrium but remains close to it for a finite time. A specific example of this phenomenon was constructed in [6]. This phenomenon was later generalized in works [7–8]. In all these works, cases were considered where the first-approximation matrix function (the linearized part with respect to the unknown vector function) has complex conjugate eigenvalues that determine the stability of the equilibrium position.

In [9], singularly perturbed equations with analytic functions in some domain D of the complex plane are studied. Using the method developed in [9], new concepts of boundary layer lines, boundary layer regions, regular and singular regions are introduced, and their existence is proven. In [10], to address the limit transition problem, the concept of attraction regions was introduced. These are regions where the solutions of the considered SPEs are attracted to the solutions of unperturbed equations. Their existence and interrelation were proven. It was found that attraction regions do not exist for all solutions of the unperturbed equation. In all the listed works, cases were considered where the eigenvalues of the first-approximation matrix function do not have poles. In [11], specific cases were considered where the first-approximation coefficient matrix has eigenvalues with poles $\lambda_{1,2}(t) = \frac{1}{(t \pm i)n}$. Thus, the study of such cases is highly relevant. In this work, a linear SPE with a single logarithmic pole will be considered.

Object of study and problem statement

Consider the equation

$$\varepsilon x' = \frac{a'(t)}{a(t)} x + \varepsilon b(t), \tag{6}$$

with the initial condition

$$x(t_0, \varepsilon) = x^0 \tag{7}$$

and the conditions

C1. $a(t), b(t) \in Q(D)$ - the space of analytic functions in D¸ where D is some disk in the complex plane.

C2. $\exists!\, T_0 \in D(\, a(T_0), a'(T_0) \neq 0)$.

According to condition C2, we say that equation (6) has a logarithmic pole.

Problem. Investigate the influence of the logarithmic pole on the asymptotic behavior of the solution to the problem (6)–(7).

Solution to the problem

Replace the problem (7)–(8) with the following

$$x = x^0 exp \frac{1}{\varepsilon} ln \frac{a(t)}{a(t_0)} + \int_{t_0}^{t} b(\tau) exp \frac{1}{\varepsilon} ln \frac{a(t)}{a(\tau)} d\tau. \tag{8}$$

To simplify (9), map the domain D ((in the t -plane), to some domain H in the u, such that

$$u = a(t). \tag{9}$$

By conditions C1–C2, the mapping (9) is conformal and bijective. Assume that (10) maps D to some disk H, in the plane with center at the point (0; 0) and radius r_0. Under this mapping, the point T_0 is mapped to the point (0; 0).

From (10), we determine the function $t = \varphi(u)$. Then (8) can be rewritten as

$$y = x^0 exp\frac{1}{\varepsilon} ln\frac{u}{u^0} + \int_{u^0}^{u} b_1(\tilde{u}) exp\frac{1}{\varepsilon} ln\frac{u}{\tilde{u}} d\tilde{u}, \qquad (10)$$

$$y \equiv x(\varphi(u), \varepsilon), b_1(\tilde{u}) = b\big(\varphi(\tilde{u})\big) \cdot \varphi'(\tilde{u}), u^0 = a(t_0).$$

Note that concentric circles centered at the point (0; 0) are level lines of the function $u_1^2 + u_2^2$. All level lines are attracted to the point (0; 0) i.e., to the logarithmic pole.

Draw a circle of radius $r_1 = \sqrt{r_0^2 + \varepsilon ln\varepsilon}$, denoted by H_1. Introduce the notation $H \backslash H_1 = H_0$, assuming that the boundary of H_1 does not belong to H_0.

For u^0 take the point (r_0; 0) as the initial point and, for convenience of calculations, rewrite (11) in polar coordinates, setting

$$u_1 = rcos\alpha, u_2 = rsin\alpha, 0 \le r \le r_0, 0 \le \alpha \le 2\pi.$$

We have

$$y = x^0 \left(\frac{re^{i\alpha}}{r_0}\right)^{\frac{1}{\varepsilon}} + \int_{(r_0;0)}^{(r;\,\alpha)} b_1(\tilde{r}, \tilde{\alpha}) \left(\frac{re^{i\alpha}}{\tilde{r}e^{i\tilde{\alpha}}}\right)^{\frac{1}{\varepsilon}} d\big(\tilde{r}(cos\tilde{\alpha} + isin\tilde{\alpha})\big). \qquad (11)$$

Choose the integration path for the integral in (12). The disk H is divided by the real axis u_1 into two parts: the upper and lower parts, denoted as

$$H_1^1 = H_{01} \cup H_{11}, H_1^2 = H_{02} \cup H_{12},$$

H_{01} and H_{02}, H_{11} and H_{12} are, respectively, the upper and lower parts of the regions H_0 and H_1.

First, consider the regions $H_{01} \cup H_{11}$. In this case, the path goes from (r_0; 0) to (r_0; $\tilde{\alpha}$) (this point lies on the boundary of H_1^1), then along the ray $\alpha = \tilde{\tilde{\alpha}}$ from the point (r_0; $\tilde{\tilde{\alpha}}$) to (r_0; α).

Consider the following cases:

1. (r; α)∈l_1- the boundary of H_1^1; 2. (r; α)∈H_{01}; 3. (r; α)∈H_{11}.

1. Let (r; α)∈l_1. Then, considering the chosen path, from (12) we have

$$y = x^0\big(e^{i\alpha}\big)^{\frac{1}{\varepsilon}} + \int_0^{\alpha} b_1(r_0, \tilde{\alpha})(e^{\frac{i(\alpha-\tilde{\alpha})}{\varepsilon}} r_0 i e^{i\tilde{\alpha}} d\tilde{\alpha}.$$

Integrating the right-hand side by parts and passing to the modulus, we obtain

$$\big||x^0| - C_1\varepsilon\big| \le |y| \le |x^0| + C_1\varepsilon, (r;\ \alpha) \in l_1. \qquad (12)$$

2. Let $(r;\alpha)\in H_{01}$. In (11), perform the following transformation

$$y=\left(\frac{r}{r_0}\right)^{\frac{1}{\varepsilon}}\left[x^0\left(e^{i\tilde{\alpha}}\right)^{\frac{1}{\varepsilon}}+\int_0^{\tilde{\tilde{\alpha}}} b_1(r_0,\tilde{\alpha})e^{\frac{i(\tilde{\tilde{\alpha}}-\tilde{\alpha})}{\varepsilon}}r_0 ie^{i\tilde{\alpha}}d\tilde{\alpha}\right]+$$

$$+\int_r^{r_0} b_1(\tilde{r},\tilde{\tilde{\alpha}})\cdot\left(\frac{r}{\tilde{r}}\right)^{\frac{1}{\varepsilon}}\left(cos\tilde{\tilde{\alpha}}+isin\tilde{\tilde{\alpha}}\right)d\tilde{r}. \qquad (13)$$

In (14), the expression in brackets [...] given $y(r_0,\tilde{\tilde{\alpha}})((r_0,\tilde{\tilde{\alpha}})\in l_1$. Considering this, rewrite (14) as

$$y=y(r_0,\tilde{\tilde{\alpha}})\left(\frac{r}{r_0}\right)^{\frac{1}{\varepsilon}}+\int_r^{r_0} b_1(\tilde{r},\tilde{\tilde{\alpha}})\left(\frac{r}{\tilde{r}}\right)^{\frac{1}{\varepsilon}}\left(cos\tilde{\tilde{\alpha}}+isin\tilde{\tilde{\alpha}}\right)d\tilde{r}. \qquad (14)$$

$\sqrt{r_0^2+\varepsilon ln\varepsilon}<r\le r_0 \vee \sqrt{1+\frac{\varepsilon ln\varepsilon}{r_0^2}}<\frac{r}{r_0}<1.$

Then, passing to the modulus in (14), we obtain

$$\left|\left|y(r_0,\tilde{\tilde{\alpha}})\right|-C_1\varepsilon\right|<|y|\le\left|y\left(r_0,\tilde{\tilde{\alpha}}\right)\right|+C_1\varepsilon,(r,\alpha)\in H_{01}. \qquad (15)$$

3. Let $(r,\alpha)\in H_{11}$. Then, passing to the modulus in (14), considering that $\frac{r}{r_0}<1$, we obtain

$$|y|\le o(\varepsilon^n)+C_1\left|\int_r^{r_0}(r/r')^{1/\varepsilon}dr'\right|=o(\varepsilon^n)+C_1\varepsilon\left(1-(r/r_0)^{1/\varepsilon}\right)\le C_2\varepsilon,$$

$$|y|\le C_2\varepsilon,(r,\alpha)\in H_{01}. \qquad (16)$$

Separately, consider the case when $r\to0$. From (15), we have

$|y|\le C_1(r/r_0)^{\frac{1}{\varepsilon}}+C_2\left|\int_{r^0}^{r}(r/r')^{\frac{1}{\varepsilon}}dr'\right|=C_1\left(\frac{r}{r_0}\right)^{\frac{1}{\varepsilon}}+C_2\frac{\varepsilon}{1-\varepsilon}\left(1-(r/r_0)^{\frac{1}{\varepsilon}}\right)\le$

$\le C_2\frac{\varepsilon}{1-\varepsilon}+\left(C_1-C_2\frac{\varepsilon}{1-\varepsilon}\right)(r/r_0)^{\frac{1}{\varepsilon}}.$

Thus, as $r\to0$ we have $|y|\le C_3\frac{\varepsilon}{1-\varepsilon}$. Similarly, the cases are considered:

1. $(r;\alpha)\in l_2$ - the boundary of H_1^2 ; 2. $(r;\alpha)\in H_{02}$; 3. $(r;\alpha)\in H_{12}$.

For these cases, estimates analogous to those in (12), (15), and (16) hold.

Conclusion. Thus, for the function (10) there exist a boundary layer region H_0, a boundary layer line (the boundary of H_1) and an attraction region H_1. Since the mapping (9) is conformal and bijective, analogous regions exist in *D*. As the calculations show, the logarithmic pole does not affect the asymptotic behavior of the solution to the problem (6)–(7).

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MSC 34E05

ON DELAYED SOLUTIONS IN THE THEORY OF SINGULARLY PERTURBED EQUATIONS

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Abstract. We investigate singularly perturbed dynamical systems consisting of several fast variables coupled with a single slow equation. Particular emphasis is placed on situations where the matrix of the principal approximation of the fast subsystem admits more than one pair of complex-conjugate eigenvalues. The study focuses on the stability of equilibrium states and on the evolution of the real parts of the eigenvalues under variation of the slow variable. It is demonstrated that a solution delay phenomenon occurs in parameter intervals where all eigenvalues simultaneously change sign. The analysis combines the Laplace method, the method of stationary phase, and successive approximation techniques. The principal results are rigorously established and stated in the form of theorems.

Key words: equilibrium position, stability, analytical, harmonic function, delayed solutions.

Аннотация. В статье обобщаются результаты исследований динамических систем, включающих несколько быстрых и одно уравнение медленных переменных. Особое внимание уделено случаям, когда у матрицы-функции первого приближения быстрых переменных имеется более одной пары комплексно-сопряжённых собственных значений. Рассматривается устойчивость положений равновесия и поведение действительных частей собственных значений при изменении медленной переменной. Показано, что задержка решения возникает на отрезках, где все собственные значения меняют знак. Для доказательства использованы методы Лапласа, стационарной фазы и последовательных приближений; результаты сведены в виде теорем.

Ключевые слова. положение равновесия, устойчивость, аналитическая, гармоническая функция, задержка решений.

Аннотация. Бул макалада бир нече тез өзгөрмөлөрдү жана жай өзгөрмөлөр үчүн бир теңдемени камтыган динамикалык системаларды изилдөөнүн натыйжалары жалпыланган. Тез өзгөрмөлөрдүн биринчи жакындашуусунун матрица-функциясы бирден көп комплекстик-түйүндөш өздүк маанилерге ээ болгон учурларга өзгөчө көңүл бурулат. Жай өзгөрмө өзгөргөндө тең салмактуулук абалдын туруктуулугу жана өздүк маанилердин чыныгы бөлүктөрүнүн жүрүмү каралат. Чыгарылыштын кармалышы бардык өздүк маанилер белгисин өзгөрткөн аралыктарда пайда болоору көрсөтүлгөн. Далилдөө үчүн Лапластын усулу, стационардык фаза жана удаалаш жакындаштыруу усулдары колдонулат; натыйжалар теоремалар катары жалпыланган.
Урунттуу сөздөр: тең салмактуулук абалы, туруктуулук, аналитикалык, гармоникалык функция, чыгарылыштын кармалышы.

Introduction

In the 1970s, under the guidance of Academician L.S. Pontryagin, the phenomenon of delayed solution (DS) in singularly perturbed equations (SPE) near newly arising unstable equilibrium position (EP) was first identifed [1]. In these studies, the system was considered

$$\begin{aligned} \varepsilon x_1'(t,\varepsilon) &= y(x_1(t,\varepsilon)-y)-x_2(t,\varepsilon)+\gamma[(x_1(t,\varepsilon)-y)^2+x_2^2(t,\varepsilon)], \\ \varepsilon x_2'(t,\varepsilon) &= x_1(t,\varepsilon)-y+yx_2(t,\varepsilon)+\gamma[(x_1(t,\varepsilon)-y)^2+x_2^2(t,\varepsilon)], \end{aligned} \tag{0.1}$$

$$y' = 1 \tag{0.2}$$

where $\gamma\ is\ const$.

System (0.1) is referred to as the system of fast motions, while equation (0.2) a describes the slow variable, in the terminology introduced by L.S. Pontryagin [2].

From the results of studies conducted [3], it follows that if an EP loses stability at a certain bifurcation value of the slow variables, then the reduced system the solution should immediately depart a finite distance from the newly unstable EP. However, this is not always the case. In particular, for the system (0.1) – (0.2), it was proved that when the focus equilibrium loses stability, the solution of the full singularly perturbed system (0.1), satisfying the initial condition

$$|x_1(-1,\varepsilon)| = O(\varepsilon), |x_2(-1,\varepsilon)| = O(\varepsilon) \tag{0.3}$$

remains in the vicinity of the focus for values of the slow variable $y \in [-1; 1]$.

Thus, a new phenomenon was discovered in the theory of singularly perturbed equations, which was subsequently termed the delay of stability loss or the delay of solutions (DS) near an unstable equilibrium position. Henceforth, we will use the term DS.

DS in [1] was established by analytically continuing (0.1) – (0.2) into the complex plane (CP). Solution (0.2) was represented as $y = t$, and a square K with vertices at the points (-1; 0), (0; 1), (1; 0), (0; -1) was defined in the complex plane. System (0.1) was replaced by integrals. Taking into account (0.3) and applying the method of successive approximations (MSAs), the existence of a solution on the entire K and the phenomenon of DSs were proven.

S. Karimov generalized the results of [4] to a wider class of systems of the form (0.1). The methodology used remained the same as in [1]. A.I. Neishtadt [5-6] considered a more general system of autonomous singularly perturbed equations (ASPEs) under the condition that the matrix-function of the first approximation has one pair of complex conjugate eigenvalues (EVs) that intersect the imaginary axis with a nonzero velocity, while the remaining EVs have negative real parts. It was proven that, for analytical systems, the phenomenon of DSs occurs under certain conditions. This work did not consider even a single pair of complex conjugate eigenvalues for which the axis intersects with zero velocity.

K.S. Alybaev [7] developed a general methodology for studying ASPEs under the loss of stability of a EP. He introduced the concept of labeled sets and formulated sufficient conditions for DSs in terms of sets. Whether these conditions are also necessary has not yet been investigated.

The asymptotic behavior of solutions to ASPEs is an important topic in the theory of differential equations and dynamic systems. Early works explored various aspects related to the behavior of solutions as the stability of the equilibrium position changes.

1. Problem statement

The object of study of this work is the following types of ASPEs:

$$\varepsilon x'(t,\varepsilon) = A(y)\tilde{x}(t,\varepsilon) + \mathrm{V}^k\big(\tilde{x}(t,\varepsilon)\big)\tilde{x}(t,\varepsilon), \tag{1.1}$$

$$y' = 1, \tag{1.2}$$

with initial condition

$$x(t_0, \varepsilon) = x^0, y(t_0) = t_0 \tag{1.3}$$

where $0 < \varepsilon$ is small real parameter; $k = 1, 2$,

$t \in \Omega \subset C$ – set of complex numbers, a $\Omega = \{t \in C, |t| < r_0, r_0 \epsilon R -$ set of real numbers $and\ r_0 \gg |t_0|\}$,

$x = colon(x_1, \ldots, x_n)$, $\tilde{x} = colon(x_1 - y,\ x_2\ ,\ x_3 - y\ , x_4\ , \ldots, x_{2n-1} - y\ , x_{2n})$,

$A(y) = diag\{A_i : i = 1..n\}$,

$\mathrm{V}(\tilde{x}(t, \varepsilon)) = ((x_1 - y)^2 + x_2^2 + (x_3 - y)^2 + x_4^2 + \cdots + (x_{2n-1} - y)^2 + x_{2n}^2)$.

2. Results obtained:

The following cases were considered in works [8-14]:

C1. $A(y) = \begin{pmatrix} A_1(y) & 0 \\ 0 & A_2(y) \end{pmatrix}$, $A_1(y) = \begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}$, $A_2(y) = \begin{pmatrix} y & -2 \\ 2 & y \end{pmatrix}$, $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The matrix function $A(y)$ has eigenvalues: $\lambda_{1,2}(y) = y \pm i,\ \lambda_{3,4}(y) = y \pm 2i$.

C2. $A_j(y) = \begin{pmatrix} y & -\alpha_j \\ \alpha_j & y \end{pmatrix}$, $(j = 1, \ldots, n)$,

The matrix function $A(y)$ has $2n$ pairwise complex conjugate eigenvalues of the form: $\lambda_{2j-1}(y) = y + i\alpha_j,\ \lambda_{2j}(y) = y - i\alpha_j,\ j = 1, \ldots, n\ , \alpha_1 < \alpha_2 < \ \ldots < \alpha_n$.

C3. $A(y) = \begin{pmatrix} y & -\alpha_1 & 0 \\ \alpha_1 & y & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}$ c $\lambda_{1,2}(y) = y \pm i\alpha_1, \lambda_3(y) = y - \alpha_2, 0 < \alpha_1, 0 < \alpha_2$,

C4. $A(y) = \begin{pmatrix} A_1(y) & 0 \\ 0 & A_2(y) \end{pmatrix}$, $A_1(y) = \begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}$, $A_2(y) = \begin{pmatrix} 1 - y^2 & 2y \\ -2y & 1 - y^2 \end{pmatrix}$,

$\mathrm{V}^2 = (x_1 - y)^2 + x_2^2 + (x_3 - y)^2 + x_4^2$.

The matrix function $A(y)$ has eigenvalues: $\lambda_{1,2} = y \pm i$, $\lambda_{3,4} = -(y \pm i)^2$, $i = \sqrt{-1}$.

C5. The matrix function $A(y)$ has eigenvalues of the form: $\lambda_{2j-1}(y) = y - \alpha_j + i$, $\lambda_{2j}(y) = y - \alpha_j - i,\ j = 1, 2, \ldots, n, 0 = \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$.

A distinctive feature of the cases under consideration is that system (1.1)-(1.2) has an equilibrium position in the space of fast variables at the point $(y, 0, y, 0, \ldots, y, 0)$ and the stability of the equilibrium position is influenced by all eigenvalues of the matrix function $A(y)$.

The equilibrium position is stable for values of y, when the real parts of the eigenvalues are negative.

In cases C1 and C2, the real parts of the eigenvalues $(\lambda_1(y), \lambda_2(y))$ vanish at $y = 0$ and change sings from negative to positive.

For C3 the real parts of the two eigenvalues $(\lambda_1(y), \lambda_2(y))$ vanish at $y = 0$, and $\lambda_3(y)$ vanishes at $y = \alpha_2 > 0$ $(0 < \alpha_2 < \alpha_1)$ and at $\alpha_1 > 0$ $(0 < \alpha_1 < \alpha_2)$.

For C4 the real parts of the two eigenvalues $\lambda_{1,2}(y)$ vanish at $y = 0$, and $\lambda_{3,4}(y)$ vanish at $y = \pm 1$.

If we consider C5, then the real parts of the two eigenvalues $\lambda_{2j-1}(y), \lambda_{2j}(y)$ vanish at the points $y = \alpha_j$ $(j = 1, \dots, n)$, $0 = \alpha_1 < \alpha_2 < \dots < \alpha_n$.

Definition 1. If the stability of the EP is lost at certain values of y, but the solution to equation (1.1) does not immediately leave the resulting unstable EP, but remains near the resulting unstable EP for a finite time. We say that a DS occurs near the unstable EP (DSs).

Problem 1. For all the cases under consideration, investigate problem (1.1)-(1.2) for the DSs and solve all problems related to the DSs (the influence of the DS eigenvalues and determining the DSs boundary, etc.).

For case C1 [8], new unknown functions (NUF) are introduced in (1.1)

$$u_1 = x_1 - t, u_2 = x_2, u_3 = x_3 - t, u_4 = x_4;$$

$$z_1 = u_1 + iu_2, z_2 = u_1 - iu_2, z_3 = u_3 + iu_4, z_4 = u_3 - iu_4$$

and the equation is reduced to the following form:

$$\varepsilon z_j' = (t + k_j i) z_j + V^2(z_1, z_2, z_3, z_4) z_j - \varepsilon, \tag{2.1}$$

$$z_j(t_0, \varepsilon) = z_j^0,\ z_j^0 \leq M\varepsilon, \tag{2.2}$$

where $k_1 = 1, k_2 = -1, k_3 = 2, k_4 = -2$; $V(z_1, z_2, z_3, z_4) = z_1 z_2 + z_3 z_4$.

Denote $E(t,\tau) = \exp\left(\frac{(t+k_j i)^2 - (t_0 + k_j i)^2}{2\varepsilon}\right)$.

(2.1)-(2.2) is replaced by the following integral equation

$$z_j = F(z_j); F(z_j) := z_j^0 E(t, t_0) + \frac{1}{\varepsilon}\int_{t_0}^{t} (-\varepsilon + V^2) z_j E(t,\tau) d\tau \tag{2.3}$$

and the method of successive approximations was applied:

$$z_{jm} = F(z_{jm-1}), z_{j\,0} \equiv 0, \tag{2.4}$$

$V_{m-1} = z_{1\,m-1} z_{2\,m-1} + z_{3\,m-1}\, z_{4\,m-1},\ m = 1,2,\ldots.$

Successive approximations (SA) (2.4) are considered in the constructed domain $\mathcal{D}$ [8, 15 p.]. An estimate is made, and the uniform convergence of SA (2.4) is proved.

The following is proved:

Theorem 1. There exists a domain $\mathcal{D}_0 \subset \mathcal{D}$ and a solution to problem (2.1)-(2.2) $z(t,\varepsilon)$ defined in this domain, and for this solution the following estimate holds [8, p.18]:

$$\|z(t,\varepsilon)\| \le M_1 \begin{cases} \varepsilon,\ t \in \mathcal{D}_1 \cup \mathcal{D}_2, \\ \sqrt{\varepsilon},\ t \in \bigcup_{j=3}^{5}(\mathcal{D}_j \cup \bar{\mathcal{D}}_j) \cup \mathcal{D}_6 \end{cases} \tag{2.5}$$

According to the transformations carried out, the estimate follows from (2.5)

$$\|\tilde{x}(t,\varepsilon)\| \le M_1 \begin{cases} \varepsilon,\ t \in \mathcal{D}_1 \cup \mathcal{D}_2, \\ \sqrt{\varepsilon},\ t \in \bigcup_{j=3}^{5}(\mathcal{D}_j \cup \bar{\mathcal{D}}_j) \cup \mathcal{D}_6, \end{cases} \tag{2.6}$$

If we consider (2.6) on the real axis, we obtain

$$\|\tilde{x}(t,\varepsilon)\| \le M_1 \begin{cases} \varepsilon,\ t_0 \le t \le 1 - \sqrt{\varepsilon}; \\ \sqrt{\varepsilon},\ 1 - \sqrt{\varepsilon} \le t \le 1. \end{cases} \tag{2.7}$$

Corollary 1. (2.7) implies that the solution of problem (1.1)-(1.2)-.1.3) is delayed near the unstable EP that has arisen on the segment [0;1].

For case C2 [9] in (1.1) using the coordinate notation of vectors we have

$$\varepsilon u'_{2j-1} = t\, u_{2j-1} - \alpha_j\, u_{2j} + \mathrm{V}(u_1, \ldots, u_{2n})\, u_{2j-1} - \varepsilon,$$

$$\varepsilon u'_{2j} = \alpha_j\, u_{2j-1} + t\, u_{2j} + \mathrm{V}(u_1, \ldots, u_{2n})\, u_{2j},\ j = 1, \ldots, n.$$

From here we got the system

$$\varepsilon z'_{2j-1} = (t + \alpha_j i)\, z_{2j-1} + \mathrm{V}_0(z_1,\ z_2, \ldots, z_{2n})\, z_{2j-1} - \varepsilon, \tag{2.8}$$

$$\varepsilon z'_{2j} = (t - \alpha_j i)\, z_{2j} + \mathrm{V}_0(z_1, z_2, \ldots, z_{2n})\, z_{2j} - \varepsilon,$$

with initial condition

$$z_{2j-1}(t_0,\varepsilon) = z^0_{2j-1},\ z_{2j}(t_0,\varepsilon) = z^0_{2j},\ j = 1, \ldots, n, \tag{2.9}$$

where $V_0(z_1, z_2, \ldots, z_{2n}) = z_1 \cdot z_2 + \cdots + z_{2n-1} \cdot z_{2n}$.

The following is proved:

Theorem 2. There exists a domain $\mathcal{D}_0 \subset \mathcal{D}$ and a solution to problem (2.8)-(2.9) $z(t,\varepsilon)$ defined in this domain, and the following estimate holds [13, p.14]:

$$\|z(t,\ \varepsilon)\| \le M_2 \begin{cases} \varepsilon,\ \ t \in \mathcal{D}_1 \cup \mathcal{D}_2, \\ \sqrt{\varepsilon},\ \ t \in \left(\bigcup_{k=3}^{5}(\mathcal{D}_k \cup \bar{\mathcal{D}}_k)\right). \end{cases} \tag{2.10}$$

According to the transformations carried out, the estimate follows from (2.10)

$$\|\tilde{x}(t,\ \varepsilon)\| \le M_2 \begin{cases} \varepsilon,\ \ t \in \mathcal{D}_1 \cup \mathcal{D}_2, \\ \sqrt{\varepsilon},\ \ t \in \left(\bigcup_{k=3}^{5}(\mathcal{D}_k \cup \bar{\mathcal{D}}_k)\right). \end{cases} \tag{2.11}$$

If we consider (2.11) on the real axis, we obtain

$$\|\tilde{x}(t,\ \varepsilon)\| \le M_2 \begin{cases} \varepsilon,\ \ t_0 \le t_1 < \alpha_1 - \sqrt{\varepsilon} \\ \sqrt{\varepsilon},\ \alpha_1 - \sqrt{\varepsilon} \le t_1 \le \alpha_1. \end{cases} \tag{2.12}$$

Corollary 1. (2.12) implies that the solution of problem (1.1)-(1.2)-(1.3) is delayed near the unstable EP that has arisen on the segment $[0; \alpha_1]$.

For case C3 [10] in (1.1) let's introduce the NUF (for convenience, we will omit the arguments of the unknown function) $x_1 - y = u_1,\ x_2 = u_2,\ x_3 - y = u_3$ and we get the system

$$\begin{aligned} &\varepsilon u_1' = tu_1 - \alpha_1 u_2 + V^2(u_1, u_2, u_3)u_1 + \varepsilon, \\ &\varepsilon u_2' = \alpha_1 u_1 + tu_2 + V^2(u_1, u_2, u_3)u_2, \\ &\varepsilon u_3' = (t - \alpha_2)u_3 + V^2(u_1, u_2, u_3)u_3 + \varepsilon, \end{aligned} \tag{2.13}$$

In (2.13), multiplying the second equation by $(\pm i)$, then adding the first equation to the second, we obtain

$$\begin{aligned} &\varepsilon z_1' = (t + i\alpha_1)z_1 + V^2(z_1, z_2, z_3)\ z_1 + \varepsilon, \\ &\varepsilon z_2' = (t - i\alpha_1)z_2 + V^2(z_1, z_2, z_3)\ z_2 + \varepsilon, \\ &\varepsilon z_3' = (t - \alpha_2)z_3 + V^2(z_1, z_2, z_3)\ z_3 + \varepsilon, \end{aligned} \tag{2.14}$$

$$z_1(t_0, \varepsilon) = z_1^0, z_2(t_0, \varepsilon) = z_2^0, z_3(t_0, \varepsilon) = z_3^0, \tag{2.15}$$

where $z_1 = u_1 + iu_2,\ z_2 = u_1 - iu_2,\ z_3 = u_3, V(z_1, z_2, z_3) = z_1 \cdot z_2 + z_3^2$.
The following is proved:

Theorem 3. If $0 < \alpha_1 < \alpha_2$ then there exists a domain $\mathcal{D}_0 \subset \mathcal{D}$ and a solution to problem (2.14)-(2.15) $z(t,\varepsilon)$ defined in this domain, and the following estimate holds [10, p.408]:

$$\|z(t,\varepsilon)\| \le M_3 \begin{cases} \varepsilon, & t \in \mathcal{D}_{01} \cup \mathcal{D}_{02}, \\ \sqrt{\varepsilon}, & t \in \bigcup_{j=3}^{5}\left(\mathcal{D}_{0j} \cup \overline{\mathcal{D}}_{0j}\right) \cup \mathcal{D}_{06}, \end{cases} \quad (2.16)$$

According to the transformations carried out, the estimate follows from (2.16)

$$\|\tilde{x}(t,\varepsilon)\| \le M_3 \begin{cases} \varepsilon, & t \in \mathcal{D}_{01} \cup \mathcal{D}_{02}, \\ \sqrt{\varepsilon}, & t \in \bigcup_{j=3}^{5}\left(\mathcal{D}_{0j} \cup \overline{\mathcal{D}}_{0j}\right) \cup \mathcal{D}_{06}, \end{cases} \quad (2.17)$$

If we consider (2.17) on the real axis, we obtain

$$\|\tilde{x}(t,\varepsilon)\| \le M_3 \begin{cases} \varepsilon, & t_0 \le t \le \alpha_1 - \sqrt{\varepsilon}; \\ \sqrt{\varepsilon}, & \alpha_1 - \sqrt{\varepsilon} \le t \le \alpha_1. \end{cases} \quad (2.18)$$

Corollary 3.1. (2.18) implies that the solution of problem (1.1)-(1.2)-(1.3) is delayed near the unstable EP that has arisen on the segment $[0; \alpha_1]$.

The following is proved:

Theorem 3.2. If $0 < \alpha_2 < \alpha_1$ then there exists a domain $\mathcal{D}_0 \subset \mathcal{D}$ and a solution to problem (2.14)-(2.15) $z(t,\varepsilon)$ defined in this domain, and the following estimate holds [10, p.410]:

$$\|z(t,\varepsilon)\| \le M_3 \begin{cases} \varepsilon, & t \in \mathcal{D}_0^1, \\ \sqrt{\varepsilon}, & t \in \mathcal{D}_0^2 \cup \mathcal{D}_0^3. \end{cases} \quad (2.19)$$

According to the transformations carried out, the estimate follows from (2.19)

$$\|\tilde{x}(t,\varepsilon)\| \le M_3 \begin{cases} \varepsilon, & t \in \mathcal{D}_0^1, \\ \sqrt{\varepsilon}, & t \in \mathcal{D}_0^2 \cup \mathcal{D}_0^3. \end{cases} \quad (2.20)$$

If we consider (2.20) on the real axis, we obtain

$$\|\tilde{x}(t,\varepsilon)\| \le M_3 \begin{cases} \varepsilon, & t_0 \le t \le \alpha_2 - \sqrt{\varepsilon}, \\ \sqrt{\varepsilon}, & \alpha_2 - \sqrt{\varepsilon} < t \le \alpha_2 \end{cases} \quad (2.21)$$

Corollary 1. (2.21) implies that the solution of problem (1.1)-(1.2)-(1.3) is delayed near the unstable EP that has arisen on the segment $[0; \alpha_2]$.

For case C4 [11] in (4) let's introduce the NUF:

$$x_1(t,\varepsilon) - t = u_1(t,\varepsilon),\ x_2(t,\varepsilon) = u_2(t,\varepsilon),$$
$$x_3(t,\varepsilon) - t = u_3(t,\varepsilon),\ x_4(t,\varepsilon) = u_1(t,\varepsilon) \quad (2.22)$$

$u(t,\varepsilon) = colon(u_1, u_2, u_3, u_4)$, where $u_j(t,\varepsilon)$ $(j = 1, 2, 3, 4)$ is NUF.

Substituting (2.22) into (1.1) we obtain

$$\varepsilon u' = A(t)u + V^2 u - \varepsilon a,\ a = colon(1, 0, 1, 0) \quad (2.23)$$

$$u(t_0,\varepsilon) = u^0. \quad (2.24)$$

In (2.24), multiply the first and third equations by i, then add the first equation to the second, and the third to the fourth, we have

$$\varepsilon z' = \Lambda(t)z + V^2 z - \varepsilon a_1, \quad (2.25)$$

$$z(t_0,\varepsilon) = z^0, |z^0| \le M_1\varepsilon, \quad (2.26)$$

where $z = colon(z_1 = u_1 + iu_2, z_2 = u_1 - iu_2, z_3 = u_3 + iu_4, z_4 = u_3 - iu_4)$, $\Lambda(t) = diag(t+i, t-i, -(t+i)^2, -(t-i)^2), V = z_1 \cdot z_2 + z_3 \cdot z_4$, $a_1 = (1,1,1,1)$.

The following is proved:

Theorem 4. There exists a domain $\mathcal{D}_0 \subset \mathcal{D}$ and a solution to problem (2.25)-(2.26) $z(t,\varepsilon)$ defined in this domain, and the following estimate holds [11, 14c.]:

$$\|z(t,\varepsilon)\| \le M_4 \begin{cases} \varepsilon, & t \in \mathcal{D}_1; \\ \varepsilon^{1/3}, & t \in \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4 \cup \overline{\mathcal{D}}_4. \end{cases} \quad (2.27)$$

According to the transformations carried out, the estimate follows from (2.27)

$$\|\tilde{x}(t,\varepsilon)\| \le M_4 \begin{cases} \varepsilon, & t \in \mathcal{D}_1; \\ \varepsilon^{1/3}, & t \in \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4 \cup \overline{\mathcal{D}}_4. \end{cases} \quad (2.28)$$

If we consider (2.28) on the real axis, we obtain

$$\|\tilde{x}(t,\varepsilon)\| \le M_4 \begin{cases} \varepsilon, t_0 \le t \le -q; \\ \varepsilon^{1/3}, \quad -q \le t \le 0. \end{cases} \quad (2.29)$$

Corollary 4. (2.29) implies that the solution of problem (1.1)-(1.2)-(1.3) is delayed near the unstable EP that has arisen on the segment [-1;0].

In case C5 [12], the NNFs are introduced in (1.1) as follows:

$$x_{2j-1} - y = u_{2j-1}, x_{2j} = u_{2j}, \qquad j = 1,2,\dots,n.$$

Received

$$\begin{aligned} \varepsilon u'_{2j-1} &= (t-\alpha_j)u_{2j-1} - u_{2j} + V^2(u_1,\ u_2,\dots,u_{2n})u_{2j-1} - \varepsilon, \\ \varepsilon u'_{2j} &= u_{2j-1} + (t-\alpha_j)u_{2j} + V^2(u_1,\ u_2,\dots,u_{2n})u_{2j}, \end{aligned} \quad (2.30)$$

$$u_{2j-1}(t_0,\varepsilon) = x^0_{2j-1} - t_0, u_{2j}(t_0,\varepsilon) = x^0_{2j} \quad (2.31)$$

In (2.30), multiplying the second equation by $(\pm i)$, then adding the first to the second, we obtain

$$\begin{aligned} \varepsilon z'_{2j-1} &= (t-\alpha_j+i)z_{2j-1} + V^2(z_1,\dots,z_{2n})z_{2j-1} - \varepsilon, \\ \varepsilon z'_{2j} &= (t-\alpha_j-i)z_{2j} + V^2(z_1,\dots,z_{2n})z_{2j} - \varepsilon \end{aligned} \tag{2.32}$$

with a condition

$$z_{2j-1}(t_0,\varepsilon) = z^0_{2j-1}, z_{2j}(t_0,\varepsilon) = z^0_{2j} \tag{2.33}$$

where $V(z_1, z_2, \dots, z_{2n}) = z_1 \cdot z_2 + \ \dots + z_{2n-1} \cdot z_{2n}$.

$$\left|z^0_{2j-1}\right| \le M_1\varepsilon, \left|z^0_{2j}\right| \le M_2\varepsilon.$$

Theorem 5. There exists a domain $\mathcal{D}_0 \subset \mathcal{D}$ and a solution to this problem (2.32)-(2.33) $z(t,\varepsilon)$ defined in $\mathcal{D}_0$ and for this solution the following estimate holds [12, 44c.]:

$$\|z(t,\varepsilon)\| \le M_5 \begin{cases} \varepsilon, \ \ t \in \mathcal{D}_1 \cup \mathcal{D}_2, \\ \sqrt{\varepsilon}, \ t \in \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5 \cup \mathcal{D}_6 \cup \overline{\mathcal{D}}_3 \cup \overline{\mathcal{D}}_4 \cup \overline{\mathcal{D}}_5 \end{cases}, \tag{2.34}$$

$$t \in \left(\cup_{j=3}^{5}\left(\mathcal{D}_j \cup \overline{\mathcal{D}}_j\right)\right) \cup \mathcal{D}_6.$$

According to the transformations carried out, from (2.34) the estimate follows

$$\|\tilde{x}(t,\varepsilon)\| \le M_5 \begin{cases} \varepsilon, \ \ t \in \mathcal{D}_1 \cup \mathcal{D}_2, \\ \sqrt{\varepsilon}, \ t \in \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5 \cup \mathcal{D}_6 \cup \overline{\mathcal{D}}_3 \cup \overline{\mathcal{D}}_4 \cup \overline{\mathcal{D}}_5 \end{cases}, \tag{2.35}$$

If we consider (2.35) on the real axis, we obtain

$$\|\tilde{x}(t,\varepsilon)\| \le M_5 \begin{cases} \varepsilon, \ t_0 \le t \le 1-\sqrt{\varepsilon}, \\ \sqrt{\varepsilon}, \ 1-\sqrt{\varepsilon} < \ t \le 1. \end{cases} \tag{2.36}$$

Corollary 5. (2.36) implies that the solution of problem (1.1)-(1.2)-(1.3) is delayed near the unstable EP that has arisen on the segment [0; 1].

Conclusion

The following methods [14] were used in studying the asymptotic behavior of solutions to the ASPEs [8-13]:

1. Transformation of equations and their replacement with integral ones;
2. Geometric constructions and a justified choice of integration paths using level lines and the properties of harmonic functions.
3. Methods: successive approximations, Laplace one, stationary phase, integration by parts, and successive replacement of initial conditions with variable initial conditions on certain lines.

The obtained results are presented as theorems, where ASPEs, MSAs, which have a different number (more than two) eigenvalues, and all eigenvalues affect the stability of the EP, the main point of the study being geometric constructions using level lines and the properties of harmonic functions.

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MSC 35G35

INVERSE PROBLEM FOR A THIRD-ORDER PARTIAL INTEGRO-DIFFERENTIAL EQUATION WITH POWER SINGULARITIES

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Abstract. The existence of a solution of the inverse problem for a third-order partial integral-differential equation with power singularities is proven in this work. The function in the initial condition is considered to be unknown along with the unknown function in the posed problem.

Бул макалада үчүнчү тартиптеги даражалуу өзгөчөлүгү менен жекече туундулуу дифференциалдык теңдеме үчүн тескери маселенин чечиминин жашашы далилденген. Коюлган маселеде белгисиз функция менен бирге баштапкы шарттагы функция да белгисиз деп каралган.

В данной работе доказано существования решения обратной задачи для дифференциального уравнения в частных производных третьего порядка со степенными особенностями. В поставленной задаче вместе с неизвестной функцией неизвестной считается функция в начальном условии.

Key words. Inverse problem, third order, partial derivative, integro-differential, equation, singularity, method of additional argument.

Тескери маселе, үчүнчү тартип, жекече туунду, дифференциалдык, теңдеме, өзгөчөлүк, кошумча аргумент кийирүү усулу.

Обратная задача, третий порядок, частная производная, дифференциальное, уравнение, особенность, метод дополнительного аргумента.

Introduction. The application areas of the method of additional argument (MAA) are currently expanding. In this paper there is discussed the existence of a solution of the inverse problem for a third-order partial integral-differential equation with power singularities.

In this work there is considered an integral-differential equation (IDE) with partial derivatives (PD) of the form:

$$D[t^m,-x^n]D^2[t^m,x^n]u(t,x) = f(t,x,u) + \int_0^t K\big(t,s,u(s,x)\big)d\,s, \quad (1)$$

$$(t,x) \in R_1^2 = [0,T] \times R,$$

where $D[\omega,\vartheta] = \omega\partial/\partial t + \vartheta\partial/\partial x$ and n,m are real numbers.

MAA was applied to the IDEs of first and second order:

1) the existence of a unique and continuous solution of the first-order problem was proven in the case $m < 1$ in the work [1];

2) the existence of a unique solution of the second-order problem in the case $0 \leq m < 1$ was proven in the work [2].

Purpose of the Study. Our goal is using MAA to determine the values m, n for which a solution of (1) equation exists.

Material and research methods.

Let us consider (1) equation with the initial conditions

$$\partial^{\mathrm{k}} \mathrm{u}(\mathrm{t}, \mathrm{x}) / \partial \mathrm{x}^2|_{\mathrm{t}=0} = \varphi_{\mathrm{k}}(\mathrm{x}), \mathrm{k} = 0,1,2. \quad (2)$$

Functions are still unknown functions that we will determine in the course of solving the problem and $\varphi_k(x), k = 1,2,\ \varphi_0(x) \in \overline{C}^{(3)}(R)$.

In this work we use the classes of functions introduced in [3].

THEOREM. Let $K(t,s,u) \in \overline{C}^{(3)}((0 \leq s \leq t \leq T) \times R^2) \cap Lip(M)|_u$,

$$f(t,x,u) \in \overline{C}^{(3)}([0,T] \times R^2) \cap Lip(L)|_u),$$

where L, M are positive constants.

Then there exists a continuous and bounded solution to problem (1)-(2) in the domain

$Q_t = \begin{cases} 0 \leq t \leq T_*, x \in R_+, n \neq 1 \\ 0 \leq t \leq T_*, x \in R, n = 1 \end{cases}$, where they are determined from the initial data, in the case when $T_* \leq T, 1/2 \leq m < 1$.

When $m \geq 1$, $m < 1/2$, equation (1) is an equation with an essential singularity.

Proof. Let's introduce the following notations:

$$z(t,x) = D^2[t^m, x^n]u(t,x) = t^{2m}u_{tt}(t,x) + 2t^m x^n u_{tx}(t,x) + x^{2n}u_{xx}(t,x) +$$
$$+ m t^{2m-1}u_t(t,x) + n x^{2n-1}u_x(t,x), \quad (3)$$

$z(t,x)|_{t=0} = \psi_0(x)$, in case $1/2 \leq m < 1$.

$D[t^m, x^n)]u(t,x)|t = 0 = \psi_1(x)$, in case $0 \leq m < 1$.

The equation (1) also can be written in the following form:

$$t^m\partial[t^{2m}u_{tt}+2t^mx^nu_{tx}+x^{2n}u_{xx}+mt^{2m-1}u_t+nx^{2n-1}u_x]/\partial t-$$
$$-x^n\partial[t^{2m}u_{tt}+2t^mx^nu_{tx}+x^{2n}u_{xx}+mt^{2m-1}u_t+nx^{2n-1}u_x]/dx=$$
$$=f(t,x,u)+\int_0^t K\big(t,s,u(s,x)\big)ds. \tag{4}$$

Putting the introduced notation (3) into (4) we have:

$$t^m\partial z(t,x)/dt-x^n\partial z(t,x)/dx=f(t,x,u)+\int_0^t K(t,s,u(s,x))ds \tag{5}$$

In case $1/2\le m<1$ by applying MAA to equation (5) with initial conditions (2), we obtain

$$z(t,x)=\psi_0(q(0,t,x))+\int_0^t f(s,q(s,t,x),u(s,q(s,t,x))/s^m ds+$$
$$+\int_0^t\int_0^s K(s,\nu,u(\nu,q(s,t,x))/s^m d\nu ds=F(t,x;u), \tag{6}$$

where

$$q(s,t,x)=[(m-1)/(\frac{n-1}{t^{m-1}}+\frac{m-1}{x^{n-1}}+\frac{n-1}{s^{m-1}})]^{1/(n-1)},$$
$$(s,t,x)\in {D_T}^+=\{0\le s\le t\le T, x\in R_+\}.$$

If $n\ne 1$, then

$$q(s,t,x)=exp[1/\{(m-1)s^{m-1}\}-1/\{(m-1)t^{m-1}\}],$$

$(s,t,x)\in D_T=\{0\le s\le t\le T,\ x\in R,$ If $n=1$.

Function $q(s,t,x)$ is a solution of the differential equation $t^mq_t(s,t,x)-x^nq_x(s,t,x)=0$, with a condition $q(s,s,x)=x$.

Indeed, we have

$$q_t(s,t,x)=q^n(s,t,x)/t^m, \tag{7}$$
$$q_x(s,t,x)=q^n(s,t,x)/t^n, \tag{8}$$

Multiplying (7) by (8) we get that t^m, x^n,

$$t^mq_t(s,t,x)-x^nq_x(s,t,x)=0.$$

Now we apply MAA to problem (6), (2):

$$D[t^m,x^n]u(t,x)=\psi_1(p(0,t,x)+\int_0^t F(s,p(s,t,x);u)/s^m ds. \tag{9}$$

From (9), (2) by using MAA again we obtain:

$$u(t,x)=\varphi_0(p(0,t,x))+t^{1-m}\psi_1(p(0,t,x)/(1-m)+$$
$$+\int_0^t\int_0^s F(\rho,p(\rho,t,x);u)/(s\rho)^m d\rho ds \tag{10}$$

where $p(s,t,x) = [(m-1)/(\frac{n-1}{s^{m-1}} + \frac{m-1}{x^{n-1}} + \frac{n-1}{t^{m-1}})]^{1/(n-1)}$,

$(s,t,x) \in D_T{}^{+}$, lf $n \neq 1$,

$$p(s,t,x) = exp[1/\{(m-1)t^{m-1}\} - 1/\{(m-1)s^{m-1}\}],$$

$(s,t,x) \in D_T$, lf $n = 1$,

The function $\rho(s,t,x)$ satisfies the equality:

$$\rho(s,\tau,\rho(\tau,t,x)) = \rho(s,t,x). \tag{11}$$

For the integral equation (10), we construct successive approximations, assuming

$$u^0(t,x) = \varphi_0(p(0,t,x)) + t^{1-m}\psi_1(p(0,t,x)/(1-m) +$$
$$+\int_0^t\int_0^s \psi_0\,(q(0,\rho,p(\rho,t,x)))/(s\rho)^m ds d\rho,$$

$$u^N(t,x) = \int_0^t\int_0^s\int_0^\rho f(v,q(v,\rho,p(\rho,t,x)),u^{N-1}(v,q(v,\rho,p)))(v\rho s)^{-m} dv d\rho ds +$$
$$+\int_0^t\int_0^s\int_0^\rho\int_0^\eta K(\eta,\xi,u^{N-1}(\xi,q(\eta,\rho,p(\rho,t,x))(\eta v\rho s)^m d\xi d\eta dv d\rho ds,$$
$$N = 1,2,3\ldots \tag{12}$$

The following inequalities are valid from (12) for $N = 1,2,3\ldots$:

$$\|u^N(t,x) - u^{N-1}(t,x)\| \le \Omega(t)\|u^{N-1}(t,x) - u^{N-2}(t,x)\|, \tag{13}$$
$$\Omega(t) = (L/3!)t^{3(1-m)}/(1-m)^3 + (M/4!)t^{4(1-m)}/(1-m)^4.$$

From inequality (13) we define $T_* \le T$, so that $\Omega(t) \le 1$.

Therefore, when $T_* \le T$ successive approximations converge to a continuous and bounded function $u(t,x)$.

The theorem is proven.

Now, after finding the unknown function, we can determine the unknown functions given in the initial conditions by taking the derivative from formula (9), then equating $t = 0$.

Example. Let $f(t,x,u) = f(t)$, $K(t,x,u) = 0$ in equation (1) .

Then from formula (10) we obtain the solution of the problem in the form:

$$u(t,x) = \varphi_0(p(0,t,x)) + t^{1-m}\psi_1(p(0,t,x)/(1-m) +$$
$$+\int_0^t\int_0^s \psi_0\,(q(0,\rho,p(\rho,t,x)))/(s\rho)^m d\rho ds. \tag{14}$$

Let us now define the functions $\varphi_k(x)$, $k = 1,2$, from (14).

$$u_t(t,x) = \varphi_0(p(0,t,x)) + t^{1-m}\psi_1(p(0,t,x)/(1-m) +$$
$$+\int_0^t \int_0^s \psi_0(q(0,\rho,p(\rho,t,x)))/(s\rho)^m d\rho ds.$$

Conclusion. There is proposed the application of MAA to inverse problems. The function in the initial condition is considered unknown together with the unknown function in the problem. It is also proven that this equation has a power singularity.

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LOCAL OPERATORS AND EQUATIONS IN UNIFORM SPACES

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Supra, the authors gave definitions of local operators for various functions defined on topological spaces: coinciding of two functions in any neighbor of a point implies a same value of the operator on such functions in the point. The notion of derivative was extended to all functions in metrical spaces. A local equation for a scalar fractal function was constructed. In the paper these results are extended for "almost coinciding" functions in any neighbor of a point.

Keywords: functional relation, local operator, local equation, equation, fractal, uniform space.

Мурда авторлор каалаган функциялар үчүн жергилик операторлор үчүн аныктоолор берди: чекиттин каалаган айланасында эки функциянын дал келиши чекиттеги мындай функциялар боюнча оператордун бирдей маанисин билдирет. Туунду түшүнүгү метрикалык мейкиндиктердеги бардык функцияларга жайылтылды. Фрактал функциясы үчүн жергилик теңдеме курулду. Бул макалада мындай жыйынтыктар функциянын чекиттин каалаган айланасында "дээрлик дал келишине" жалпыланат.

Урунттуу сөздөр: байланыш, жергилик оператор, жергилик теңдеме, кадимки теңдеме, фрактал, бир калыптуу мейкиндик

Ранее авторы дали определения локальных операторов для различных функций: совпадение двух функций в любой окрестности точки влечет за собой одинаковое значение оператора над такими функциями в этой точке. Понятие производной было распространено на все функции в метрических пространствах. Было построено локальное уравнение для фрактальной функции. В данной статье эти результаты распространены на «почти совпадающие» в любой окрестности точки функции.

Ключевые слова: функциональное соотношение, локальный оператор, локальное уравнение, уравнение, фрактал, равномерное пространство.

Introduction

We use the notion "local" in meaning "infinitesimal". This notion in the paper does not relate to "local operators, local variables" in programming and other branches of mathematics where it stands for "not global, in a bounded, definite domain."

Considered a general task on relations between equations with global and local operators for a same function.

Previously, definitions of local operators involved any kind of smoothness of a function defined on a domain with locally linear structure.

For example, definition [1], briefly: Local operator refers to the definition of a local product of fields and their space-time derivatives, and other definitions which meant any "smoothness".

We proposed more general definitions. In this paper we extend these definitions.

1. Definition for topological spaces

Definition 1 [2]. A general local operator Q of the continuous (with additional conditions) function $w:U\to V$, where $U\to V$ are topological spaces, at the point $u\in U$ satisfies the condition: if for some neighborhood D of the point u the restrictions $w_1|_D = w_2|_D$, then $Q(w_1,u) = Q(w_2,u)$.

Remark. This definition did not demand any structure in the range V.

Example 1. Φ of a general local operator defined for all continuous functions and not related to any "differentiation". Consider functions $w\in C(R)$.

$Q(w,u) = limsup\{measure\ \{x\in[u-h, u+h]: w(x)\geq 0\}/(2h): h\to 0\}$.

Remark. The statement *"$Q(w,u)=0$"* does not imply the statement *"u is a point of a local maximum / minimum of the function w"*.

Example. Define $w(x):=\Sigma\{max\{0,\ 4^{-n}-|x-2^{-n}|\}: n\in N\}$**.** The function $w(x)$ has positive values in any neighbor of $x=0$ but the measure (the sum of their "lengths") is estimated as (for $h=2^{-n}$) $O(4^{-n}/2^{-n})= O(2^{-n})$ tends to 0 as n tends to ∞.

2. Definitions of derivatives of all functions

Probably, the first example of local operators over non-smooth scalar functions was

Definition 2 [3]. The upper right derivative number in a point x_0 is

$$limsup\{(f(x)-f(x_0))/(x-x_0)|\ x\to x_0+0\} \quad (1)$$

(if it exists). Similarly, the lower right, the upper left and lower left derivative number were determined.

We [4] modified (1) to extend to all functions:

Definition 3. We proposed to change (1) to

$$limsup,\ liminf\ \{(f(x)-f(x_0))/\ (x-x_0)/\ (|\ f(x)-f(x_0)|\ /|x-x_0|+1)\ |\ x\to x_0\pm 0\}. \quad (2)$$

Such four real numbers exist for all functions of real variable. We called them “reduced derivative numbers”.

We generalized it to functions from metrical spaces to metrical spaces.

Definition 4. “Lower” and “upper” estimations of “rate of changing” of the function.

$$liminf,\ limsup\{\rho(f(x),\ f(x_0))/\rho\ (x,\ x_0)/(\ \rho(f(x),\ f(x_0))/\rho\ (x,\ x_0)+1)\ |\ x\to x_0\}. \quad (3)$$

3. Examples of local equations for fractals

Example 2. Define the function:

$f(x)=1\ (-2\le x<1);\ f(x)=\ -1\ (1<x\le 2);\ f(0)=0.$

The functional relation $f(x)=\ f(-2x)/2$ extends the function to all the axis R.

This function fulfils the following local equations:

$liminf\{mes\{f(x)>0|-h\le x<0\}/h\ |\ h\to +0\}=1/3;$

$limsup\{mes\{f(x)>0|-h\le x<0\}/h\ |\ h\to +0\}=2/3;$

$liminf\{mes\{f(x)>0|\ 0<x\le h\}/h\ |\ h\to +0\}=1/3;$

$limsup\{mes\{f(x)>0|\ 0<x\le h\}/h\ |\ h\to +0\}=2/3.$

Example 3. Consider the functional relation for a continuous function $u(x,y)\in C(R^2\to R)$ (fractal, concentric circles)

$$u(2x,2y)=\ ku(x,y),\ (x,y)\in R^2. \quad (4)$$

Theorem 1. Solutions of the equation (4) for $k>1$ fulfil the local equation

$Q(u,0,0)=a_u \in R,$ (5)

$Q(u,x,y) := \limsup\{measure\ \{|(x-x', y-y')|\le h \mid u(x',y')\ge 0\}/(\pi h^2)|\ h\to +0\};$ (6)

$a_u := \sup\{\ measure\ \{|(x,y)| \in[1-t, 2-t]: u(x,y)\ge 0\}|0 \le t \le 1/2\}/(\pi(3-2t)).$ (7)

Proof. The assumption "$u(0,0)\neq 0$" implies contradiction: we have $u(1,1) =k^n u(2^{-n},2^{-n})$ for all $n\in N$, thus $|u(1,1)|$ is unbounded. Hence, $u(0,0)=0$.

Theorem 2. Solutions of the equation (4) for $k<-1$ fulfil the local equation (5) with

$a_u := \sup\{\ measure\ \{|(x,y)| \in[1-t, 2-t]: u(x,y)\ge 0\}|0 \le t \le 3/4\}/(\pi(3-2t)).$

Example 4. The functional relation for a continuous function

$u(2x,2y)= ku(-y, x), (x,y)\in R^2, k>1$ (8)

presents a spiral fractal.

4. Definition for uniform range of functions

Definition 1 does not cover the known

Example 5 of discontinuous everywhere but differentiable at $x=0$ function

$w_1(x)=\{0$, x is a rational number; x^2, x is an irrational number$\}$ and $w_2(x)\equiv 0$: $w_1'(0)=0;\ w_2'(0)=0$.

We propose

Definition 5. A general local operator Q on functions $w:U\to U$, where U is a uniform space, at the point $u\in U$ satisfies the condition:

if for any $n\in N$ there exist such entourages W and W_n in U that $((u,x) \in W)\Rightarrow ((w_1 (x), w_2 (x)) \in W_n)$

then $Q(w_1,u) = Q(w_2 ,u)$.

Corresponding definition for metrical spaces:

Definition 6. A general local operator Q on functions $w:U\to U$, where U is a metric space, at the point $u\in U$ satisfies the condition:

if for any $n\in N$ there exists such $\varepsilon>0$ that $(\rho(u,x) <\varepsilon)\Rightarrow (\rho (w_1 (x), w_2 (x)) <\varepsilon/n)$

then $Q(w_1,u) = Q(w_2 ,u)$.

For Example 5 we may take $\varepsilon=1/n$:

If $|x-0|<1/n$ then $|w_1(x) - w_2(x)| \le |x^2-0|<1/n^2 = \varepsilon/n$.

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MSC 49M37

THE PROBLEM OF OPTIMAL DISTRIBUTION OF SOWN AND RENTED AREA FOR AGRICULTURAL CROPS

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The paper formulates an economic and mathematical model of the problem of determining the optimal size of the sown area for each type of crop on one's own and rented plots, taking into account financial credit for expanding agricultural production and with possible minimum and possibly maximum restrictions on rented plots. For the problem under consideration, in accordance with the given conditions, the maximum income was obtained.

Key words: Peasant farm, sown area, rent, profit, market conditions, mathematical model, price, unit of weight, products, costs, yield.

Макалада айыл чарба өндүрүшүн кеңейтүү үчүн каржылык кредитти эске алуу менен, ошондой эле ижарага алынган участокторго мүмкүн болгон минималдуу жана мүмкүн болгон максималдуу чектөөлөрдү эске алуу менен өздүк жана ижарага алынган участоктордо айыл чарба өсүмдүктөрүнүн ар бир түрү боюнча эгин аянтынын оптималдуу өлчөмүн аныктоо маселесинин экономикалык-математикалык модели түзүлгөн. Бул маселеде берилген шарттарга ылайык максималдуу киреше алынган.

Урунттуу сөздөр: Дыйкан чарбасы, айдоо аянты, рента, пайда, рынок шарты, математикалык модел, баа, салмак бирдиги, продукция, чыгаша, түшүм.

В статье сформулирована экономико-математическая модель задачи определения оптимального размера посевной площади под каждый вид культуры на своих и арендуемых участках с учетом финансового кредита для расширения производства сельхоз продукции и с возможными минимальными и возможно максимальными ограничениями на арендуемые участки. Для рассматриваемой задачи в соответствии с приведенными условиями получен максимальный доход.

Ключевые слова: Крестьянское хозяйство, посевные площади, аренда, прибыль, рыночные условия, математическая модель, цена, единица веса, продукция, затраты, урожайность.

Introduction

Agriculture in Kyrgyzstan is one of the main sources of wealth for the population of our country. As a result of the reform carried out in the agricultural sector, rural residents became owners of land plots, but with different soil qualities in different plots. Having received land plots, many owners began to engage in agricultural production, dynamically forming numerous peasant (farming) households, cooperatives, business partnerships and joint-stock companies [6]. At present, a significant part of the gross agricultural output is produced by the non-state sector, and peasant farms. The goods they produce are based on small-scale land and a weak material and technical base. This circumstance determines the low incomes of peasant farms and forces them to look for ways to increase production capacity and expand their plots.

Farmers or agricultural enterprises with limited land areas, planning to increase production capacity, need additional territory to expand crop areas and increase yields.

The revival of lease relations in the use and expansion of crop areas of farms is an effective organizational form of production in the agricultural sector of the rural industry.

Lease is becoming an economically advantageous option for expanding crop areas [5,7], without the need to purchase land in ownership and, accordingly, increase yields and profits. Leased lands allow farmers and agricultural enterprises to quickly adapt to changing market conditions and requirements, for example, if there is a demand for certain a crop, you can lease land in suitable regions for their cultivation.

This practice is of key importance in modern agriculture, especially for those farmers or agricultural enterprises who do not have enough of their own land to conduct business.

In the works [1,2,3] the authors developed an economic and mathematical model for expanding the sown area in accordance with their financial capabilities.

In this work, we will formulate an economic and mathematical model for the problem of determining the optimal size of the sown area for each crop using leased lands, where possible lower and upper boundaries of leased sown areas are indicated, taking into account the received loan.

Statement of the problem.

Let the farm have n different types of sown areas with different productivity per unit area, where it is planned to grow k types of agricultural crops, $k \in K=\{1..p\}$. The sizes of each category of sown areas in the farm are known and equal to S_j, $j=1..n$.

To increase production capacity, the farm plans to expand its existing sown areas by renting sown areas of varying productivity per unit area. For this purpose, the farm plans to take out a loan in the amount of T monetary units. The sizes of the rented sown areas y_j, $j=1..n$, are limited by upper and lower limits, i.e. $R_j^{'} \leq y_j \leq R_j^{''}, \quad j = 1..n$.

It is assumed that the following are known: the yield of the *k*-th crop per unit of the *j*-th sown area, $k \in K$, the costs of obtaining the *k*-th product per unit of the *j*-th sown area, $j=1..n$, the market price (forecast) per unit of weight of the *k*-th agricultural product, $k \in K$.

The payment r_j per unit of the *j*-th rented sown area with different productivity per unit of area is also known.

It is required to determine the optimal size of the sown area for each crop so that the gross profit of the farm is maximum.

For the formulation of the mathematical model of the problem, we give known parameters.

c_0^k is the market price per unit of weight of the *k*-th agricultural crop, $k \in K$;

c_j^k is costs of obtaining the *k*-th product from a unit of the *j*-th sown area, $j=1..\ n$, $k \in K$;

a_j^k is yield of the *k*-th crop from a unit of the *j*-th sown area, $k \in K$, $j = 1..n$;

α is interest rate of the loan received by the farm;

T is the maximum amount of the loan received by the farm;

r_j is the payment for a unit of the j-th leased sown area, $j = 1..n$;

S_j is size of the j-th sown area of the farm, $j = 1..n$;

$R_j^{''}, R_j^{'}$ is the maximum and minimum possible sizes of the j-th rented sown area, $j = 1..n$;

Sought variables:

x_j^k is the size of the j-th sown area for the k-th crop, $j=1..n$, $k \in K$;

y_j is the size of the rented sown area of the j-th type, $j=1..n$;

z_j is the size of the loan received by the farm at interest for the acquisition of sown areas for rent for growing agricultural crops.

According to the accepted notations, the mathematical model of the problem has the form.

Find the maximum

$$L(x,y) = \sum_{k \in K} \sum_{j=1}^{n} (a_j^k c_0^k - c_j^k) x_j^k - \sum_{j=1}^{n} r_j y_j - (1+\alpha) z_j \quad (1)$$

under the conditions

$$\sum_{k \in K} x_j^k = S_j + y_j, \quad j = 1..n, \quad (2)$$

$$R_j^{'} \leq y_j \leq R_j^{''}, \quad j = 1..n, \quad (3)$$

$$\sum_{k \in K} c_j^k x_j^k + r_j y_j = z_j, \quad j = 1..n, \quad (4)$$

$$\sum_{j=1}^{n} z_j \leq T, \quad (5)$$

$$x_j^k \geq 0, \quad y_j \geq 0, \quad z_j \geq 0, \quad j=1..n, \quad k \in K, \quad (6)$$

where $x = \left| x_j^k \right|_{n,|K|}$, $y = (y_1, y_2, \ldots, y_n)$.

The objective function (1) determines the maximum net income of the farm;

Constraint (2) requires that the total size of the sown and rented area of each type should be allocated to all types of agricultural crops;

Constraint (3) shows that the volume of rented sown area of each type is limited by upper and lower limits;

Constraint (4) determines the size of the financial loan received by the farm;

Constraint (5) requires that the total size of the financial loan received by the farm should not exceed the maximum issuance capacity;

Constraint (6) requires that the variables are not negative.

To solve problem (1)-(6), we will use the methods for solving the linear problem given in [4].

We will write the economic and mathematical model of the problem (1)-(6) of determining the optimal size of the sown area for each crop using leased lands in the form of Table 1, where for compactness of notation the notations

$$d_j^k = a_j^k c_0^k - c_j^k, \quad j=1..n, \; k \in K, \qquad (*)$$

are introduced.

Table 1.

x_1^1	…	x_1^p	…	x_n^1	…	x_n^p	y_1	…	y_n	z_j	…	z_n		
1	…	1	…		…		-1	…					=	S_1
													=	S_2
	…		…		…			…					…	…
	…		…	1	…	1		…	-1				=	S_n
							1						≤	R_1''
							1						≥	R_1'
					…			…					…	…
									1				≤	R_n''
	…		…		…			…	1				≥	R_n'
c_1^1	…		…	c_n^1			R_1			-1	…		=	0
	…		…		…			…			…		…	
		c_1^k	…		…	c_n^k			R_n			-1	=	0
										1	…	1	≤	T
d_1^1	…	d_1^p	…	d_n^1	…	d_n^p	$-r_1$	…	$-r_n$				→	*max*

From the solution of problem (1)-(6) we determine the volume of the sown area y_j^* for each type of crop j = $1..n$, the size of the financial loan received by the farm to expand the sown area and the size of the rented sown area.

Conclusion

Rented sown areas are an important tool in managing agricultural business. They provide farmers and agricultural enterprises with economic flexibility, reduce risks and allow them to adapt to changing market and environmental conditions. Thanks to this, agricultural enterprises can develop effectively and maintain high profitability.

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MSC 45D05

REGULARIZATION AND UNIQUENESS OF SOLUTIONS FOR ONE CLASS OF VOLTERRA LINEAR INTEGRAL EQUATIONS OF THE THIRD KIND ON THE HALF-LINE

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Annotation. In this paper, we construct Lavrent'ev-type regularizing operators for a specific class of linear Volterra integral equations of the third kind defined on the half-line. Under appropriate conditions, the uniqueness of the solution is proved, and estimates for the solution are obtained. The results are supported by a concrete example demonstrating the applicability of the theoretical findings.

Keywords. Volterra integral equation, third kind, regularization, uniqueness, formula, space, half-line.

Аннотация. Бул макалада жарым окто аныкталган Вольтеранын интегралдык теңдемелеринин үчүнчү түрүнүн бир классы үчүн Лаврентьев ыкмасы боюнча регуляризациялоочу операторлор түзүлгөн. Айрым шарттарда чыгарылыштын жалгыздыгы далилденип, алар үчүн баалар алынган. Теориялык натыйжалардын колдонулушу мисал аркылуу көрсөтүлгөн.

Урунттуу сөздөр. Вольтерранын интегралдык теңдемеси, үчүнчү түр, регуляризациялоо, жалгыздык, формула, мейкиндик, жарым ок.

Аннотация. В данной работе построены регуляризующие операторы в смысле Лаврентьева для одного класса линейных интегральных уравнений Вольтерра третьего рода на полуоси. При выполнении определённых условий доказана единственность решения и получены оценки решения. Представлен пример, подтверждающий применимость теоретических результатов.

Ключевые слова. интегральное уравнение Вольтерра, третий род, регуляризация, единственность, формула, пространство, полуось.

Various questions of the theory of integral equations have been studied in many papers. In particular, the monograph [1] contains studies of linear integral equations of the second kind and systems of such equations on finite and infinite intervals. In [2], a survey of results on Volterra integral equations of the second kind is given. In [3], the existence of a multiparameter family of solutions was proved for linear Volterra integral equations of the first and third kind with smooth kernels. Lavrent'ev regularizing operators were constructed in [4] for solving linear Fredholm integral equations of the first kind. M.I. Imanaliev [5–7] proved uniqueness theorems and constructed Lavrent'ev regularizing operators for the following systems: systems of linear and nonlinear Volterra integral equations of the first kind with non-smooth matrix kernels [5], systems of nonlinear Volterra integral equations of the third kind [6], and systems of linear Fredholm integral equations of the third kind [7]. In [8], on the basis of a new approach, the existence and uniqueness of a solution of system (1) on a finite interval were investigated.

In [9], using a modification of the approach proposed in [8], one class of Fredholm integral equations of the third kind on a finite interval was studied. On the basis of the approaches proposed in [8] and [9], the papers [10] and [11] developed an improved approach to the study of systems of linear and nonlinear Fredholm integral equations of the third kind on a finite interval with finitely many multipoint singularities. In [12], using a modification of the approach proposed in [10] and [11] uniqueness and existence theorems for a solution of systems of linear Fredholm integral equations of the third kind on the real line with finitely many multipoint singularities are proved.

In this work, regularizing operators in the sense of M.M. Lavrent'ev for one class of linear integral Volterra equations of the third kind are constructed on the semi-axis, and uniqueness theorems are proved.

The following linear integral equations are considered at the same time:

$$m(t)V(t) + \int_{t_0}^{t} a(t)b(s)V(s)ds = f(t), \quad t \in [t_0, \infty), \tag{1}$$

$$(\varepsilon + m(t))V(t,\varepsilon) + \int_{t_0}^{t} a(t)b(s)V(s,\varepsilon)ds = f(t) + \varepsilon V(t_0), \quad t \in [t_0, \infty) \tag{2}$$

where *m(t), a(t)* and *f(t)* are known continuous functions on $[t_0, \infty)$, $m(t_0)=0$, $b(t) \in L_1[t_0, \infty)$, is a known function, $V(t)$ is a solution of equation (1), $\varepsilon > 0$ is a small parameter.

Let us denote by $C[t_0, \infty)$ the space of all bounded continuous functions $V(t)$ defined on $[t_0, \infty)$, denote by $C_{\varphi}^{\gamma}[t_0, \infty)$ the space of all functions from $C[t_0, \infty)$ and satisfying the condition

$|u(t) - u(s)| \leq M|\varphi(t) - \varphi(s)|^{\gamma}$, where $0 < \gamma \leq 1$,

M is a positive constant that depends on $V(t)$, but does not depend on *t* and *s*,

$\varphi(t) = \int_{t_0}^{t} a(s)b(s)ds + m(t)$, $t \in [t_0, \infty)$.

In space $C[t_0, \infty)$ we define the norm $\|u(t)\|_c = \sup_{t \geq t_0} |u(t)|$. Then the space $C[t_0, \infty)$ will be a Banach space.

Let's assume that the following conditions are met:

a) $a(t), f(t) \in C[t_0, \infty)$, $m(t_0)=0$, $m(t) \geq 0$, $a(t) \geq 0$ and $b(t) \geq 0$ at $t \in [t_0, \infty)$, $m(t) \in C[t_0, \infty)$ and is nondecreasing, $b(t) \in L_1[t_0, \infty)$;

b) for $t \geq \tau$, for all $t, \tau \in [t_0, \infty)$ the estimation holds:

$|a(t) - a(\tau)| \leq C_0 \left[\int_{\tau}^{t} a(\tau)\, b(\tau)d\tau + m(t)\right]^{\gamma_1}$, where $0 < \gamma_1 \leq 1$.

Theorem. Let conditions a), b) be satisfied and *V(t)* is the solution of the equation (1) satisfying the conditions $V(t) \in C_{\varphi}^{\gamma}[t_0, \infty)$, $0 < \gamma \leq 1$, $|V(t) - V(t_0)| \leq C_1 a(t)$, $t \in [t_0, \infty)$, where constant $C_1 > 0$. Then the solution *V(t,ε)* of equation (2) converges by the norm $C[t_0, \infty)$ to *V(t)* as $\varepsilon \to 0$.

At the same time, the assessment is valid:

$$\|V(t,\varepsilon)-V(t)\|_{c}\le M(M_1+M_3)\varepsilon^{\gamma}+M_2\varepsilon^{\gamma_1} \qquad (3)$$

where $M=\sup\limits_{t,s\in[t0,\infty)}|V(t)-V(s)|\,/\,|\varphi(t)-\varphi(s)|^{\gamma}$,

$M_1=e\sup\limits_{\nu\ge 0}(\nu^{\gamma}e^{-\nu})$, $M_2=C_0C_1e\int_0^{\infty}e^{-\nu}\,\nu^{\gamma_1}d\nu$, $M_3=e\int_0^{\infty}e^{-\nu}\,\nu^{\gamma}d\nu$.

Proof. In an integral equation of the second kind (2) we substitute

$$V(t,\varepsilon)=V(t)+\xi(t,\varepsilon),\quad t\in[t_0,\infty). \qquad (4)$$

Substituting (4) in (2) we have:

$$[\varepsilon+m(t)]\xi(t,\varepsilon)+\int_{t_0}^{t}a(t)b(s)\xi(s,\varepsilon)ds=-\varepsilon[V(t)-V(t_0)\,],\qquad t\in[t_0,\infty)$$

From this we get

$$\xi(t,\varepsilon)=-\frac{a(t)}{\varepsilon+m(t)}\int_{t_0}^{t}\mathrm{b(s)}\xi(\mathrm{s},\varepsilon)\mathrm{ds}-\frac{\varepsilon[V(t)-V(t_0)]}{\varepsilon+m(t)},\ t\in[t_0,\infty) \qquad (5)$$

Denote $E(t,s,\varepsilon)=exp\left(-\int_s^t\frac{a(\tau)b(\tau)d\tau}{\varepsilon+m(\tau)}\right)$, $E_1(t,\varepsilon)=\exp\left(-\frac{m(t)}{\varepsilon+m(t)}\right)$.

Find the solution of equation (5) using the resolvent

$$R(t,s,\varepsilon)=-\frac{a(t)}{\varepsilon+m(t)}\,b(s)E(t,s,\varepsilon),\qquad (t,s)\in G \qquad (6)$$

the kernel $K(t,s)=-\frac{a(t)}{\varepsilon+m(t)}\,b(s)$, $\quad (t,s)\in G$.

Then we have from (5)

$$\xi(t,\varepsilon)=-\frac{\varepsilon[V(t)-V(t_0)]}{\varepsilon+m(t)}+\frac{a(t)}{\varepsilon+m(t)}\int_{t_0}^{t}\mathrm{b(s)}E(t,s,\varepsilon)\frac{\varepsilon[V(s)-V(t_0)]}{\varepsilon+m(s)}ds,\ t\in[t_0,\infty)\ (7)$$

It is not difficult to show the validity of identity

$$\int_{t_0}^{t}\frac{a(s)b(s)}{\varepsilon+m(t)}E(t,s,\varepsilon)\frac{\varepsilon[V(t)-V(t_0)]}{\varepsilon+m(s)}ds=$$

$$=\frac{\varepsilon[V(t)-V(t_0)]}{\varepsilon+m(t)}-\frac{\varepsilon[V(t)-V(t_0)]}{\varepsilon+m(t)}E(t,t_0,\varepsilon)\,,\quad t\in[t_0,\infty). \qquad (8)$$

Given (8), we get from (7)

$$\xi(t,\varepsilon)=\psi_1(t,\varepsilon)+\psi_2(t,\varepsilon)+\psi_3(t,\varepsilon),\quad t\in[t_0,\infty). \qquad (9)$$

where

$$\psi_1(t,\varepsilon)=-\frac{\varepsilon[V(t)-V(t_0)]}{\varepsilon+m(t)}E(t,t_0,\varepsilon), \qquad (10)$$

$$\psi_2(t,\varepsilon)=\int_{t_0}^{t}\frac{[a(t)-a(s)]}{\varepsilon+m(t)}b(s)E(t,s,\varepsilon)\frac{\varepsilon[V(s)-V(t_0)]}{\varepsilon+m(s)}ds, \qquad (11)$$

$$\psi_3(t,\varepsilon) = -\int_{t_0}^{t} \frac{a(s)b(s)}{\varepsilon+m(t)} E(t,s,\varepsilon) \frac{\varepsilon[V(t)-V(s)]}{\varepsilon+m(s)} ds. \tag{12}$$

By virtue of the condition of the theorem from (10):

$$|\psi_1(t,\varepsilon)| \le \frac{\varepsilon \mathrm{M}}{\varepsilon+m(t)} |\varphi(t)|^{\gamma} e E_1(t,\varepsilon) e^{-\frac{1}{\varepsilon+m(t)}\int_{t_0}^{t} a(\tau)b(\tau)d\tau} =$$

$$= \mathrm{M}\varepsilon^{\gamma} \left(\frac{\varepsilon}{\varepsilon+m(t)}\right)^{1-\gamma} \left[\frac{\varphi(t)}{\varepsilon+m(t)}\right]^{\gamma} \times eE_1(t,\varepsilon) \le \mathrm{M}\varepsilon^{\gamma} e \sup_{\nu\ge 0}(\nu^{\gamma}e^{-\nu}).$$

Hence we have

$$\|\psi_1(t,\varepsilon)\|_{\mathrm{c}} \le \mathrm{M}\mathrm{M}_1\varepsilon^{\gamma}. \tag{13}$$

By virtue of the conditions of the theorem from (11), we get:

$$|\psi_2(t,\varepsilon)| \le \int_{t_0}^{t} \frac{|a(t)-a(s)|}{\varepsilon+m(t)} b(s) E(t,s,\varepsilon) \frac{\varepsilon[V(s)-V(t_0)]}{\varepsilon+m(s)} ds \le$$

$$\le \int_{t_0}^{t} \frac{C_0\left[\int_s^t a(\tau)b(\tau)d\tau + m(t)\right]^{\gamma_1}}{\varepsilon+m(t)} b(s)\, eE_1(t,\varepsilon)E(t,s,\varepsilon) \frac{\varepsilon C_1 a(s)}{\varepsilon+m(s)} ds$$

$$\le C_0C_1\varepsilon^{\gamma_1}\left(\frac{\varepsilon}{\varepsilon+m(t)}\right)^{1-\gamma_1} e\int_{t_0}^{t} E(t,s,\varepsilon)\, E_1(t,\varepsilon)\left[\frac{m(t)}{\varepsilon+m(t)} + \int_s^t \frac{a(\tau)b(\tau)d\tau}{\varepsilon+m(\tau)}\right]^{\gamma_1} \times$$

$$\frac{a(s)b(s)}{\varepsilon+m(s)} ds \le C_0C_1e\varepsilon^{\gamma_1}\int_0^{\infty} e^{-\nu}\,\nu^{\gamma_1}d\nu.$$

Hence we have

$$\|\psi_2(t,\varepsilon)\|_{\mathrm{c}} \le \mathrm{M}_2\varepsilon^{\gamma_1}. \tag{14}$$

By virtue of the conditions of the theorem from (12), we get:

$$|\psi_3(t,\varepsilon)| \le \varepsilon\int_{t_0}^{t} \frac{a(s)b(s)}{\varepsilon+m(t)} e \frac{E(t,s,\varepsilon)E_1(t,\varepsilon)}{\varepsilon+m(s)} \mathrm{M}\left[\int_s^t a(\tau)\, b(\tau)d\tau + m(t)\right]^{\gamma} ds \le$$

$$\le Me\varepsilon^{\gamma}\left(\frac{\varepsilon}{\varepsilon+m(t)}\right)^{1-\gamma}\int_{t_0}^{t} E(t,s,\varepsilon)E_1(t,\varepsilon)\left[\frac{m(t)}{\varepsilon+m(t)} + \int_s^t \frac{a(\tau)b(\tau)d\tau}{\varepsilon+m(\tau)}\right]^{\gamma}$$

$$\times \frac{a(s)b(s)}{\varepsilon+m(s)} ds \le Me\varepsilon^{\gamma}\int_0^{\infty} e^{-\nu}\,\nu^{\gamma}d\nu.$$

Hence we have

$$\|\psi_3(t,\varepsilon)\|_{\mathrm{c}} \le M\mathrm{M}_3\varepsilon^{\gamma}. \tag{15}$$

By virtue of inequalities (13), (14) and (15), a grade of (3) follows from (9). Theorem 1 has been proved.

Corollary 1. Let the conditions a), b), $V(t) \in C_{\varphi}^{\gamma}[t_0, \infty)$, $0<\gamma \leq 1$ и $0 < a_0 \leq a(t)$ for all $t \in [t_0, \infty)$, $a_0 \in R$. Then the solution $V(t, \varepsilon)$ of the equation (2) at $\varepsilon \to 0$ converges by the norm $C[t_0, \infty)$ to $V(t)$ and

$$\|V(t,\varepsilon) - V(t)\|_c \leq M(M_1 + M_3)\varepsilon^{\gamma} + M_2\varepsilon^{\gamma_1} \quad (16)$$

where $M = \sup\limits_{t,s \in [t0,\infty)} |V(t) - V(s)| \,/\, |\varphi(t) - \varphi(s)|^{\gamma}$, $M_1 = e \sup\limits_{\nu \geq 0}(\nu^{\gamma} e^{-\nu})$,

$M_2 = C_0 C_1 e \int_0^{\infty} e^{-\nu}\, \nu^{\gamma_1} d\nu$, $M_3 = e \int_0^{\infty} e^{-\nu}\, \nu^{\gamma} d\nu$, $C_1 = 2\|V(t)\|_c \frac{1}{a_0}$.

Proof. $|V(t) - V(t_0)| \leq 2\|V(t)\|_c \frac{1}{a_0} a_0 \leq C_1 a(t), \qquad t \in [t_0, \infty)$.

Hence, all the conditions of the theorem at $C_1 = 2\|V(t)\|_c \frac{1}{a_0}$ are satisfied; the assessment (16) and Corollary 1 have been proven.

Corollary 2. Let the conditions a), b), $0 < a_0 \leq a(t)$ for $t \in [t_0, \infty)$ and there is such a thing as $t_1 \in [t_0, \infty)$ that $\psi(t) = m(t) + \int_{t_0}^{t} a(s)b(s)ds > 0$ for all $t \in (t_0, t_1)$, $\lim\limits_{t \to t_0} \frac{1}{\psi(t)} \int_{t_0}^{t} b(s)ds = \alpha \in (0, \infty)$.

Then the solution of the equation (1) is unique in space $C_{\varphi}^{\gamma}[t_0, \infty)$.

Proof. Let $V(t) \in C_{\varphi}^{\gamma}[t_0, \infty)$ be a non-zero solution of equation (1) for $f(t) \equiv 0$. Then $m(t)V(t) + \int_{t_0}^{t} a(t)b(s)V(s)ds = 0$, $t \in [t_0, \infty)$. Further

$$m(t)V(t_0) + \int_{t_0}^{t} a(s)b(s)V(t_0)ds + m(t)[V(t) - V(t_0)] +$$

$$+ \int_{t_0}^{t} [a(t) - a(s)]b(s)V(s)ds + \int_{t_0}^{t} a(s)b(s)[V(s) - V(t_0)]ds = 0.$$

Hence, by virtue of conditions a) and b), for $t \in (t_0, t_1)$ we have

$$|V(t_0)| \leq \frac{m(t)}{\psi(t)}[V(t) - V(t_0)] + \left[\sup_{s \in [t_0, t]} |V(s) - V(t_0)|\right] \frac{\int_{t_0}^{t} a(s)b(s)ds}{\psi(t)} +$$

$$+ \sup_{s \in [t_0, t]} |a(t) - a(s)|\, \|V(t)\|_c \times \frac{\int_{t_0}^{t} b(s)ds}{\psi(t)} \leq |V(t) - V(t_0)| +$$

$$\sup_{s \in [t_0, t]} |V(s) - V(t_0)| + \sup_{s \in [t_0, t]} |a(t) - a(s)|\, \|V(t)\|_c \left[\frac{1}{\psi(t)} \int_{t_0}^{t} b(s)ds\right].$$

Passing to the limit at $t \to t_0$, we obtain $V(t_0) = 0$. From (16) we have

$$\|V(t)\|_c = \|V(t,\varepsilon) - V(t)\|_c \to 0 \tag{17}$$

at $\varepsilon \to 0$. Since $V(t,\varepsilon) = 0$ for all $t\in [t_{0,} \infty)$ и $\varepsilon > 0$. From (17) it follows that $V(t) = 0$ for all $t\in [t_{0,} \infty)$.

Example 1. Consider equation of type (1)

$$t\,V(t) + \int_0^t \frac{1}{(1+t)(1+s)^2} V(s)ds + f(t),\ t\in [0, \infty). \tag{18}$$

Here $t_0 = 0$, $m(t)=t$, $a(t)=\frac{1}{1+t}$, $b(t)= \frac{1}{(1+t)^2}$, $t\in [0, \infty)$.

In this case, conditions a) and b) of Theorem are fulfilled. Since at $t \geq \tau$, $t, \tau \in [0, \infty)$ the estimate is valid.

$$|a(t) - a(\tau)| = \frac{t-\tau}{(1+t)(1+\tau)} \leq t \leq m(t) \leq \left[\int_\tau^t a(\tau)b(\tau)\, d\tau + m(t)\right]^{\gamma_1}.$$

Here $\gamma_1 = 1$, $C_0=1$.

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MSC 35M10

BOUNDARY VALUE PROBLEMS FOR A MIXED PARABOLIC-HYPERBOLIC TYPE EQUATION OF THE THIRD ORDER WITH AN INTEGRAL CONDITION OF GLUING

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The existence of a unique solution to a boundary value problem with integral conjugation conditions in a rectangular domain for a mixed parabolic-hyperbolic equation of the 3rd order is proved, when the equation of characteristics has 3 multiple roots (x>0), and has 1 simple and 2 multiple roots (x<0). Using Green's functions (x>0) and Riemann function (x<0), the solution to the problem is equivalently reduced to solving the Volterra integral equation of the 2nd kind with a weak kernel, the solvability of which is proved by the method of successive approximations.

Keywords: Differential equation, third order, multiple characteristic, boundary condition, integral conjugation condition, uniqueness, existence, Green's function, Riemann function, integral equation

Мүнөздүк теңдемеси x>0 болгондо 3 эселүү тамырга, ал эми x<0 болгондо – 1 жөнөкөй жана 2 эселүү тамырга ээ болгон учурдагы 3-тартиптеги аралаш параболалык-гиперболалык типтеги теңдеме үчүн тик бурчтуу аймакта интегралдык жалгаштыруу шарты бар чек аралык маселенин жалгыз чечиминин жашашы далилденген. Маселенин чыгарылышы Гриндин функциясын (x>0) жана Риман функциясын (x<0) колдонуу менен 2-түрдөгү начар ядролуу Вольтерранын интегралдык теңдемесин чыгарууга эквиваленттүү келтирилет, анын чыгарылышы удаалаш жакындаштыруу ыкмасы менен далилденет.

Урунттуу сөздөр: Дифференциалдык теңдеме, үчүнчү тартиптеги, эселик мүнөздөмөлөр, жалгаштыруу шарттар, чектик шарттар, чечимдин жалгыздыгы жана жашашы, Грин функциясы, Риман функциясы, интегралдык теңдемелер.

Доказаны существование единственного решения краевой задачи с интегральными условиями сопряжения в прямоугольной области для уравнения смешанного параболо-гиперболического типа 3-го порядка, когда при x>0 уравнение характеристик имеет 3 кратных корня, а при x<0 имеет 1 простой и 2 кратных корня. Используя функции Грина и функции Римана, решение задачи эквивалентным образом сводится к решению интегрального

уравнения Вольтерра 2-го рода со слабым ядром, разрешимость которого доказывается методом последовательных приближений.

Ключевые слова: Дифференциальное уравнение, третий порядок, кратная характеристика, краевое условие, интегральное условие сопряжения, единственность, существование, функция Грина, функция Римана, интегральные уравнения.

1. Introduction. It is known that the solvability of boundary value problems for equations of mixed type depends on the condition of gluing the desired function and its derivative, performed on the line of change of the type of equations, while the line of change of the type of equations can be either characteristic or non-characteristic for a given equation [1]-[3].

In [4] the Tricomi problem is considered for an equation of mixed elliptic-hyperbolic type in cases, when passing through the line of parabolic degeneration, the solution and its derivative can have a discontinuity of the first kind and on this line satisfy the gluing conditions

$$u(x,-0)=\alpha(x)u(x,-0)+\gamma(x), u_y(x,-0)=\beta(x)u_y(x,+0)+\delta(x),$$

where $\alpha(x), \beta(x), \gamma(x), \delta(x)$ are the given functions.

In [5], in the problem of propagation of electrical oscillations in composite lines, when losses are neglected on a section of a semi-infinite line, and the rest of the line is considered as a cable without leakage, gluing conditions of the form were used

$$u(\ell-0,y)=u(\ell+0,y), u_x(\ell-0,y)=\frac{R}{L}\int_0^y u_x(\ell+0,\eta)d\eta,$$

where L is the self-inductance of the first section of the line, and R is the resistance of the second section.

In [6] the Tricomi problem is considered for an equation of mixed parabolic-hyperbolic type in cases when on this line of change of the equation type the gluing conditions of the form are satisfied

$$u(x,-0)=\lambda(x)u(x,-0), u_y(x,-0)=\mu(x)u_y(x,+0),$$

where $\lambda(x), \mu(x)$ are given functions.

Third-order partial differential equations arise in various fields of science and technology, describing heat exchange in soil, fluid filtration in fractured and layered media. For example, the problem of determining pressure in porous media is reduced to solving a third-order equation of the form [7, p. 194] $\frac{\partial p}{\partial x} - \eta \frac{\partial^3 p}{\partial x^2 \partial t} = \kappa \frac{\partial^2 p}{\partial x^2}$.

The equations of non-stationary filtration of liquid also reduce to a third-order partial differential equation of the form [8, p. 50] $\frac{\partial p}{\partial t} + \lambda_v \frac{\partial^2 p}{\partial t^2} = \kappa \left(\frac{\partial^2 p}{\partial x^2} + \lambda_p \frac{\partial^3 p}{\partial x^2 \partial t} \right)$

In [9], using the Riemann function method, he studied local and nonlocal boundary value problems for a third-order equation of the form

$$\frac{\partial^3 u}{\partial x^2 \partial t} + d(x,t) \frac{\partial u}{\partial t} + \eta(x,t) \frac{\partial^2 u}{\partial x^2} + a(x,t) \frac{\partial u}{\partial x} + b(x,t) u = -q(x,t)$$

In [10] the Riemann function is constructed for a third-order equation with two independent variables of the form

$$\frac{\partial^3 u}{\partial x^2 \partial y} + a(x,y) \frac{\partial^2 u}{\partial x^2} + b(x,y) \frac{\partial^2 u}{\partial x \partial y} + c(x,y) \frac{\partial u}{\partial x} + d(x,y) \frac{\partial u}{\partial y} + e(x,y) u$$
$$= f(x,y)$$

and an explicit form of the representation of the Riemann function was found in a number of special cases of this equation.

Despite the wide application of mixed-type equations in problems of gas dynamics, hydrodynamics and other applied areas, boundary value problems for third-order mixed parabolic-hyperbolic equations with discontinuous and integral gluing conditions remain insufficiently studied, which necessitates further research in this direction.

2. Statement of the problem. Let C^{n+m} be the class of functions whose derivatives are all continuous $\partial^{r+s}/\partial x^r \partial y^s (r = 0,1,\ldots,n; s = 0,1,\ldots,m)$.

In the area $D = \{(x,y): -\ell_1 < x < \ell, 0 < y < h\}$ $(\ell, \ell_1, h > 0)$ consider the equation

$$0 = \begin{cases} L_1 u \equiv u_{xxx} - u_{xy}, (x,y) \in D_1 = D \cap (x > 0) \\ L_2 u \equiv u_{xxy} + a u_x + b u_y + c u, (x,y) \in D_2 = D \cap (x < 0), \end{cases} \qquad (1)$$

where a, b, c are the given functions, and

$$a(x,y), a_x(x,y), b(x,y), b_y(x,y), c(x,y) \in C(\bar{D}_2). \qquad (2)$$

Problem 1. Find in the domain D a function $u(x,y)$ from the class $C(\bar{D}) \cap C^1(D_i) \cap [C^{3+1}(D_1) \cup C^{2+1}(D_2)] (i = 1,2)$ that satisfies equation (1) in the domain D_1

and the boundary conditions

$$u(\ell, y) = \varphi_1(y), 0 \le y \le h, \tag{3}$$

$$u_{xx}(\ell, y) = \varphi_2(y), 0 \le y \le h, \tag{4}$$

$$u(x, 0) = \psi_1(x), 0 \le x \le \ell, \tag{5}$$

satisfying equation (1) in the region D_2 and the boundary conditions

$$u(-\ell_1, y) = \varphi_3(y), 0 \le y \le h, \tag{6}$$

$$u(x, 0) = \psi_2(x), -\ell_1 \le x \le 0, \tag{7}$$

and the gluing condition

$$u(+0, y) = u(-0, y), 0 \le y \le h, , \tag{8}$$

$$u_x(+0, y) = \alpha(y) u_x(-0, y) + \int_0^y \beta(\eta) u_x(-0, \eta) d\eta \,, 0 \le y \le h, \tag{9}$$

where $\varphi_1(y), \varphi_2(y), \varphi_3(y), \psi_1(x), \psi_2(x), \alpha(y), \beta(y)$ are given functions, and

$$\varphi_i(y) \in C^1[0, h] (i = \overline{1,3}), \psi_1(x) \in C^2[0, \ell], \psi_2(x) \in C^2[-\ell_1, 0],$$

$$\alpha(y) \in C^1[0, h], \beta(y) \in C[0, h], \forall y \in [0, h]: \alpha(y) \neq 0, \tag{10}$$

$$\psi_1(0) = \psi_2(0), \psi_1(\ell) = \varphi_1(0), \psi_2'(0) = \alpha(0) \psi_1'(0),$$

$$\psi_1''(\ell) = \varphi_2(0), \psi_2(-\ell_1) = \varphi_3(0). \tag{11}$$

In this paper, the existence and uniqueness of the solution to problem 1 for a third-order mixed parabolic-hyperbolic equation with an integral gluing condition of the form (9) are proved.

Note that if $\forall y \in [0, h]: \alpha(y) \equiv 1, \beta(y) \equiv 0$, then we obtain continuous gluing conditions on the line $x = 0$. Introduce the following notations:

$$u(+0, y) = u(-0, y) = \tau(y), u_x(+0, y) = \nu_1(y), u(-0, y) = \nu_2(y), 0 \le y \le h, \tag{12}$$

where $\tau(y), \nu_1(y), \nu_2(y)$ – functions that are still unknown. Then from condition (9) we have the relation

$$\nu_1(y) = \alpha(y) \nu_2(y) + \int_0^y \beta(\eta) \nu_2(\eta) d\eta \,, 0 \le y \le h. \tag{13}$$

3. The ratio obtained from the area D_1. Consider the following auxiliary

Problem 2. Find a function $u(x, y) \in C(\bar{D}_1) \cap C^{3+1}(D_1)$, that satisfies equation (1) in the domain D_1, conditions (3), (4), (5) and

$$u_x(0, y) = \nu_1(y), 0 \le y \le h. \tag{14}$$

Integrating in the region D_1 equation (1) within the limits from x to ℓ and taking into account condition (4), we obtain the equation

$$u_{xx} - u_y = \omega(y), (x,y) \in D_1, \tag{15}$$

where $\omega(y) = \varphi_2(y) - \varphi_1'(y)$ is a known function.

Then, the solution to problem 2, satisfying equation (15), boundary conditions (3), (5) and (14), has the form

$$u(x,y) = -\int_0^y G(x,y;0,\eta)\nu_1(\eta)d\eta - \int_0^y G_\xi(x,y;\ell,\eta)\varphi_1(\eta)d\eta +$$

$$+\int_0^\ell G(x,y;\xi,0)\psi_1(\xi)d\xi - \int_0^y \omega(\eta)d\eta \int_0^\ell G(x,y;\xi,\eta)d\xi\,, (x,y) \in D_1, \tag{16}$$

where the Green's function can be represented in the form [11]:

$$G(x,y;\xi,\eta) = \frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{n=-\infty}^{n=+\infty}\left[exp\left(-\frac{(x-\xi+4n\ell)^2}{4(y-\eta)}\right) - exp\left(-\frac{(x+\xi+4n\ell)^2}{4(y-\eta)}\right) -\right.$$

$$\left.- exp\left(-\frac{(x-\xi+2(2n+1)\ell)^2}{4(y-\eta)}\right) - exp\left(-\frac{(x+\xi+2(2n+1)\ell)^2}{4(y-\eta)}\right)\right].$$

For $x = 0$ (16) we have relationships between the functions $\tau(y)$ and $\nu_1(y)$:

$$\tau(y) = \int_0^y N_1(y,\eta)\nu_1(\eta)d\eta + \Psi_1(y), \tag{17}$$

Where $N_1(y,\eta) = \frac{1}{\sqrt{\pi(y-\eta)}}\left\{-1 + exp\left(-\frac{\ell^2}{y-\eta}\right) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty}\left[exp\left(-\frac{4n^2\ell^2}{y-\eta}\right) -\right.\right.$

$\left.\left. exp\left(-\frac{(2n+1)^2\ell^2}{y-\eta}\right)\right]\right\},$

$\Psi_1(y) = -\int_0^y G_\xi(0,y;\ell,\eta)\varphi_1(\eta)d\eta + \int_0^\ell G(0,y;\xi,0)\psi_1(\xi)d\xi -$

$\int_0^y \omega(\eta)d\eta \int_0^\ell G(0,y;\xi,\eta)d\xi.$

Eliminating from (13) and (17) $\nu_1(y)$ we have

$$\tau(y) = \int_0^y N_2(y,\eta)\nu_2(\eta)d\eta + \Psi_1(y), \tag{18}$$

where $N_2(y,\eta) = \alpha(\eta)N_1(y,\eta) + \beta(\eta)\int_\eta^y N_1(y,\eta_1)d\eta_1$.

4. Construction of the solution to Problem 1 in the domain D_2. Let us consider the following auxiliary problem.

Problem 3. Find in the domain D_2 a function $u(x,y) \in C(\bar{D}_2) \cap C^{2+1}(D_2)$, that satisfies equation (1), conditions (7) and

$$u(0,y) = \tau(y), 0 \le y \le h, \tag{19}$$

$$u_x(0,y) = \nu_2(y), 0 \le y \le h. \tag{20}$$

We will construct the solution to Problem 3 using the Riemann function method [10], [11]. Let $w(x,y;\xi,\eta)$ be an arbitrary function defined in the domain

$D_2^* = \{(\xi,\eta): x < \xi < 0, 0 < \eta < y\}$ with the border $\partial D_2^* = MP \cup PA \cup AQ \cup QM$, where $MP = \{(\xi,\eta): \xi = x, 0 < \eta < y\}$, $PA = \{(\xi,\eta): x < \xi < \ell, \eta = 0\}$, $AQ = \{(\xi,\eta): \xi = 0, 0 < y < h\}$, $QM = \{(\xi,\eta): x < \xi < 0, \eta = y\}$.

We require that the function $w(x,y;\xi,\eta)$ satisfy the equation with respect to variables (ξ,η):

$$L_2^*(w) \equiv -w_{\xi\eta\xi} - (aw)_\xi - (bw)_\eta + cw = 0 \tag{21}$$

Consider the identity

$$\mathrm{wL_2(u)\text{-}uL_2^*(w)} = [\mathrm{wu_{\xi\eta} + w_{\xi\eta}u + awu}]_\xi\text{-}[\mathrm{w_\xi u_\xi\text{-}bwu}]_\eta. \tag{22}$$

Let $M(x,y)$ be an arbitrary point in a region in D_2. Consider a region $D_2^* = \{(\xi,\eta): x < \xi < 0, 0 < \eta < y\}$ with boundary $\partial D_2^* = MP \cup PA \cup AQ \cup QM$, where $MP = \{(\xi,\eta): \xi = x, 0 < \eta < y\}$, $PA = \{(\xi,\eta): x < \xi < \ell, \eta = 0\}$, $AQ = \{(\xi,\eta): \xi = 0, 0 < y < h\}$, $QM = \{(\xi,\eta): x < \xi < 0, \eta = y\}$.

Integrating identity (21) over the domain D_2^* we have

$$\iint_{D_2^*} [wL_2(u) - uL_2^*(w)]d\xi d\eta = \int_{\partial D_2^*} (w_\xi u_\xi - bwu)d\xi +$$

$$(wu_{\xi\eta} + w_{\xi\eta}u + awu)d\eta. \tag{23}$$

Let $w(x,y;\xi,\eta) \in C(\bar{D}_2^*) \cap C^{2+1}(D_2^*)$ be a solution of equation (21) satisfying the following boundary conditions:

$$w(x,y;\xi,\eta)|_{\xi=x} = 0, 0 \le \eta \le y, \;\; w_\xi(x,y;\xi,\eta)\big|_{\xi=x} = 1, 0 \le \eta \le y,$$

$$w(x,y;\xi,\eta)|_{\eta=y} = \theta(x,y;\xi), x \le \xi \le 0, \tag{24}$$

where $\theta(x,y;\xi)$ is defined as the solution to the Cauchy problem:

$$w_{\xi\xi}(x,y;\xi,y) - b(\xi,y)w(x,y;\xi,y) = 0, \tag{25}$$

$$w(x,y;\xi,y)|_{\xi=x} = 0, w_\xi(x,y;\xi,y)\big|_{\xi=x} = 1. \tag{26}$$

Using the integration method from equation (21), we obtain a Volterra-type integral equation

$$w(x,y;\xi,\eta) = \xi - x - \int_x^\xi (\xi - \xi_1)b(\xi_1,\eta)w(x,y;\xi_1,\eta)d\xi_1 +$$

$$+\int_x^{\xi} d\xi_1 \int_{\eta}^{y} [a(\xi_1,\eta_1) - (\xi-\xi_1)c(\xi_1,\eta_1)] w(x,y;\xi_1,\eta_1) d\eta_1. \qquad (27)$$

which has a unique solution.

Note that if $a(x,y) \equiv b(x,y) \equiv 0, c(x,y) \equiv c$, where $c = const$, then the Riemann function $w(x,y;\xi,\eta)$ is defined explicitly:

$$w(x,y;\xi,\eta) = \sum_{n=0}^{+\infty} \frac{c^n}{n!(2n+1)!} (\xi - x)^{2n+1} (\eta - y)^n = \xi - x + \frac{c}{3!} (\xi - x)^3 (\eta - y) +$$

$$+\frac{c^2}{2!5!} (\xi - x)^5 (\eta - y)^2 + \frac{c^3}{3!7!} (\xi - x)^7 (\eta - y)^3 + \ldots + \frac{c^n}{n!(2n+1)!} (\xi - x)^{2n+1} (\eta - y)^n + \ldots \qquad (28)$$

Integrating identity (23) over the domain $D_2^* = \{(\xi,\eta): x < \xi < 0, 0 < \eta < y\}$ and taking into account the properties of the function $w(x,y;\xi,\eta)$, we obtain the solution to problem 3 in the form:

$$u(x,y) = w_{\xi}(x,y;0,y)\tau(y) + w_{\xi}(x,y;x,0)\psi_2(x) - w_{\xi}(x,y;0,0)\psi_2(0) -$$

$$-w(x,y;0,y)\nu_2(y) + w(x,y;0,0)\psi_2'(0) + \int_0^y w_{\eta}(x,y;0,\eta)\nu_2(\eta) d\eta -$$

$$-\int_0^y [w_{\xi\eta}(x,y;0,\eta) + a(0,\eta) w(x,y;0,\eta)]\tau(\eta)] d\eta. \qquad (29)$$

Using condition (6), from (29), we have

$$\varphi_3(y) = w_{\xi}(-\ell_1,y;0,y)\tau(y) + w_{\xi}(-\ell_1,y;-\ell_1,0)\psi_2(-\ell_1)$$

$$- w_{\xi}(-\ell_1,y;0,0)\psi_2(0) -$$

$$-w(-\ell_1,y;0,y)\nu_2(y) + w(-\ell_1,y;0,0)\psi_2'(0) + \int_0^y w_{\eta}(-\ell_1,y;0,\eta)\nu_2(\eta) d\eta -$$

$$-\int_0^y [w_{\xi\eta}(-\ell_1,y;0,\eta) + a(0,\eta) w(-\ell_1,y;0,\eta)]\tau(\eta)]\, d\eta. \qquad (30)$$

We rewrite equality (30) in the form:

$$w(-\ell_1,y;0,y)\nu_2(y) = w_{\xi}(-\ell_1,y;-\ell_1,y)\tau(y) + \int_0^y B_1(y,\eta)\tau(\eta) d\eta +$$

$$+\int_0^y w_{\eta}(-\ell_1,y;0,\eta)\nu_2(\eta) d\eta + \Phi_1(y), \qquad (31)$$

where $B_1(y,\eta) = -w_{\xi\eta}(-\ell_1,y;0,\eta) - a(0,\eta) w(-\ell_1,y;0,\eta)$,

$$\Phi_1(y) = -\varphi_3(y) + w_{\xi}(-\ell_1,y;-\ell_1,y)\psi_2(-\ell_1) - w_{\xi}(-\ell_1,y;0,0)\psi_2(0)$$

$$+ w(-\ell_1,y;0,0)\psi_2'(0).$$

Lemma 1. If

$$\forall (x,y) \in D_2: b(x,y) \le 0, \tag{32}$$

then

$$\forall y \in [0,h]: w(-\ell_1, y; 0, y) \ge \ell_1. \tag{33}$$

Proof. From equation (27) $x = -\ell_1, \eta = y$ we have

$$w(-\ell_1, y; 0, y) = \ell_1 + \int_{-\ell_1}^{0} \xi_1 b(\xi_1, y) w(-\ell_1, y; \xi_1, y) d\xi_1. \tag{34}$$

If condition (32) is satisfied, from equality (34) we conclude that $\forall y \in [0,h]: w(-\ell_1, y; 0, y) \ge \ell_1$.

If $a(x,y) \equiv b(x,y) \equiv 0, c(x,y) \equiv c = const$, then from (28) we obtain that $w(-\ell_1, y; 0, y) = \ell_1$. Consequently, condition (33) is satisfied. Lemma 1 is proved.

Taking into account inequality (32), we rewrite equation (30) in the form

$$\nu_2(y) = \Phi_2(y) + A_1(y)\tau(y) + \int_0^y B_2(y,\eta)\tau(\eta)d\eta + \int_0^y K_1(y,\eta)\nu_2(\eta)d\eta, \tag{35}$$

where $A_1(y) = \frac{w_\xi(-\ell_1, y; -\ell_1, y)}{w(-\ell_1, y; 0, y)}, B_2(y,\eta) = \frac{B_1(y,\eta)}{w(-\ell_1, y; 0, y)}, K_1(y,\eta) = \frac{w_\eta(-\ell_1, y; 0, \eta)}{w(-\ell_1, y; 0, y)}$, $\Phi_2(y) = \frac{\Phi_1(y)}{w(-\ell_1, y; 0, y)}$. Considering equation (35) as a Volterra integral equation of the second kind with respect to $\nu_2(y)$, we represent its solution through the resolvent $R_1(y,\eta)$ of the kernel $K_1(y,\eta)$ in the form

$$\nu_2(y) = \Phi_3(y) + A_1(y)\tau(y) + \int_0^y B_3(y,\eta)\tau(\eta)d\eta, \tag{36}$$

where $B_3(y,\eta) = B_2(y,\eta) + R_1(y,\eta)A_1(\eta) + \int_\eta^y R_1(y,\eta_1)B_2(\eta_1,\eta)d\eta_1$,

$$\Phi_3(y) = \Phi_2(y) + \int_0^y R_1(y,\eta)\Phi_2(\eta)d\eta.$$

Substituting the value from (18) into (36), we have

$$\nu_2(y) = \Phi(y) + \int_0^y K(y,\eta)\nu_2(\eta)d\eta, \tag{37}$$

where $K(y,\eta) = A_1(y)N_2(y,\eta) + \int_\eta^y B_3(y,\eta_1)N_2(\eta_1,\eta)d\eta_1$, $\Phi(y) = \Phi_3(y) + A_1(y)\Psi_1(y) + \int_0^y B_3(y,\eta)\Psi_1(\eta)d\eta$.

Equation (37) is a Volterra integral equation of the second kind with a weak kernel: $|K(y,\eta)| \le \frac{M}{\sqrt{y-\eta}}$, M is a constant number that admits a unique solution [12].

After determining $\nu_2(y)$, we find sequentially $\nu_1(y)$ using formula (13), then $\tau(y)$ using formula (18). Then, the solution to problem 1 in the region D_1 is determined using formula (16), and in the region D_2 – using formula (28).

Thus, the following is proved

Theorem 1. If conditions (2), (10), (11) and (32) are satisfied, then the solution to Problem 1 in the domain exists and is unique.

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THEORY OF CATEGORIES AND EFFECTS WHILE INVESTIGATION OF ASYMPTOTIC OF SOLUTIONS OF INITIAL VALUE PROBLEMS FOR DYNAMIC SYSTEMS

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Supra, the author proved existence and some properties of “special“ (slowly varying) solutions of initial value problems for scalar delay-differential equations and corresponding systems of difference equations. In the paper such equations are presented as the category *Equa-Func-Volt* and their properties are done as consequences of effects.

Keywords: delay-differential equation, special solution, difference equation, initial value problem, category, effect.

Мурда автор аргументинин кечигүүсү болгон дифференциалдык теңдемелер менен дал келген айырма теңдемелери системалары үчүн баштапкы маселелердин «атайын» чыгарылыштарынын жашоосун жана алардын касиеттерин далилдеди. Бул макалада мындай теңдемелер *Equa-Func-Volt* категориясы, касиеттери эффекттердин натыйжалары катары көрсөтүлөт.

Урунттуу сөздөр: кечигүүчү аргументтүү дифференциалдык теңдеме, атайын чыгарылыш, айырмалык теңдеме, баштапкы маселе, категория, эффект.

Ранее автор доказала существование и наличие свойств «специальных» решений начальных задач для дифференциальных уравнений с запаздыванием аргумента и соответствующих систем разностных уравнений. В статье такие уравнения представлены, как категория *Equa-Func-Volt,* а их свойства – как следствие эффектов.

Ключевые слова: дифференциальное уравнение с запаздывающим аргументом, специальное решение, разностное уравнение, начальная задача, категория, эффект.

1. Introduction

In accordance with the concept being developed in Kyrgyzstan (see [1]), each direction in mathematical research is associated with some effect, and the results obtained, especially the phenomenon, are consequences of it.

By the approach of category theory, objects of study can be represented in various equivalent forms, and equations are classified not according to recording methods, but according to the properties of solutions.

The *Equa-Func-Volt* category of "initial problems for dynamic systems" is considered as a subcategory of the known category of equations for the *Equa-Func* functions, which, in turn, is a subcategory of the category of equations for the *Equa* functions.

It is known that shift operators along trajectories of delay differential equations with smooth coefficients "smooth" solutions.

Formerly, in order to conduct the in-depth study of differential equations with delay, the author proposed the method of splitting the solution space reducing such equations to the systems of operator-difference equations. Using this method, the author assumed new conditions, i.e. the absolute domains for coefficients sufficient for the existence of special (slowly changing) solutions, and proved the presence of approximating and asymptotically approximating properties in them, as well as the asymptotic one-dimensional space of solutions of the initial problems for linear scalar differential equations with small delay of argument and the corresponding difference equation systems (special solutions correspond, to the solutions with a slowly changing first component and a relatively small second component). For the purposes of the single-point representation of the obtained results and other data related to the theory of dynamic systems (the distance between the solution values tends to zero alongside the unlimited increase in argument).

Section 2 introduces the required definitions and denotations.

Section 3 presents some results.

Some results of this paper were delivered at the V Borubaev's Readings, Bishkek, June 2024 and published [7].

2. Denotatons and definitions

Let us denote as follows: $\boldsymbol{N}_0=\{0,1,2,3,\ldots\}$, $\boldsymbol{N}=\{1,2,3,\ldots\}$, $\boldsymbol{R}=(-\infty,\infty)$, $\boldsymbol{R}_+=[0,\infty)$, $\boldsymbol{R}_{++}=(0,\infty)$, $C^{m(k)}D$ is the space of functions $u: D\to\boldsymbol{R}^m$ continuous with the derivatives till the order k;

$\|\{u_1,u_2,...u_q\}\|_1=\|u_1\|+\|u_2\|+...+\|u_q\|$ is the norm in $\boldsymbol{R}^q$,

D is the domain in $\boldsymbol{R}^q$, $0\in D$, $m\in\mathbf{N}$, $k\in\mathbf{N}_0$;

$C^{*m(k)}D$ is the sub-space of functions meeting the condition $u(0)=0\in\boldsymbol{R}^m$.

We will use the categories: *Set* of sets;

Ord of ordered sets (with the subcategory *Ord-Dyn* of ordered sets Λ that have a minimum element (0) but do not have a maximum element (usually $\Lambda=\boldsymbol{R}_+$ or $\Lambda=\mathbf{N}_0$);

Func of functions (with the subcategory *Func-Volt* of functions with domain Λ of the category *Ord-Dyn*);

Equa of equations. This category contains objects as tuples

{Non-empty sets X, Y, predicate P(x) on X, transformation $B:X\to Y$*}.*

If $(\exists x\in X)(P(x)\wedge(y=B(x)))$ then $y\in Y$ is said to be a solution of the equation $\{X, Y, P, B\}$. Particularly, if B is the identity operator I, then we obtain the equation *"P(x)"*only. *Mor(Equa)* are such transformations of tuples $\{X, Y, P, B\}$ that solutions (or their absence) preserve.

We propose

Definition 1. *Ob(Equa-Func-Volt)* are tuples

{ M; Z,Y$\in$*Ob(Func-Volt), predicate Q(z(0)), predicate P(z,t) on* $Z\times\Lambda$, $B:Z\to Y$ $\in$ *Ob(Func)}.*

A predicate Q presents an initial condition. The main property of a predicate $P(z,t)$: $(z_1(s)=z_2(s)\ (0\le s\le t))$ implies $(P(z_1,t)=P(z_2,t))$.

This paper discusses the initial problems only. Assuming that the initial problem *Equa-Func-Volt* always has some solution, then the solution is the only and global one, and, thus, it covers the entire set Λ; so, then the space of solutions of any particular dynamic system with the initial condition φ may be written as the operator $W(t,\varphi):\Lambda\times\Phi\to Z$, with Z being the topological space of the solution values. In the case of $\Lambda=\boldsymbol{R}_+$, let us assume that $W(t,\varphi)$ is continuous with respect to t.

Definition 2. The following relation of equivalence to space Φ is said to be asymptotic equivalence: If Z is a Banach space, then

$$(\varphi_1 \sim \varphi_2) \Leftrightarrow (\lim\{ \|W(t,\varphi_1)-W(t,\varphi_2)\|_Z : t\to\infty\} = 0).$$

Definition 3. Solutions to $W(t,\varphi)$, $\varphi\in\Phi_0$, are *special* solutions, if

S1) $(\forall t_1 \in \Lambda)\ (\forall z_1 \in Z)\ (\exists \varphi_1 \in \Phi_0)(W(t_1,\varphi_1)=z_1);$

S2) $(\exists\lambda > 0)(\forall \varphi_1, \varphi_2 \in \Phi_0)(\forall t_1, t_2 \in \Lambda)$

$$\left(\|W(t_1,\varphi_1) - W(t_1,\varphi_2)\| \geq \|W(t_2,\varphi_1) - W(t_2,\varphi_2)\| exp(-\lambda|t_1 - t_2|)\right).$$

The following effect is known. "Natural" initial value problems for dynamic systems have solutions, unlike boundary value problems and other classes of mathematical problems.

Let's consider a differential equation with one delay h. $\Lambda=\boldsymbol{R}_+$, the required function is defined on $[-h,\infty)$. The functions $p(t)\in C(\Lambda)$, $q(t)\in C([-h,0])$ are given.

$$Q(z)=(\forall t\in[-h,0])\ (z(t)=q(t));\ P(z,t)=(\forall s\leq t)(z'(s)=p(s)z(s-h));B=I \qquad (1)$$

I is the identity operator. This equation is equivalent to the integral equation

$$P_1(z,t)=(\forall s\leq t)(z(s)=q(0)+\textstyle\int_0^s p(v)z(v-h)dv);\ B=I. \qquad (2)$$

3. Morphisms in *Equa-Func-Volt* and results

Denote $w_m(t) = z(t + mh), -h \leq t \leq 0$, $w=\{w_m | m\in N_0\}$,

$S_m\zeta(\cdot)(t) = \zeta(0) + \int_{-h}^{t} p(s + mh + h)\,\zeta(s)ds,$

- integral shift operators along the trajectories of equation (1) by step h. We obtain an equivalent equation (system of difference equations)

$\Lambda_1=\boldsymbol{N_0}$; $Q_1(w_0)=$ "$w_0=q$"; $P(w,m) = (\forall k\leq m)(w_{k+1} = S_k w_k)$,

$B_1(w,t)=w_{[t/h]}(t- t/h).$ (3)

By such way we obtained [6] some results on existence and properties of special solutions extending results [2-5] on equations with small delay.

For equations with a constant delay, a smoothing effect occurs. Its consequence is the phenomenon of "oscillation slowdown":

Theorem. If $p(t)>0$ and the solution to equation (1) has no more than $2n$ zeros inside the interval $[a, a+h]$, then it has no more than $2n$ zeros inside the interval $[a+h, a+2h]$.

In the *Equa-Func-Volt* category, equations (1) and (3) are considered as different forms of the same equation. But for an equation of the form (3) the expression "small delay" is no longer suitable. Therefore, we applied splitting the space *C([-h,0])*. This, accordingly, leads to the splitting of the operator S_m into a matrix operator. We have shown that the condition of "small delay" corresponds to the condition that one element of the matrix is significantly larger than the rest by norm. Since the concept of a matrix is the broadest, we called this property "dominance", and the effect as a whole "domination effect". This made it possible to identify a class of "dominated" processes.

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MSC 54D05

MOTION OF STRETCHED OBJECTS IN CARTESIAN PRODUCTS OF KINEMATICAL SPACES

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Supra algorithms of controlled motion of points in kinematical spaces were implemented. We proposed controlled motion of stretched objects in kinematical spaces. An algorithm of controlled motion of a square on topological torus was proposed. Cartesian products of kinematical spaces and motion in them are considered in the paper.

Key words: axiomatization, topological space, kinematical space, velocity, motion, topological torus.

Мурда кинематикалык мейкиндиктердеги чекиттердин башкарылуучу кыймылынын алгоритмдери ишке ашырылган. Биз кинематикалык мейкиндиктерде созулган объектилердин башкарылуучу кыймылын сунуш кылдык. Топологиялык шакек боюнча чарчынын башкарылуучу кыймылынын алгоритми сунуш кылынган. Макалада кинематикалык мейкиндиктер жана алардын декарттык көбөйтүндүсүндөгү кыймыл каралат.

Урунттуу сөздөр: аксиомалаштыруу, топологиялык мейкиндик, кинематикалык мейкиндик, ылдамдык, кыймылдоо, топологиялык шакек.

Ранее были реализованы алгоритмы управляемого движения точек в кинематических пространствах. Мы предложили управляемое движение растянутых объектов в кинематических пространствах. Был предложен алгоритм управляемого движения квадрата по топологическому тору. В статье рассматривается декартово произведение кинематических пространств и движение в нем.

Ключевые слова: аксиоматизация, топологическое пространство, кинематическое пространство, скорость, движение, топологический тор.

1. Introduction

Definition of a kinematical space M (the metric ρ_K is the minimal time of passing between points) with controlled motion of points was introduced in [1] (see also [2]), see Section 2 below.

We proposed controlled motion of stretched sets [3].

Section 3 contains an algorithm to implement such motion on a topological torus with obstacles [4]. Section 4 considers motion in Cartesian products of kinematical spaces.

2. Definitions for motion with bounded velocity

As a codifying the ancient idea of controlled motion with bounded velocity, the notion of kinematical space was introduced [1].

D e f i n i t i o n 1. A computer program is said to be a **presentation** of a computer kinematical space if:

P1) there is an (infinite) metrical space X of points and a set X_1 of display-presentable points being sufficiently dense in X;

P2) the user can pass from any point x_1 in X_1 to any other point x_2 by a sequence of adjacent points in X_1 by their will;

P3) the minimal time to reach x_2 from x_1 is (approximately) equal of the minimal time to reach x_2 from x_1.

The space X is said to be a **kinematic space**; the space X_1 is said to be a **computer kinematic space**; this minimal time is said to be the **kinematical distance** ρ_X between x_1 and x_2; a sequence of adjacent points is said to be a **route**. Passing to a limit as X_1 tends to X we obtain the following.

There is a set K of **routes**; each route M, in turn, consists of the positive real number T_M (**time** of route) and the function $m_M : [0, T_M] \to X$ (**trajectory** of route);

(K1) For $x_1 \neq x_2 \in X$ there exists such $M \in K$ that $m_M(0) = x_1$ and $m_M(T_M) = x_2$, and the set of values of such T_M is bounded with a positive number below;

(K2) If $M = \{T_M, m_M(t)\} \in K$ then the pair $\{T_M, m_M(T_M - t)\}$ is also a route of K (the reverse motion with same speed is possible); (cf. P3).

(K3) If $M = \{T_M, m_M(t)\} \in K$ and $T^* \in (0, T_M)$ then the pair: T^* and function $m^*(t) = m_M(t) \quad (0 \leq t \leq T^*)$ is also a route of K (one can stop at any desired moment);

(K4) concatenation of routes for three distinct points x_1, x_2, x_3.

If there exists a kinematic consistent with the given metric then the metric space is said to be **kinematizable**.

R e m a r k. Artificial intelligence *chat.deepseek.com* noted that for real control of motion by person any structure (such as local Euclidean one) must be in the space.

Consider the practical task. Let there be a thing and obstacles. It is necessary to move the thing to another place. Is possible? If yes then in what minimal time it can be done?

D e f i n i t i o n 2. There is a family P of connected subsets of the set X (we will call them **passes**); each pass has the positive **length (time)** and a family Q of connected subsets of the set X (we will call them **things**).

(It means that a thing moves along a pass).

The space X is said to be a **generalized kinematic space**.

(G1) For each $x \in p \in P$ there exists such $q \in Q$ that $x \in q$ [a thing can be in each place of a pass].

(G2) For each $x_1 \neq x_2 \in X$ there exists such pass $p \in P$ that $x_1, x_2 \in P$ and the set of lengths of such p is bounded with a positive number below; this infimum is said to be the **generalized kinematical distance** ρ_X between points x_1 and x_2.

(G3) For each $q_1 \neq q_2 \in Q$ there exists such pass $p \in P$ that $q_1, q_2 \in P$ and the set of lengths of such p is bounded with a positive number below; this infimum is said to be the **generalized kinematical distance** ρ_X between things p_1 and p_2.

(G4) If $x_1, x_2 \in p_1$ and $x_2, x_3 \in p_2$ then there exists such pass $p_3 \in P$ that $x_1, x_2, x_3 \in p_3$ and $length(p_3) \leq length(p_1) + length(p_2)$.

If

(G5) For each $x_1 \neq x_2 \in X$ there exists such pass $p_{12} \in P$ that $length(p_{12}) = \rho_X(x_1, x_2)$

then the generalized kinematical space X is said to be **flat** (with respect to P).

If there exists a generalized kinematic (families P and Q) consistent with the given Hausdoff metric for Q then the metric space is said to be K-kinematizable.

D e f i n i t i o n 2. Let a set K^* of routes in the set X be given. If

(K6) for all $x_1 \neq x_2 \in X$ there exists such $M \in K^*$ that $m_M(0) = x_1$ and $m_M(T_M) = x_2$ and

(K7) sums of times of concatenations from x_1 to x_2 is bounded with a positive number below

then the set K^* is a **base of kinematic**.

T h e o r e m 1. The set of routes being concatenations of elements of K^* is a kinematic. It is said to be **generated** by K^*.

R e m a r k. (K1) instead of (K7) is not sufficient to be a basic of kinematic.

E x a m p l e. Let $X=R$ and the only $M(u,v)$ has the time $T_M(u,v)=|u-v|^2$. Then for any $n \in N$ the concatenation from 0 to 1: $M(0,1/n)+M(1/n,2/n)+\ldots+M((n-1)/n,n/n)$ has the sum $n\cdot(1/n)^2=1/n$ which is not bounded below.

T h e o r e m 2. The Cartesian product of several kinematical spaces $\{X_1,K_1\}$, $\{X_2,K_2\},\ldots,\{X_p,K_p\}$ with the following base of kinematics: $\{T \text{ of } M_j;\ x_1\in X_1,\ldots,x_{j-1}\in X_{j-1}, M_j\in K_j, x_{j+1}\in X_{j+1},\ldots,x_p\in X_p\}, j=1..p$ is a kinematical space.

P r o o f. If $(u_1,u_2,\ldots,u_p)\neq(v_1,v_2,\ldots,v_p)$ then $(\exists j)(u_j\neq v_j)$ and any concatenation contains a trajectory between u_j and v_j in X_j.

3. Algorithm of motion of square on topological torus

All numbers are integer.

Given a topological torus obtained by gluing sides of $N\times N$-square: $(N+1)\equiv 0;$ a list of obstacles $X[M], Y[M], M=1..K.$ The initial position of the controlled square Q is $[0;L]\times[0;L], L<M.$ The goal is (U,V) for the point of the controlled square Q with initial position $(0,0)$. All obstacles are out of the initial position of Q.

The commands of control are $C=\{N, E, S, W\}$.

Algorithm.

Define function [gluing]

F(Z):= Z; If Z<0 then F(Z):= Z+N+1; If Z>N then F(Z):= Z-N-1;

Cycle { Cycle { XQ[P,T]:=P; YQ[P,T]:=T | T=1..N}| P=1..N};

[initial position of Q]

Repeat {Input C; DX:=0; DY:=0;

If C=N then DY:= 1; If C=E then DX:=1;

If C=S then {DY:=-1}; If C=W then {DX:=-1};

Cycle { Cycle { XQ1[P,T]:= F(XQ[P,T]+DX);

YQ1[P,T]:=F(YQ[P,T]+DY) | T=1..N}| P=1..N};

[attempt to shift to direction C];

OB:=true;

Cycle { Cycle { Cycle { If (XQ1[P,T]= X[M] and

YQ1[P,T]:= Y[M]) then OB:=false | T=1..N}| P=1..N} | M=1..K};

If OB then Cycle { Cycle { XQ[P,T]:= XQ1[P,T]);

YQ[P,T]:= YQ1[P,T] | T=1..N}| P=1..N} else

Output ("Obstacle!");

If (X[0,0]=U and Y[0,0]=V) then (Output ("Goal!"); Exit) }

4. Motion in Cartesian products of kinematical spaces

We consider a media in conditions of Theorem 2. Motion and controlled motion of a point is obvious. It can be implemented by consecutive motions in each component.

T h e o r e m 3. If a stretched object is the Cartesian product of stretched objects in each component of the Cartesian product of spaces and motions in each component are possible then motion of such stretched object within the Cartesian product is possible.

R e m a r k. In [5] consecutive rotations in each Euclidean R^2 as a subspace of Euclidean R^4 were implemented on computer. The well-known fact that rotation of any 3D-object in R^4 can change its orientation in R^3 was illustrated.

5. Conclusion

We hope that new facts of properties of multidimensional and non-Euclidean spaces will be found by means of controlled motion in kinematical spaces with obstacles.

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MSC 35A25

APPLYING THE METHOD OF ADDITIONAL ARGUMENT TO SOLVE AN INITIAL VALUE PROBLEM FOR A QUASILINEAR DIFFERENTIAL EQUATION OF THE FIRST ORDER

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Кошумча аргумент ыкмасын колдонуу менен баштапкы Коши шарты коюлган квазисызыктуу дифференциалдык теңдеменин чыгарылышын изилдөө.

Макаланын максаты: кошумча аргумент ыкмасын колдонуу менен баштапкы Коши шарты койулган квазисызыктуу дифференциалдык теңдеменин чыгарылышын изилдөөнүн негизинде каралган маселени интегралдык теңдемелер системасына келтирү ыкмасын киргизүү.

Урунттуу сөздөр: кошумча аргумент, баштапкы Коши шарты, квазисызыктуу дифференциалдык теңдеме, интегралдык теңдеме.

Исследование с применением метода дополнительного аргумента квазилинейных дифференциальных уравнений первого порядка с начальным условием Коши.

Цель статьи: С помощью метода дополнительного аргумента свести квазилинейные дифференциальные уравнения первого порядка с начальным условием Коши к системам интегральных уравнений.

Ключевые слова: дополнительный аргумент, начальное условие Коши, квазилинейное дифференциальное уравнение, интегральное уравнение.

Investigation using the method of additional argument of first order quasilinear differential equations with the initial Cauchy condition.

Purpose of the article: Using the method of additional argument, reduce quasilinear differential equations of the first order with the initial Cauchy condition to systems of integral equations.

Keywords: additional argument, initial Cauchy condition, quasilinear differential equation, integral equation.

In this paper we apply the method of additional argument (MAA) developed in Kyrgyzstan [1, 2, 3] to a Cauchy problem for a quasilinear partial differential equation with three arguments. The result was briefly published in [4].

The initial condition

$u(0,x,y)=\varphi(x,y)$ (1)

for the equation

$$u_t'(t,x,y) + f_1\big(t,x,y,u(t,x,y)\big)u_x'(t,x,y) + f_2\big(t,x,y,u(t,x,y)\big)u_y'(t,x,y) = f\big(t,x,y,u(t,x,y)\big), \quad (2)$$

$t\in [0,T]$, $T=const$, $(x,y)\in R^2$.

It is assumed, that the functions $f(t,x,y,u)$, $f_1(t,x,y,u)$, $f_2(t,x,y,u)$ and $\varphi(x,y)$ are continuously differentiable with respect to all their arguments.

Using the MAA, we reduce the problem (1) - (2) to the following system of partial differential equations:

$$\begin{cases} p_{1s}'(s,t,x,y) = f_1(s,p_1(s,t,x,y),p_2(s,t,x,y),w(s,t,x,y)) \\ p_{2s}'(s,t,x,y) = f_2(s,p_1(s,t,x,y),p_2(s,t,x,y),w(s,t,x,y)) \\ w_s'(s,t,x,y) = f(s,p_1(s,t,x,y),p_2(s,t,x,y),w(s,t,x,y)) \end{cases} \quad (3)$$

with additional conditions:

$$\begin{cases} p_1(t,t,x,y) = x; \\ p_2(t,t,x,y) = y \end{cases} \quad (4)$$

$$w(0,t,x,y) = \varphi(p_1(0,t,x,y),p_2(0,t,x,y)). \quad (5)$$

In this paper, we show that the problem (3) - (5) can be reduced to a system of integral equations and prove their equivalence.

Having integrated the first two equations of system (3) with respect to *s* from *s* to *t*, we obtain, taking into account the conditions (4), two integral equations:

$$p_1(s,t,x,y) = x - \int_s^t f_1(v,p_1(v,t,x,y),p_2(v,t,x,y),w(v,t,x,y))dv,$$

$$p_2(s,t,x,y) = y - \int_s^t f_2(v,p_1(v,t,x,y),p_2(v,t,x,y),w(v,t,x,y))dv. \quad (6)$$

We integrate the last equation of system (3) with respect to *s* from *0* to *s* and, taking into account (5), (6), we obtain the integral equation:

$$w(s,t,x,y)=\varphi\begin{pmatrix} x - \int_0^t f_1(v,p_1(v,t,x,y),p_2(v,t,x,y),w(v,t,x,y))dv, \\ y - \int_0^t f_2(v,p_1(v,t,x,y),p_2(v,t,x,y),w(v,t,x,y))dv \end{pmatrix}+$$

$$+\int_0^s f(v,p_1(v,t,x,y),p_2(v,t,x,y),w(v,t,x,y))dv. \quad (7)$$

So, we have got that the system of integral equations (6), (7) follows from the problem (3) - (5).

By direct differentiation with respect to equations (6), (7), we obtain system (3). Substituting $s=t$ in (6), we obtain the conditions (4), and substituting $s=0$, we obtain the conditions (5).

Thus, the Cauchy problem (3) - (5) and the system of integral equations (6), (7) are completely equivalent.

Now let a function $u(t,x,y)$ be a solution to the Cauchy problem (1), (2). Having determined $p_1(s,t,x,y)$, $p_2(s,t,x,y)$ from the system of partial differential equations

$$p_{1s}'(s,t,x,y) = f_1(s, p_1(s,t,x,y), p_2(s,t,x,y), u(s, p_1(s,t,x,y), p_2(s,t,x,y)),$$

$$p_{1s}'(s,t,x,y) = f_2(s, p_1(s,t,x,y), p_2(s,t,x,y), u(s, p_1(s,t,x,y), p_2(s,t,x,y)).$$

With the conditions (4), we obtain that the function $w(s,t,x,y) = u(s, p_1(s,t,x,y), p_2(s,t,x,y))$ will satisfy the last equation of the system (3) and the initial condition.

This proves that each solution $u(t, x, y)$ to the Cauchy problem (1), (2) gives a solution to the Cauchy problem (3) - (5), and hence does a solution to the system of integral equations (6), (7).

We denote:

$$N_\phi = max\left\{\sup_{R^2}|\varphi|, \sup_{R^2}|\partial_x\varphi|, \sup_{R^2}|\partial_y\varphi|\right\},$$

$$N_i = max\left\{\sup_{\Omega}|f_i|, \sup_{\Omega}|\partial_{p_1}f_i|, \sup_{\Omega}|\partial_{p_2}f_i|, \sup_{\Omega}|\partial_w f_i|\right\},\ i=1,2,$$

$$N_f = max\left\{\sup_{\Omega}|f|, \sup_{\Omega}|\partial_{p_1}f|, \sup_{\Omega}|\partial_{p_2}f|, \sup_{\Omega}|\partial_w f|\right\}.$$

Lemma 1. Let the functions $(p_1(s,t,x,y),\ p_2(s,t,x,y),\ w(s,t,x,y)) \in C^{1,1,1,1}(R^4)$ and $\Omega_1=\{(s,t,x,y):\ 0 \le s \le t,\ (x,y)\in R^2\}$, then the solution to the system of integral equations (6), (7) gives the solution to the Cauchy problem (1), (2) for $0 < t < T_0 \le T$, where

$T_0 < 1/\big(N_f + (1+N_\phi)(N_1+N_1)\big) - \delta$, $\delta > 0$ is an arbitrarily small number.

Proof. We denote the differential operator

$$D^* := \frac{\partial^*}{\partial t} + a_1(t,x,y,u)\frac{\partial^*}{\partial x} + a_2(t,x,y,u)\frac{\partial^*}{\partial y},\ \text{where } u(t,x,y)=w(t,t,x,y). \qquad (8)$$

We apply this operator sequentially to all equations of system (6), (7).

We have:

$Dp_1 = f_1(t,x,y,u) - f_1(t,x,y,u) - \int_s^t [\partial_{p_1} a_1 \cdot Dp_1 + \partial_{p_2} a_1 \cdot Dp_2 + \partial_w a_1 \cdot Dw]\, dv,$

$$Dp_1 = -\int_s^t [\partial_{p_1} f_1 \cdot Dp_1 + \partial_{p_2} f_1 \cdot Dp_2 + \partial_w f_1 \cdot Dw]\, dv. \quad (9)$$

Similarly, we get

$$Dp_2 = -\int_s^t [\partial_{p_1} a_2 \cdot Dp_1 + \partial_{p_2} a_2 \cdot Dp_2 + \partial_w a_2 \cdot Dw]\, dv. \quad (10)$$

Applying the introduced differential operator *D** to equation (7) and taking into account (9), (10) we obtain:

$$Dw = -\partial_x \phi[-\int_0^s [\partial_{p_1} f_1 \cdot Dp_1 + \partial_{p_2} f_1 \cdot Dp_2 + \partial_w f_1 \cdot Dw]\, dv] -$$
$$-\partial_y \phi[-\int_0^s [\partial_{p_1} f_2 \cdot Dp_1 + \partial_{p_2} f_2 \cdot Dp_2 + \partial_w f_2 \cdot Dw]\, dv] +$$
$$+\int_s^t [\partial_{p_1} f \cdot Dp_1 + \partial_{p_2} f \cdot Dp_2 + \partial_w f \cdot Dw]\, dv. \quad (11)$$

The constants N_1, N_2, N_f are determined taking into account the fact that the functions p_1, p_2, w included in f_1, f_2, f are known.

Using the introduced denotations, we obtain from (9), (10), (11):

$|Dp_i| \leq tN_1(||Dp_1||_\Omega + ||Dp_2||_\Omega + ||Dw||_\Omega$, $i=1,2$,

$|Dw| \leq tN_\phi(N_1 + N_1)(||Dp_1||_\Omega + ||Dp_2||_\Omega + ||Dw||_\Omega) + tN_f(||Dp_1||_\Omega + ||Dp_2||_\Omega + ||Dw||_\Omega)$.

Since the last inequalities are valid for all $0 \leq s \leq t$, $(x, y) \in R^2$, it follows from them:

$||Dp_i||_\Omega \leq tN_i(||Dp_1||_\Omega + ||Dp_2||_\Omega + ||Dw||_\Omega$, $i=1,2$,

$||Dw||_\Omega \leq tN_\phi(N_1 + N_2)(||Dp_1||_\Omega + ||Dp_2||_\Omega + ||Dw||_\Omega) + tN_f(||Dp_1||_\Omega + ||Dp_2||_\Omega + ||Dw||_\Omega)$.

Adding these inequalities, we get:

$||Dp_1||_\Omega + ||Dp_2||_\Omega + ||Dw||_\Omega \leq t(N_f + (1 + N_\phi)(N_1 + N_2))(||Dp_1||_\Omega + ||Dp_2||_\Omega + ||Dw||_\Omega)$.

Hence it follows that for $t < 1/\big(N_f + (1 + N_\phi)(N_1 + N_1)\big)$ the identity $||Dp_1||_\Omega + ||Dp_2||_\Omega + ||Dw||_\Omega = 0$ is true.

We denote $T_0 = 1/\big(N_f + (1 + N_\phi)(N_1 + N_1)\big) - \delta$, where δ is an arbitrarily small number.

From the above tabs it follows that for $0 < s < T_0 \leq T$ the following equalities are true:

$$p_{it}{}' + f_1(t,x,y,u)p_{1x}{}' + f_2(t,x,y,u)p_{1y}{}' = 0,$$

$$w_t{}' + f_1(t,x,y,u)w_x + f_2(t,x,y,u)w_y = 0. \tag{12}$$

Substitute now $u(t, x, y) = w(t, t, x, y)$ into the equation (2). We will have

$$w_s'(s,t,x,y)|_{s=t} + w_t'(s,t,x,y)|_{s=t} + f_1 w_x'(t,t,x,y) + f_2 w_y'(t,t,x,y) = f.$$

Taking into account the equalities (12) and $w_s'(s,t,x,y)|_{s=t} = f(t,x,y,u)$ we obtain the correct equality $f(t, x, y, u) = f(t, x, y, u)$.

Substituting $s=t=0$ in (7), we obtain $u(0, x, y)=w(0,0, x, y)=\varphi(x, y)$.

Thus, it has been proved that the function $w(s, t, x, y)$ for $s=t$ gives a solution to the initial value problem (1)-(2).

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BOUNDEDNESS OF SOLUTIONS OF A CLASS OF LINEAR DIFFERENTIAL EQUATIONS OF THE THIRD ORDER ON THE SEMIAXIS

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Annotation. In this paper, basing on a formula for one class of third order linear differential equations with variable coefficients, sufficient conditions for the boundedness of solutions for one class of third order linear differential equations on the semi axis were established.

Keywords. Formula, linear differential equations, third order, boundedness, semi axis.

Аннотация. Бул макалада өзгөрүлмө коэффициенттери бар экинчи тартиптеги сызыктуу дифференциалдык теңдемелердин бир классы үчүн формуланын негизинде жарым октогу үчүнчү тартиптеги сызыктуу дифференциалдык теңдемелердин бир классынын чыгарылыштарынын чектелишинин жетиштүү шарттары көрсөтүлгөн.

Урунттуу сөздөр. Формула, сызыктуу дифференциалдык теңдемелер, үчүнчү тартиптеги, чектелгендик, жарым ок.

Аннотация. В данной работе на основе формулы для одного класса линейных дифференциальных уравнений второго порядка с переменными коэффициентами установлены достаточные условия ограниченности решений одного класса линейных дифференциальных уравнений третьего порядка на полуоси.

Ключевые слова. Формула, линейные дифференциальные уравнения, третьего порядка, ограниченность, полуось.

Consider the linear differential equation

$$y''' + a_1(t)y'' + a_2(t)y' + [a_3(t) + q(t)]y = f(t), t \in I = [t_0, \infty), \quad (1)$$

where $a_1(t), a_2(t), a_3(t), q(t), f(t)$ are the known continuous functions on I.

The questions of boundedness and asymptotic stability of solutions on the semiaxis for differential and integro-differential equations were studied in [1-9]. In particular, in [5], the issues of boundedness and quadratic integrability of solutions and their first derivatives of a class of second-order linear integro-differential equations of the Volterra type on a semi-axis are studied. In [7], the issues of asymptotic stability of solutions of a linear homogeneous integro-differential equation of the third order of the Volterra type on a semiaxis are investigated. In [9], based on a formula for one class of second order linear differential equations with variable coefficients were established sufficient conditions of exponential estimate of solutions for one class of second order linear differential equations on the semi axis. In this paper, based on the formula for solving a third-order linear differential equation obtained in [10], sufficient conditions for the boundedness of solutions of third-order linear differential equations (1) on the semiaxis are established. Assume that the following conditions are met:

a) Let the functions $a_1(t), a_2(t), a_3(t)$ are represented as

$$a_1(t) = \alpha(t) + p(t), \quad a_2(t) = p'(t) + \alpha(t)p(t) +$$

$$+a(t)[p(t) - a(t)] - l^2(t) + a'(t), a_3(t) = a''(t) + a'(t)[p(t) -$$

$$-2a(t) + \alpha(t)] + a(t)[p'(t) + \alpha(t)p(t) - \alpha(t)a(t)] -$$

$$-l^2(t)[4a(t) - 2p(t) + \alpha(t)], l(t) = r\, exp\left\{\int_{t_0}^{t}[2a(s) - p(s)]ds\right\}, t \in I, \qquad (2)$$

where $\alpha(t), a(t), a'(t), a''(t), p(t), p'(t), f(t) \in C(I), r \in (0, \infty)$;

$b)$ $\alpha(t) \geq p(t) + \beta,\ a(t) + l(t) \geq 0,\ a(t) - l(t) \geq 0$ for all $t \in I,\ 0 < \beta -$ $const,$

$$|f(t)|e^{\int_{t_0}^{t}\alpha(\nu)d\nu} \leq l_1(t), |q(\tau)|e^{\int_{t_0}^{t}\alpha(\nu)d\nu} \leq l_2(t)$$

for all $t \in I$, where $l_1(t), l_2(t) \in L_1[t_0, \infty), l_1(t) \geq 0$ and $l_2(t) \geq 0$ for all $t \in I.$

Taking into account a) and by virtue of Theorem 2 from [10], from (1) we obtain

$$y(t) = -\int_{t_0}^{t}[\int_{t_0}^{s} e^{-\int_{\tau}^{s}\alpha(\nu)d\nu}\, q(\tau)y(\tau)d\tau](2l(s)y_1(s)y_2(s))^{-1}[y_1(s)y_2(t) -$$

$$y_1(t)y_2(s)]ds + y_4(t) + \sum_{i=1}^{3} c_i\, y_i(t), t \in I. \quad (3)$$

wherec_1, c_2, c_3 are arbitrary constants,

$$y_1(t) = exp\left[-\int_{t_0}^{t}(a(s) + l(s))ds\right], y_2(t) = exp\left[-\int_{t_0}^{t}(a(s) - l(s))ds\right], \quad (4)$$

$$y_3(t) = \int_{t_0}^{t} e^{-\int_{t_0}^{s}\alpha(\nu)d\nu}(2l(s)y_1(s)y_2(s))^{-1}[y_1(s)y_2(t) - y_1(t)y_2(s)]ds =$$

$$= \int_{t_0}^{t}\frac{1}{2r}e^{-\int_{t_0}^{s}[\alpha(\nu)-p(\nu)]d\nu}[y_1(s)y_2(t) - y_1(t)y_2(s)]ds, \quad (5)$$

$$y_4(t) = \int_{t_0}^{t}[\int_{t_0}^{s} e^{-\int_{\tau}^{s}\alpha(\nu)d\nu}\, f(\tau)d\tau](2l(s)y_1(s)y_2(s))^{-1}[y_1(s)y_2(t) -$$

$$y_1(t)y_2(s)]ds\,, t \in I. \quad (6)$$

Применяя формулы Дирихле, из (3) и (6) имеем

$$y(t) = -\int_{t_0}^{t}\int_{\tau}^{t} e^{-\int_{\tau}^{s}\alpha(\nu)d\nu}\,(2l(s)y_1(s)y_2(s))^{-1}[y_1(s)y_2(t) -$$

$$y_1(t)y_2(s)]q(\tau)y(\tau)dsd\tau + y_4(t) + \sum_{i=1}^{3} c_i\, y_i(t), t \in I, \quad (7)$$

$$y_4(t) = \int_{t_0}^{t}\int_{\tau}^{t} e^{-\int_{\tau}^{s}\alpha(\nu)d\nu}\,(2l(s)y_1(s)y_2(s))^{-1}[y_1(s)y_2(t)$$

$$- y_1(t)y_2(s)]f(\tau)dsd\tau =$$

$$= \frac{1}{2r}\int_{t_0}^{t} f(\tau)\int_{\tau}^{t} e^{-\int_{t_0}^{s}[\alpha(\nu)-p(\nu)d\nu}\,[y_1(s)y_2(t) - y_1(t)y_2(s)]dsd\tau. \quad (8)$$

We introduce the notation:

$$F(t) = \sum_{i=1}^{3} c_i\, y_i(t) + y_4(t), t \in I \quad (9)$$

$$K(t,\tau) = -\int_{\tau}^{t} e^{-\int_{\tau}^{s}\alpha(\nu)d\nu}\,(2l(s)y_1(s)y_2(s))^{-1}[y_1(s)y_2(t) -$$

$$y_1(t)y_2(s)]q(\tau)ds =$$

$$= \frac{(-1)}{2r}q(\tau)e^{\int_{t_0}^{\tau}\alpha(\nu)d\nu}\int_{\tau}^{t} e^{-\int_{t_0}^{s}[\alpha(\nu)-p(\nu)]d\nu}\,[y_1(s)y_2(t) - y_1(t)y_2(s)]ds, \quad (10)$$

$$(t,s) \in G = \{(t,s): t_0 \le s \le t < \infty\}.$$

Taking into account notations (9) and (10), we write equation (7) in the form

$$y(t) = F(t) + \int_{t_0}^{t} K(t,\tau)y(\tau)d\tau, \quad t \ge t_0, \quad (11)$$

Theorem. Let conditions a) and b) be satisfied. Then the solution of the differential equation problem (1) satisfies the following estimate:

$$|y(t)| \le c, \quad t \ge t_0, \tag{12}$$

where

$$c = c_4 r\, exp\left[\frac{1}{\beta r}\int_{t_0}^{\infty} l_2(s)ds\right], c_4 = |c_1| + |c_2| + |c_3| + \frac{1}{\beta r}\int_{t_0}^{\infty} l_1(s)ds.$$

Proof. Considering condition b), from (9) and (10) we have:

$$|F(t)| \le c_4, \quad t \ge t_0, \tag{13}$$

$$|K(t,\tau)| \le \frac{l_2(s)}{\beta r}, \quad (t,\tau) \in G. \tag{14}$$

Then, by virtue of (13) and (14) from (11) we obtain:

$$|y(t)| \le c_4 + \frac{1}{\beta r}\int_{t_0}^{t} l_2(s)|y(s)|ds, \quad t \ge t_0. \tag{15}$$

Further, due to the Gronwall-Bellman inequality, from (15) we have the estimate (12). Theorem is proven.

Example. Consider problems the differential equation (1) for $t_0 = 0$, $a_1(t) = 6 + 4t$, $a_2(t) = 5t^2 + 14t + 11$, $a_3(t) = 2t^3 + 8t^2 + 12t + 6$,

$$q(t) = 2\, exp[-t^2 - 6t]\,,\ f(t) = 3\, exp[-t^2 - 5t]\,,\ t \in [0, \infty).$$

That is, consider the following linear differential equation

$$y''' + (6 + 4t)y'' + (5t^2 + 14t + 11)y' + \{2t^3 + 8t^2 + 12t + 6 + 2\, exp[-t^2 - 6t]\}y =$$
$$= 3\, exp[-t^2 - 5t], \quad t \ge 0 \tag{16}$$

In this case, conditions a) and b) are satisfied for

$$a(t) = t + 1,\ p(t) = 2(t + 1),\ r = 1,\ l(t) = 1,\ \alpha(t) = 2t + 4,\ \beta = 2,$$
$$l_1(t) = 3e^{-t},\ \ l_2(t) = 2e^{-2t}, \quad t \in [0, \infty).$$

First, let's check the fulfillment on condition a):

$$a_1(t) = \alpha(t) + p(t) = 2t + 4 + 2(t + 1) = 4t + 6,$$
$$a_2(t) = p'(t) + \alpha(t)p(t) + a(t)[p(t) - a(t)] - l^2(t) + a'(t) =$$
$$= 2 + (2t + 4)2(t + 1) + (t + 1)[2(t + 1) - t - 1] - 1 + 1$$
$$= 5t^2 + 14t + 11,$$

$$a_3(t) = a''(t) + a'(t)[p(t) - 2a(t) + \alpha(t)] + a(t)[p'(t) + \alpha(t)p(t) - \alpha(t)a(t)] -$$
$$-l^2(t)[4a(t) - 2p(t) + \alpha(t)]$$
$$= 2t + 4 + (t+1)[2 + (2t+4)(t+1)] - (2t+4) =$$
$$= (t+1)(2t^2 + 6t + 6) = 2t^3 + 8t^2 + 12t + 6.$$

Thus, the condition a) is fulfilled. Next, we will check the fulfillment of condition b):

$$\alpha(t) = 2t + 4 = p(t) + \beta,\ a(t) + l(t) = t + 2 \geq 0,\ a(t) - l(t) = t \geq 0\ for\ all\ t \in [0,\infty),$$

$$|f(t)|e^{\int_0^t \alpha(\tau)d\tau} = 3e^{-t^2-5t}e^{\int_0^t (2s+4)ds} = 3e^{-t} = l_1(t) \in L_1(0,\infty),$$

$$|q(t)|e^{\int_0^t \alpha(\tau)d\tau} = 2e^{-t^2-6t}e^{\int_0^t (2s+4)ds} = 2e^{-2t} = l_2(t) \in L_1(0,\infty)..$$

Therefore, by virtue of the theorem, for all solutions of the differential equation (16) the estimate (12) is valid, where

$$c = c_4 e^{\frac{1}{2}},\ c_4 = |c_1| + |c_2| + |c_3| + \frac{3}{2}.$$

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AXIOMATIC AND TOPOLOGICAL PROPERTIES OF QUASIVARIETY GENERATED BY CERTAIN FINITE MODULAR LATTICE

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An axiomatizable class of algebras that is closed under subalgebras and direct products is a *quasivariety*. An algebra A is *profinite with respect to class of algebras* K if it is isomorphic to the inverse limit of finite algebras from K. A topological algebra $A_\tau = \langle A, \sigma, \tau\rangle$ is *Boolean* if its topological reduct $\langle A, \tau\rangle$ is a totally disconnected compact Hausdorff space. A finite algebra A equipped with the discrete topology τ generates a topological quasivariety $\boldsymbol{Q}_\tau(A)$ consisting of all topologically closed subalgebras of non-zero direct powers of A, endowed with the product topology. A topological quasivariety $\boldsymbol{Q}_\tau(A)$ is *standard* if every Boolean topological algebra with the algebraic reduct in $\mathbf{Q}(A)$ is profinite with respect to $\mathbf{Q}(A)$. The state “the quasivariety generated by one specific lattice is neither finitely axiomatizable nor standard” is proven in the paper.

Keywords: lattice, quasivariety, finite basis of quasi-identities, standard quasivariety.

Ички алгебраларга жана түз көбөйтүүлүккө карата жабык болгон, аксиомалаштырылуучу алгебралар классы *көп түспөлдүксымал* деп аталат. Алгебра A K алгебралар классына карата *профиниттик* деп аталат, эгерде ал K класстагы чектелген алгебралардын тескери чегине изоморфтук болсо. Топологиялык алгебра $A_\tau = \langle A, \sigma, \tau\rangle$ *булдук* деп аталат, эгер анын топологиялык редукциясы $\langle A, \tau\rangle$ толугу менен ажыратылган компакттуу Хаусдорф мейкиндиги болсо. Дискреттик топология τ менен жабдылган чектелген A алгебрасы топологиялык көп түспөлдүксымалды $\boldsymbol{Q}_\tau(A)$ жаратат, ал A алгебрасынын нөл эмес түз көбөйтүүлөрдү топологиялык жактан жабык ички алгебралардын бардыгынан турат жана көбөйтүлүшүнүн топология менен жабдылган. Топологиялык көп түспөлдүксымал $\boldsymbol{Q}_\tau(A)$ *стандарттуу* деп аталат, эгерде $\mathbf{Q}(A)$ ичиндеги алгебралык редукцияга ээ болгон ар бир булевалык топологиялык алгебра $\mathbf{Q}(A)$ классына карата профиниттик болсо. Макалада белгилүү бир тор тарабынан түзүлгөн көп түспөлдүксымал чектелген аксиомалар менен аныкталбай турганы, жана стандарттуу эместиги далилденген.

Уруунттуу сөздөр: торчо, көп түспөлдүксымал, теңдешсымалдардын чектүү негизи, стандарттуу көп түспөлдүксымал.

Аксиоматизируемый класс алгебр, замкнутый относительно подалгебр и прямых произведений, называется *квазимногообразием*. Алгебра A называется *проконечной относительно класса алгебр* K, если она изоморфна обратному пределу конечных алгебр из K. Топологическая алгебра $A_\tau = \langle A, \sigma, \tau \rangle$ называется *булевой*, если её топологический редукт $\langle A, \tau \rangle$ является вполне несвязным компактным хаусдорфовым пространством. Конечная алгебра A, снабжённая дискретной топологией τ, порождает топологическое квазимногообразие $\boldsymbol{Q}_\tau(A)$, состоящее из всех топологически замкнутых подалгебр ненулевых прямых произведений A, оснащённых топологией произведения. Топологическое квазимногообразие $\boldsymbol{Q}_\tau(A)$ называется *стандартным*, если любая булева топологическая алгебра, алгебраический редукт которой принадлежит $\mathbf{Q}(A)$, является проконечной относительно $\mathbf{Q}(A)$. В статье доказано, что квазимногообразие, порождённое одной конкретной решёткой, не является ни конечно аксиоматизируемым, ни стандартным.

Ключевые слова: решётка, квазимногообразие, конечный базис квазитождеств, стандартное квазимногообразие.

1. Introduction

Two fundamental and closely connected questions arise in universal algebra and lattice theory:

1. Which finite lattices generate finitely axiomatizable quasivarieties?
2. Which finite lattices generate standard topological quasivarieties?

The first question, known as the finite basis problem for quasivarieties of algebras, was posed by V. A. Gorbunov et al [1]. The second, referred to as the standardness problem for quasivarieties of algebras, was suggested by D. M. Clark et al [2].

R. McKenzie [3] showed that every finite lattice has a finite basis of identities. V. P. Belkin [4] constructed the first example of a finite lattice without a finite basis of quasi-identities. This naturally led to the problem formulated by V. A. Gorbunov et al [3] in 1979: Which finite lattices generate finitely axiomatizable quasivarieties? Sufficient condition consisting of two parts under which the locally finite quasivariety of lattices has no finite (independent) basis of quasi-identities was found by V.I.Tumanov [4]. He conjectured that a finite modular lattice has a finite basis of quasi-identities if and only if a quasivariety generated by this lattice is a variety. In general, this conjecture is false. W. Dziobiak [6] provided a counterexample: a finite non-modular lattice generating a finitely based proper quasivariety. Nevertheless, the modular case remains unresolved to this day.

This problem is closely connected to a question raised in [2]: Which finite lattices generate standard topological quasivarieties? In particular, they demonstrated that Belkin's lattice generates to a non-standard topological quasivariety. It is also well-established that every finite lattice generates a standard topological variety — a result originally proved by D. M. Clark et al [7] in 2005. Recentlly, A. M. Nurakunov et al [8] extended this line of research by showing that the lattices constructed by Tumanov generate non-standard topological quasivarieties.

We prove that the quasivariety generated by one specific lattice is neither finitely axiomatizable nor standard.

2. Basic definitions and preliminaries

We start by briefly reviewing some fundamental definitions and results on quasivarieties that will be employed throughout this paper. For further details on the basic concepts of universal algebra and lattice theory referenced here and applied in our work are available in [9,10,11].

A quasivariety is a class K of algebras closed under subalgebras, direct products and ultraproducts. Equivalently, a quasivariety is a class K of algebras which can be axiomatized by quasi-identities. A quasi-identity is a universal Horn sentence with non-empty positive part, that is of the form

$$(\forall \bar{x})[p_1(\bar{x}) \approx q_1(\bar{x}) \wedge \dots \wedge p_n(\bar{x}) \approx q_n(\bar{x}) \rightarrow p(\bar{x}) \approx q(\bar{x})]$$

where $\bar{x} = (x_1, x_2, \dots x_m)$ and $p, q, p_1, q_1, \dots p_n, q_n$ are terms of the signature σ. A class K of algebras is called a variety if it is closed under subalgebras, homomorphic images, and direct products. By Birkhoff theorem, a variety is a class of similar algebras axiomatized by a set of identities, where by identity we mean a sentence of the form $(\forall \bar{x})[s(\bar{x}) \approx t(\bar{x})]$ for some $s(\bar{x})$ and $t(\bar{x})$.

Let $V(K)(Q(K))$ denote the smallest variety (quasivariety) containing a class K. We say that $V(K)(Q(K))$ is the variety (quasivariety) generated by K. If K is a finite family of finite algebras then $Q(K)$ is called finitely generated. If K has a single member A we write simply $Q(A)$.

A triple $\langle A;\ \sigma,\ \tau\rangle$ is called a topological algebra if $\langle A,\ \sigma\rangle$ is an algebra of the signature σ, $\langle A,\ \tau\rangle$ is a topological space and every operation from σ is continuous with respect to topology τ . A topology τ is called Boolean if it is compact, Hausdorff and totally disconnected. A topological algebra $A_\tau = \langle A,\ \sigma,\ \tau\rangle$ is Boolean if it's topology is Boolean. A finite algebra A equipped with the discrete topology τ generates a topological quasivariety $\boldsymbol{Q}_\tau(A)$ consisting of all topologically closed subalgebras of non-zero direct powers of A, endowed with the product topology. A topological quasivariety $\boldsymbol{Q}_\tau(A)$ is standard if every Boolean topological algebra with the algebraic reduct in Q(A) is profinite with respect to Q(A).

A quasivariety K has a finite basis of quasi-identities if there is a finite set Σ of quasi-identities such that $K = Mod(\Sigma) = A|A \vDash \Sigma$. A lattice A has a finite basis of quasi-identities if the quasivariety $Q(A)$ generated by A is finitely based, and if $Q(A)$ is not finitely based then lattice A has no finite basis of quasi-identities.

A congruence θ on an algebra A is trivial if it is the least or the greatest element in the congruence lattice $Con\ A$. A homomorphism $h\colon A \to B$ from algebra A to B is proper if the kernel congruence ker$h$ is not trivial.

We say that X is *pointwise non-separable* with respect to quasivariety R if the following condition holds: there exist $a, b \in X,\ a \neq b$, such that, for each $n \in N$, each finite structure $M \in R$ and each homomorphism $\varphi\colon X_n \to M$, we have $\varphi(a) = \varphi(b)$.

The following lemma and theorem provide criteria for non-finite axiomatizability and non-standardness of quasivarieties, respectively.

Lemma 2.1. A locally finite quasivariety K is not finitely axiomatizable if for any positive integer n there is a finite algebra A_n such that $A_n \notin K$ and every proper subalgebra of A_n belongs to K.

Theorem 2.2. [2] (Second inverse limit technique).

Let $X = \varprojlim\{X_n \mid n \in N\}$ be a surjective inverse limit of finite structures, and let K be a quasivariety. Suppose that $X_n \notin K,\ X \in K$ is pointwise non-separable with respect to K and each substructure of X_n that is generated by at most n elements belongs to K for all $n \in N$. Then K is non-standard.

3. Construction of lattices

Let $M_3, M_{3,3}, M_{3-3}$ be the lattices:

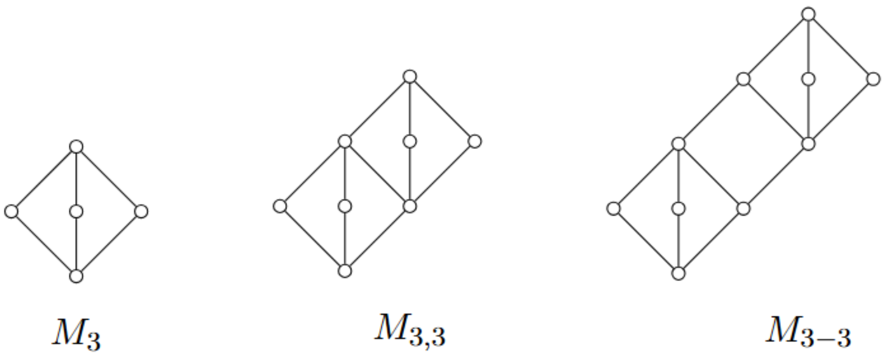

FIGURE 1. Lattices M_3, $M_{3,3}$, M_{3-3}

We construct a modular lattice L_n (see Figure 2) by induction on n:

$n = 1$. $L_1 \cong \mathrm{M}_3$ and $L_1 = \langle a_1, b_1, c_1 \rangle$.

$n > 1$. $L_n = \langle L_{n-1} \cup a_n, b_n, c_n \rangle$ where $b_n = c_{n-1}$ and $c_n, a_n \notin L_{n-1}$, $a_n \wedge c_i = c_n \wedge c_i \notin L_{n-1}$ and $a_n \vee c_i = c_n \vee c_i \notin L_{n-1}$ for all $i < n$.

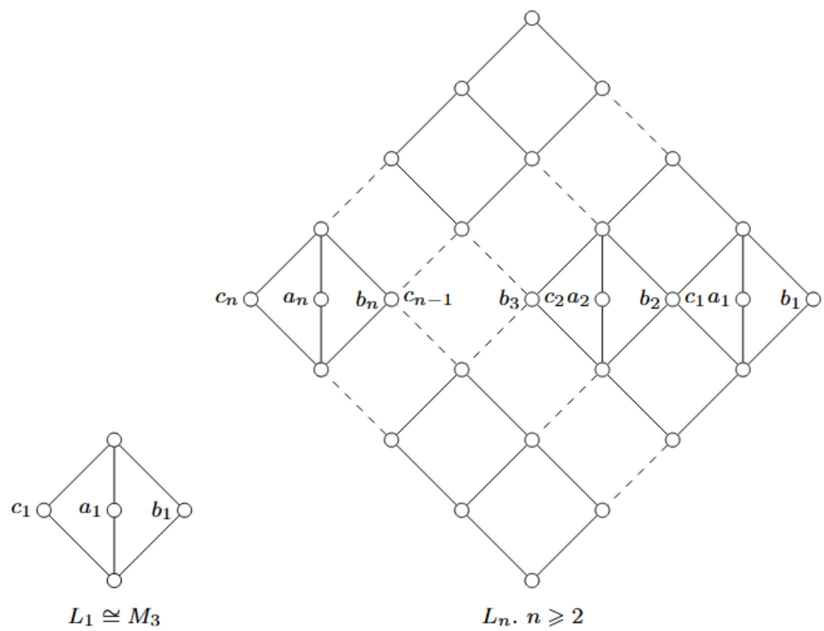

FIGURE 2. Lattices L_1, L_n

We construct a modular lattice A_n as $A_n = \langle T_n \cup a_0, b_0, c_0 \rangle$ where $\langle a_0, b_0, c_0 \rangle \cong M_3$; $a_n \wedge a_2 = c_0 \vee a_0$, $a_n \wedge a_1 = b_0$ and $1_{L_n} = 1_{A_n}$ (see Figure 3).

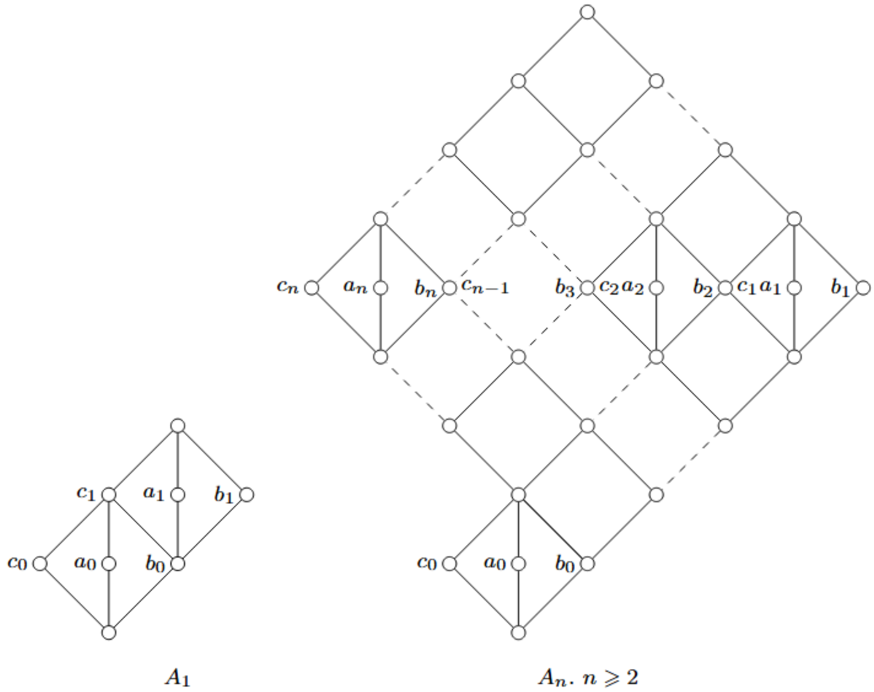

FIGURE 3. Lattices A_1, A_n

4. **Non-finite axiomatizability**

Let A be the lattice

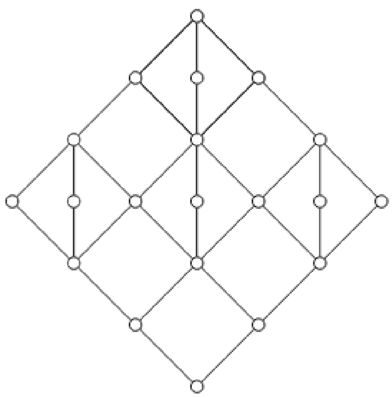

FIGURE 4. Lattice A

Theorem 4.1. Quasivariety $Q(A)$ generated by lattice A is not finitely based.

To prove this theorem, according to Lemma 2.1, it is necessary to construct an infinite set of finite lattices $\{X_i \notin K \mid i > 0\}$ such that any proper sublattice of X_i belongs to K . We will show that the lattices A_n satisfy this property.

Lemma 4.2. For any $n > 1$ and a non-trivial congruence $\theta \in ConA_n$ there is $1 < m < n$ such that $A_n/\theta \cong A_m$ or $A_n/\theta \cong M_{3,3}$ provided $(a_1, b_1) \notin \theta$, otherwise

$A_n/\theta \cong T_m$.

Proof. We prove by induction on $n > 2$. One can check that it is true for $n = 3$ because of $A_3/\theta \cong A_2$ or $A_3/\theta \cong M_{3,3}$ whether $(a_1, b_1) \notin \theta$, otherwise $A_3/\theta \cong A_2$ or $A_3/\theta \cong M_3$ for any non-trivial congruence $\theta \in ConA_3$.

Let $n > 3$ and let u cover v in A_n and $\theta(u, v) \subseteq \theta$. By construction of A_u we have $A_n/\theta(u, v) \cong A_{n-1}$ or $L_n/\theta(u, v) \cong L_{n-1}$.

Assume $(a_1, b_1) \notin \theta$. Since for every non-trivial congruence $\theta \in ConA_n$ there are $(u, v) \in A_n$ such that u covers v and $\theta(u, v) \subseteq \theta$, we get

$$A_n/\theta \cong (A_n/\theta(u, v))/(\theta/\theta(u, v))$$

Since $A_n/\theta(u, v) \cong A_{n-1}$ we obtain

$A_n/\theta \cong \big(A_n/\theta(u, v)\big)/\big(\theta/\theta(u, v)\big) \cong A_{n-1}/\theta'$,

for some $\theta' \in ConA_{n-1}$. And, by induction, $A_{n-1}/\theta' \cong A_m$ or $A_{n-1}/\theta' \cong M_{3,3}$ for some $m > 0$. Thus $A_n/\theta \cong A_m$ or $A_n/\theta' \cong M_{3,3}$.

Now assume $(a_1, b_1) \in \theta$. Then $\theta(a_1, b_1) = \theta(u, v)$ and $A_n/\theta(u, v) \cong L_n$. Hence $A_n/\theta \cong \big(A_n/\theta(u, v)\big)/\big(\theta/\theta(u, v)\big) \cong L_{n-1}/\theta'$,

for some $\theta' \in Con(L_n)$. It is not difficult to check that $L_n/\theta \cong L_m$ for some $m > 0$. Thus $A_n/\theta \cong A_m$ or $A_n/\theta \cong L_m$.

Corollary 4.3. There is no proper homomorphism from A_n to M_{3-3}, A.

Proof. We provide the proof for a homomorphism from A_n into M_{3-3}. It is not difficult to see that there is no homomorphism from A_n into A.

Assume $h: A_n \to M_{3-3}, n > 1$, is a proper homomorphism. Hence ker$h$ is not a trivial congruence on A_n. By Lemma 4.2, $A_n/\text{ker}h \cong A_m$ or $A_n/\theta \cong M_{3,3}$or $A_n/\text{ker}h \cong L_m$ for some $m > 1$. Thus $A_m \cong h(A_n) \leq M_{3-3}$. It is impossible because by definition of $A_m, |A_m| > |M_{3-3}|$ for all $m > 1$, hence A_n is not a sublattice of M_{3-3}. Obviously, $M_{3,3}$ and L_m are not sublattices of M_{3-3}. thus there is no such homomorphism h.

Let S be a non-empty subset of a lattice L. Denote $\langle S \rangle$ the sublattice of L generated by S.

Lemma 4.4. For every $n > 2$, a lattice A_n has the following properties:

i) $A_n \notin Q(A)$;

ii) Every proper subalgebra of A_n belongs to $Q(A)$.

Proof. i). Suppose $A_n \in Q(A)$ for some $n > 1$. Then A_n is a subdirect product of subdirectly $Q(A)$-irreducible algebras. Since every $Q(A)$-irreducible algebra is a subalgebra of A, we get that A_n is a subdirect product of subalgebras of A. By Corollary 4.3, there is no proper homomorphism from A_n onto M_{3-3}, A. Hence $A_n \in Q(M_3)$ for all $n > 1$. It is impossible because $M_{3-3} \leq A_n$ and $M_{3-3} \notin Q(M_3)$.

ii). We prove by induction on n. It is true for $n \leq 2$ by manual checking. Let $n > 2$ and let S be a maximal sublattice of A_n. Since the lattice A_n is generated by

the set of double irreducible elements $a_1, \dots, a_n, a_0, b_1$, there is $0 < i \leq n$ such that $a_i \notin S$ or $b_1 \notin S$ or $a_0 \notin S$.

Suppose $b_1 \notin S$. One can see that $\langle S \rangle \leq_s 2 \times M_3 \times L_{n-1}$. Since $L_{n-1} \leq_s M_3^{n-1}$ we get $\langle S \rangle \in Q(M_3) \subset Q(A)$.

Suppose $a_0 \notin S$. Then $\langle S \rangle \leq_s 2 \times L_n \leq_s 2 \times M_3^n \in Q(M_3) \subset Q(A)$.

Suppose $a_i \notin S$. Since $n > 2$ and S is a maximal sublattice, then there are $i \neq k \neq l$ such that $\theta(b_k, c_k), \theta(b_l, c_l) \in ConA, \theta(b_k, c_k) \cap \theta(b_l, c_l) = \Delta$.

And $A_n/\theta(b_k, c_k) \cong A_n/\theta(b_l, c_l) \cong A_{n-1}$ or $\{A_n/\theta(b_k, c_k), A_n/\theta(b_l, c_l)\} = \{A_{n-1}, L_{n-1}\}$ for $n \geq 3$. We proved for the case when $A_n/\theta(b_k, c_k) \cong A_n/\theta(b_l, c_l) \cong A_{n-1}$. This isomorophisms mean that $A_n \leq_s A_{n-1} \times A_{n-1}$ and $S \leq A_{n-1} \times A_{n-1}$. Let $h_k: A_n \to A_{n-1}$ and $h_l: A_n \to A_{n-1}$ are homomorphisms such that $kerh_k = \theta(b_k, c_k)$ and $kerh_l = \theta(b_l, c_l)$. Since $(a_i, b_i) \notin \theta(b_k, c_k) \cup \theta(b_l, c_l))$ then $h_k(S), h_l(S)$ are proper sublattices of A_{n-1}. By induction $h_k(S), h_l(S) \in Q(A)$. As $b_k, c_k, b_l, c_l \in S$, the restrictions of congruences $\theta(b_k, c_k)|_S$ and $\theta(b_l, c_l)|_S$ are not trivial congruences on S. More $\theta(b_k, c_k)|_S \cap \theta(b_l, c_l)|_S = \Delta_S$. Hence $S \leq_S h_k(S) \times h_l(S)$. It means $S \in Q(A)$. Since every maximal proper subalgebra of A_n belongs to $Q(A)$ then every proper subalgebra of A_n belongs to $Q(A)$. It is easy to check that for $\{A_n/\theta(b_k, c_k), A_n/\theta(b_l, c_l) = \{A_{n-1}, L_{n-1}\}$ the same arguments apply.

Proof of the Theorem. By Lemma 4.4(i), $A_n \notin Q(A)$ for all $n > 1$. By Lemma

4.4(ii), every proper subalgebra of A_n belongs to $Q(A)$. Hence, by Lemma 2.1, $Q(A)$ is not finitely axiomatizable.

5. **Non-standardness**

Theorem 5.1. The lattice A generates non-standard topological quasivariety.

Proof. We use Theorem 2.2. According to this theorem we need to construct $X = \varprojlim\{X_n \mid n \in N\}$ a surjective inverse limit of the finite lattices such that $X_n \notin Q(A)$, every n generated sublattice of X_n belongs to $Q(A)$ and $X \in Q(A)$ is pointwise non-separable with respect to $Q(A)$.

As X_n we take lattice A_n drawn in the Figure 3. By Lemma 4.4 we have:

i) $A_n \notin Q(A)$;

ii) Every proper subalgebra of A_n belongs to $Q(A)$.

Now we construct inverse limit.

Let $\varphi_{n,n-1}$ be a homomorphism from A_n to A_{n-1} such that $\ker \varphi_{n,n-1} = \theta(a_n, b_n)$, and $\varphi_{n,n}$ an identity map for all $n > 1$ and $m < n$. And let $\varphi_{n,m} = \varphi_{m+1,m} \circ \dots \circ \varphi_{n,n-1}$. It easy to verify $\{A_n; \varphi_{n,m}, N\}$ forms an inverse family, where N is the linear ordered set of positive integers. We denote $L = \varprojlim\{A_n \mid n \in N\}$ and show that $L \in Q(A)$. Let α be a quasi-identity of the following form

$$\&_{i \le r} p_i(x_0, \dots, x_{n-1}) \approx q_i(x_0, \dots, x_{n-1}) \to p(x_0, \dots, x_{n-1}) \approx q(x_0, \dots, x_{n-1}).$$

Assume that α is valid on $Q(A)$ and $L \vDash p_i(a_0, \dots, a_{n-1}) = q_i(a_0, \dots, a_{n-1})$ for all $i < r$, for some $a_0, \dots, a_{n-1} \in L$. From the definition of inverse limit we have that $L \le_s \prod_{i \in I} L_i$. Therefore

$$L_s \vDash p_i\big(a_0(s), \dots, a_{n-1}(s)\big) = q_i\big(a_0(s), \dots, a_{n-1}(s)\big) \text{ for all } i < r.$$

Each at most n generated subalgebra of L_s belongs to $Q(A)$ for all $s > n$, by Lemma 3.4. ii) [8]. Hence α is true in L_s for all $s > n$. And this in turn entails $L_s \vDash p\big(a_0(s), \dots, a_{n-1}(s)\big) = q\big(a_0(s), \dots, a_{n-1}(s)\big)$. Since $a_i(m) = \varphi_{s,m}(a_i(s))$ for all $0 \le i < n$ and $m < s$, we get

$$L_m \vDash p\big(a_0(m), \dots, a_{n-1}(m)\big) = q\big(a_0(m), \dots, a_{n-1}(m)\big) \text{ for all } m < s.$$

So $L \vDash p(a_0, \dots, a_{n-1}) = q(a_0, \dots, a_{n-1})$. Hence $L \vDash \alpha$ for every α that is valid on $Q(A)$. This proves that $L \in Q(A)$.

To finish the proof we need to show that L is point-wise separable with respect to $Q(A)$.We know that $\varphi_{n,m}(a_1) = a_1$ and $\varphi_{n,m}(b_1) = b_1$, by definition of $\varphi_{n,n-1}$. And $a = (a_1, \dots, a_1, \dots),\ b = (b_1, \dots, b_1, \dots) \in L$, by definition of inverse limit. Let $\varphi : L \to M$ be a homomorphism, $M \in Q(A)$ and M be finite. There is $n > 2$ and homomorphism $\psi_M : A_n \to M$ such that $\alpha = \varphi_n \circ \psi_M$ for some surjective homomorphism $\varphi_n: L \to A_n$ (by universal property of inverse limit). Since $\psi_M(L_n) \le M \le (A)^k$ for some $k > 0$, by Corollary 4.3 we obtain that $\psi_M(L_n)$ is trivial. That is $\psi_M(x) = 1$ for all $x \in A_n$. So we get $\alpha(a) = \alpha(b)$.

We note that there is an infinite number of lattices similar to the lattice A.

Theorem 5.2. Let L be a finite lattice such that $M_{3,3} \nleq L,\ A \le L$ and $A_n \nleq L$ for all $n > 1$. Then the topological quasivariety generated by the lattice L is not standard.

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SOME PROPERTIES OF HYPERSPACES OF CLOSED AND COMPACT SUBSETS

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The theory of hyperspace of uniform spaces is one of the main directions of uniform topology, intensively developing applications in various areas of mathematics at present. Hyperspaces of uniform spaces are considered in the works of N. Bourbaki, J. Isbell, A. Borubaev and K. Morita. There are different types of hyperspaces of uniform spaces: hyperspaces of closed subsets, hyperspaces of compact subsets and hyperspaces of finite subsets. A uniformity on the set of closed subsets is called a Hausdorff uniformity. A Hausdorff uniformity induced to the Vietoris topology. The study of the relationship between uniform spaces and hyperspaces is relevant. In this paper we study τ *-bounded,* τ *-star, strongly* τ *-star,* u *-uniformly Cech complete,* u *-uniformly plume and uniformly Rothberger properties of hyperspaces of closed and compact subsets of uniform spaces.*

Key words: *uniform space, hyperspace, cover, compact, closed subsets.*

Бир калыптуу мейкиндиктердин гипермейкиндиктеринин теориясы азыркы мезгилде тездик менен өнүгүп жаткан математиканын ар түрдүү областарында колдонулушка ээ болгон бир калыптуу топологиянын негизги багыттарынан болуп саналат. Бир калыптуу мейкиндиктердин гипермейкиндиктери Н. Бурбаки, Дж. Исбелл, А. Бөрүбаев жана К. Мориталардын эмгектеринде изилденген. Бир калыптуу мейкиндиктердин гипермейкиндиктеринин ар кандай түрлөрү бар: туюк камтылган көптүктөрдүн гипермейкиндиги, компактуу камтылган көптүктөрдүн гипермейкиндиги, чектүү камтылган көптүктөрдүн гипермейкиндиги. Туюк камтылган көптүктөрдүн көптүгүндөгү бир калыптуулук Хаусдорфтун бир калыптуулугу деп аталат. Хаусдорфтун бир калыптуулугу Вьетористин топологиясын пайда кылат. Бир калыптуу мейкиндиктер менен алардын гипермейкиндиктеринин арасындагы байланышты изилдөө актуалдуу маселе болуп саналат. Бул илимий макалада бир калыптуу мейкиндиктердин туюк жана компактуу көптүктөрдүн гипермейкиндиктеринин τ *-чектелгендик,* τ *-жылдыздуу, күчтүү* τ *-жылдыздуу, Чех боюнча* u *-бир калыптуу толуктуулук,* u *-бир калыптуу канатчалуулук жана бир калыптуу Ротбергдик касиеттер изилденет.*

Урунттуу сөздөр: *бир калыптуу мейкиндик, гипермейкиндик, жабдуу, компакт, туюк камтылган көптүк.*

Теория гиперпространства равномерных пространств является одним из основных направлений равномерной топологии, интенсивно развивающимся в настоящее время и имеющая приложения в различных областях математики. Гиперпространства равномерных пространств исследованы в работах Н. Бурбаки, Дж. Исбелла, А. Борубаева и К. Морита. Существуют разные виды гиперпространства равномерных пространств: гиперпространства замкнутых подмножеств, гиперпространства компактных подмножеств и гиперпространства конечных подмножеств. Равномерность на множестве замкнутых подмножеств называется равномерностью Хаусдорфа. Хаусдорфова равномерность порождает топологию Вьеториса. Исследование связи между равномерными пространствами и их гиперпространствами является актуальным. В настоящей работе изучаются τ -ограниченные, τ -звездные, сильно τ -звездные, u -равномерно полные по Чеху, u -равномерно перистые и равномерно Ротбергера свойства гиперпространств замкнутых и компактных подмножеств равномерных пространств.

Ключевые слова: *равномерное пространство, гиперпространство, покрытие, компакт, замкнутое подмножество.*

Throughout this work all uniform spaces are assumed to be Hausdorff and mappings are uniformly continuous.

For covers α and β of a set X, we have: $\alpha \wedge \beta = \{A \cap B : A \in \alpha, B \in \beta\}$. $\alpha(x) = \cup St(\alpha, x)$, $St(\alpha, x) = \{A \in \alpha : A \ni x\}$, $x \in X$, $\alpha(H) = \cup St(\alpha, H)$, $St(\alpha, H) = \{A \in \alpha : A \cap H \neq \varnothing\}$, $H \subset X$. For covers α and β of the set X, the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, the symbol $\alpha * \succ \beta$ means that the cover α is a strongly star refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $\alpha(A) \subset B$.

A uniformity on a nonempty set X is a family U of covers of X which satisfies the following conditions:

$(U1)$ if $\alpha \in U$ and if β is a cover of X such that $\alpha \succ \beta$, then $\beta \in U$;

$(U2)$ if $\alpha_1, \alpha_2 \in U$, then there exists $\alpha \in U$ such that $\alpha \succ \alpha_1$ and $\alpha \succ \alpha_2$;

$(U3)$ if $\alpha \in U$, then there exists $\beta \in U$ such that $\beta * \succ \alpha$;

$(U4)$ for any two distinct points $x, y \in X$ there is an $\alpha \in U$ such that no member of α contains both x and y.

The covers from U are called uniform covers, and the pair (X,U) a uniform space.

The topology on X generated by U, denoted $\tau_U = \{O \subset X : \forall x \in O \exists \alpha \in U : \alpha(x) \subset O\}$.

If U and V are two uniformities on a set X and $U \subset V$, then we say that the uniformity V is finer than the uniformity and U. The finest uniformity on the Tychonoff space X, is called the universal uniformity on the space X and is denoted by U_X.

Let $\exp X$ be the set of all non-empty closed subsets of the space (X, τ_U). For each $\alpha \in U$ we put $P(\alpha) = \{\langle \alpha' \rangle : \alpha' \subset \alpha\}$, where $\langle \alpha' \rangle = \{F \in \exp X : F \subset \cup \alpha', F \cap A \neq \varnothing \forall A \in \alpha'\}$. A uniform space $(\exp X, \exp U)$ is called the hyperspace of closed subsets of a uniform space (X,U) and the uniformity $\exp U$ is called Hausdorff uniformity on $\exp X$.

Let $\exp_c X$ be the set of all compact subsets of the space (X, τ_U). For each $\alpha \in U$ we put $K(\alpha) = \{\langle \alpha' \rangle : \alpha' \subset \alpha - \mathit{finite}\}$, where $\langle \alpha' \rangle = \{K \in \exp_c X : K \subset \cup \alpha', K \cap A \neq \varnothing \forall A \in \alpha'\}$. A uniform space $(\exp_c X, \exp_c U)$ is called the hyperspace of compact subsets of a uniform space (X,U).

Let $\exp X$ be the set of all non-empty closed subsets cardinality $\leq n$ of the space (X, τ_U) i.e. $\exp_n X = \{F \in \exp X : |F| \leq n\}$. For each $\alpha \in U$ we put $P(\alpha) = \{\langle \alpha' \rangle : \alpha' \subset \alpha\}$, where $\langle \alpha' \rangle = \{F \in \exp_n X : F \subset \cup \alpha', F \cap A \neq \varnothing \forall A \in \alpha'\}$. A uniform space $(\exp_n X, \exp_n U)$ is called the hyperspace of closed subsets cardinality $\leq n$ of a uniform space (X,U).

A uniform space (X,U) is called:

(i) precompact, if the uniformity U has a base consisting of finite covers [1], [3];

(ii) τ-bounded, if each $\alpha \in U$ has a set $H \subset X$ cardinality $|H| \le \tau$ such that $\alpha(H) = X$ [1], [3];

(iii) pre-Lindelöf or $\aleph_0$-bounded, if the uniformity U has a base consisting of countable cover [1], [3], [14];

(iv) uniformly Rothberger, if for any sequence of uniform covers $\{\alpha_n\}$ of (X,U) there exist $\{A_n\}$ such that for any $n \in N$, $A_n \in \alpha_n$ and $\bigcup_{n \in N} A_n = X$ [14].

A uniformly continuous mapping $f : (X,U) \to (Y,V)$ of a uniform space (X,U) onto a uniform space (Y,V) is called uniformly continuous if for each a uniform cover $\beta \in V$ there exist a uniform cover $\alpha \in U$ such that $f\alpha \succ \beta$ [1].

Let $f : (X,U) \to (Y,V)$ be a uniformly continuous mapping of a uniform space (X,U) onto a uniform space (Y,V). The mapping f is called:

(1) precompact, if for each $\alpha \in U$ there exist a uniform cover $\beta \in V$ and a finite uniform cover $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [1], [3];

(2) uniformly perfect, if it is both precompact and perfect [1], [3].

Let (X,U) be a uniform space.

Lemma 1. If α is a cover of X, then $K(\alpha)$ is a cover of $\exp_c X$.

Proof. Let α be a cover X. We proof that the system $K(\alpha)$ is a cover of $\exp_c X$. Let $K \in \exp_c X$ be an arbitrary point. Then there exist $\alpha' \subset \alpha$ such that $K \cap A_i \ne \varnothing$, $A_i \in \alpha'$ for each $i = 1,2,...,n.$ and $K \subset \cup\alpha'$. Hence, $K \in \langle \alpha' \rangle$, $\langle \alpha' \rangle \in K(\alpha)$. Thus, $K(\alpha)$ is a cover of $\exp_c X$.

Lemma 2. If B is a base of the space (X,U), then $K(B) = \{K(\alpha) : \alpha \in B\}$ forms a base of some uniformity $\exp_c U$ on $\exp_c X$.

Proof. Let α be a uniform cover such that $\alpha* \succ \beta$. We show that $K(\alpha)* \succ K(\beta)$. Let $\langle \alpha_0' \rangle \in K(\alpha)$ be an arbitrary element. For any $A_0' \in \alpha_0'$ there exists a set $B' \in \beta$ such that $\alpha(A_0') \subset B'$. Let $\beta' = \{B'_{A_0'} : A_0' \in \alpha_0'\}$. We show that $K(\alpha)(\langle \alpha_0' \rangle) \subset \langle \beta' \rangle$. Let $K \in K(\alpha)(\langle \alpha_0' \rangle)$ be an arbitrary element. Then there exists an element $\langle\alpha'\rangle \in K(\alpha)$ such that $K \in \langle\alpha_0'\rangle$ and $\langle\alpha_0'\rangle \cap \langle\alpha'\rangle \neq \varnothing$. Let $K \in \langle\alpha_0'\rangle \cap \langle\alpha'\rangle$. Then $K \subset (\cup\alpha) \cap (\cup\alpha_0')$, $K \cap A_0' \neq \varnothing$ for each $A_0' \in \alpha_0'$ and $K \cap A' \neq \varnothing$ for each $A' \in \alpha'$. For each $A' \in \alpha'$ there is $A_0' \in \alpha_0'$ such that $(A' \cap A_0') \cap K \neq \varnothing$. Then $A' \subset \alpha(A_0') \subset B'$ for some $B' \in \beta$. Consequently, $K \in \{\beta'\}$. Hence, $K(\alpha) * \succ K(\beta)$.

Let $\beta \succ \alpha_1 \wedge \alpha_2$, we show that $K(\beta) \succ K(\alpha_1) \wedge K(\alpha_2)$. Let $\langle\beta'\rangle \in K(\beta)$ be an arbitrary element. Since $\beta \succ \alpha_1 \wedge \alpha_2$, then for each $B' \in \beta'$ there exist $A_{1,B'}' \in \alpha_1$ and $A_{2,B'}' \in \alpha_2$ such that $B' \subset A_1'$ and $B' \subset A_2'$. Put $\alpha_i' = \{A_{i,B'}' : B' \in \beta'\}$, $i = 1,2$. We proof that $\langle\beta'\rangle \subset \langle\alpha_0'\rangle \cap \langle\alpha'\rangle$. Let $K \in \{\beta'\}$. Then $K \subset \cup\beta'$ and $K \cap A_0' \neq \varnothing$ for each $B' \in \beta'$. By definition α_i' we have $K \subset \cup\alpha_i'$ and $K \cap A_i' \neq \varnothing$ for any $A_i' \in \alpha_i'$, $i = 1,2$. Consequently, $K \in \langle\alpha_1'\rangle \cap \langle\alpha_2'\rangle$. It is easy to see that $K = \cap\{K(\alpha)(\{x\}) : \alpha \in B\}$ for any $K \in \exp_c X$.

Theorem 1. A uniform space (X,U) is τ-bounded if and only if $(\exp X, \exp U)$ is τ-bounded.

Proof. Let (X,U) be a τ-bounded uniform space and $P(\beta) \in \exp U$ be an arbitrary uniform cover. Then there exists a uniform cover α cardinality $\leq \tau$ such that $\alpha * \succ \beta$. We show that $P(\alpha)$ is a cover and $P(\alpha) \succ P(\beta)$. This follows from Lemmas 1 and 2. Now it suffices to show that the cover $P(\alpha)$ has cardinality $\leq \tau$. Since the cover α has a cardinality $\leq \tau$ and there exists a finite subfamily $\alpha' \subset \alpha$ such that $F \subset \cup\alpha'$ and $F \cap A \neq \varnothing$ for each $A \in \alpha'$ and $F \in \exp X$, the $P(\alpha)$ has cardinality $\leq \tau$. Therefore, $(\exp X, \exp U)$ is τ-bounded. Conversely, if $(\exp X, \exp U)$ is τ-bounded, then the space (X,U) is τ-bounded as a subspace of the τ-bounded space $(\exp X, \exp U)$.

Corollary 1. A uniform space (X,U) is pre-Lindelöf if and only if $(\exp X, \exp U)$ is pre-Lindelöf.

Corollary 2. A uniform space (X,U) is precompact if and only if $(\exp X, \exp U)$ is precompact.

Corollary 3. A uniform space (X,U) is τ-bounded if and only if $(\exp_c X, \exp_c U)$ is τ-bounded.

Corollary 4. A uniform space (X,U) is pre-Lindelöf if and only if $(\exp_c X, \exp_c U)$ is pre-Lindelöf.

Corollary 5. A uniform space (X,U) is precompact if and only if $(\exp_c X, \exp_c U)$ is precompact.

Recall, that a cover α of a uniform space (X,U) is called τ-star cover, if $|St(\alpha,x)| \leq \tau$ for any $x \in X$, where $St(\alpha,x) = \{A \in \alpha : A \ni x\}$. A cover α of a space (X,U) is called strongly τ-star cover, if $|St(\alpha,A)| \leq \tau$ for any $A \in \alpha$, where $St(\alpha,A) = \{A' \in \alpha : A \cap A' \neq \varnothing\}$.

A uniform space (X,U) is called τ-star space, if every uniform cover has a τ-star uniform refinement; a uniform space (X,U) is called strongly τ-star, if every uniform cover has a strongly τ-star uniform refinement.

Theorem 2. A uniform space (X,U) is τ-star if and only if $(\exp_c X, \exp_c U)$ is τ-star.

Proof. Similar to the proof of the Theorem 1.

Theorem 3. A uniform space (X,U) is strongly τ-star if and only if $(\exp_c X, \exp_c U)$ is strongly τ-star.

Proof. Similar to the proof of the Theorem 1.

Theorem 4. For a uniform space (X,U) the following condition is are equivalent:

1. (X,U) is u-uniformly complete Cech space;
2. $(\exp_c X, \exp_c U)$ is u-uniformly co Cech space.

Proof. Let (X,U) be complete and $c - ic(U) \leq \aleph_0$. Then the space (X,U) is perfectly mapped onto some separably metrizable complete space (Y,V), i.e. there exists a perfect uniformly continuous mapping $f:(X,U) \to (Y,V)$ of the uniform space (X,U) onto some complete separably metrizable space (Y,V). It is known that the mapping $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = f(K)$,

$K \in \exp_c X$, is a perfect uniformly continuous mapping “onto”. Then it is easy to see that the uniform space $(\exp_c Y, \exp_c V)$ is complete and separably metrizable. Consequently, $(\exp_c X, \exp_c U)$ is complete and $c - ic(\exp_c U) \leq \aleph_0$.

Theorem 5. For the uniform space (X,U) following conditions are equivalent:

1. (X,U) is u-uniformly plume;
2. $(\exp_c X, \exp_c U)$ - u-uniformly plume.

Theorem 6. For the uniform space (X,U) following conditions are equivalent:

1. (X,U) is uniformly Rothberger;
2. $(\exp_c X, \exp_c U)$ is uniformly Rothberger.

Proof. $1. \Rightarrow 2.$ Let (X,U) be a uniformly Rothberger and $\{K(\alpha_n)\}$, где $K(\alpha_n) = \{\langle \alpha_n' \rangle : \alpha_n' \subset \alpha_n\}, \langle \alpha_n' \rangle = \{K \in \exp X : K \subseteq \cup \alpha_n', K \cap A_n \neq \varnothing, A_n \in \alpha_n'\}$ is a sequence of uniform covers of the hyperspace of compact subsets of $(\exp_c X, \exp_c U)$. Then for the sequence $\{\alpha_n\}$ uniform covers of the space (X,U) there exists a sequence $\{A_n\}$, such that $A_n \in \alpha_n$ and the family $\{A_n\}$ is a cover of the space (X,U). We show that $\{\langle \alpha_n' \rangle\}$, $\langle \alpha_n' \rangle \in K(\alpha_n)$ is a cover of hyperspace of compact subsets $(\exp_c X, \exp_c U)$. Let $K \in \exp_c X$ be an arbitrary compact. Since $\{A_n\}$ is a cover of the space (X,U), then there exist a number $k \leq n$, such that $A_k \cap K \neq \varnothing$. Then there exists $\alpha_k' \subset \alpha_k$ such that $K \subseteq \cup \alpha_k'$ and $K \cap A \neq \varnothing$ for all $A \in \alpha_n'$, $A_k \in \alpha_k'$. Therefore, $K \in \langle \alpha_k' \rangle$. Consequently, $\exp_c X \subset \bigcup_{n \in N} \langle \alpha_n' \rangle$. So $(\exp_c X, \exp_c U)$ is a uniformly Rothberger space.

It is clear that (X,U) is a uniformly Rothberger subspace of the uniformly Rothberger space $(\exp_c X, \exp_c U)$.

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