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BOUNDARY VALUE PROBLEMS FOR A MIXED FOURTH-ORDER PARABOLIC-HYPERBOLIC EQUATION WITH DISCONTINUOUS GLUING CONDITIONS

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The theorem of existence and uniqueness of the solution of the boundary value problem for a mixed parabolic-hyperbolic equation of the 4^{th} order with discontinuous gluing conditions has been proved. By the method of integral equations of the Riemann and Green functions, the solvability of the boundary value problem is reduced to solving the Fredholm integral equation of the second kind obtained on the line of type change. Green's functions used in solving the problem have been visually constructed.

Key words: boundary value problem, boundary conditions, integral equations, uniqueness and existence of the solution, Riemann and Green functions, Fredholm integral equation of the second kind.

Үзгүлтүктүү жабыштыруу шарттары бар аралаш 4-тартиптеги парабола-гиперболалык теңдеме үчүн чек аралык маселенин чечиминин жашашы жана жалгыздыгы теоремасы далилденген. Интегралдык теңдемелер ыкмасын, Риман жана Грин функцияларын колдонуу менен, чек аралык маселенин чечилиши $y=0$ өзгөрүү сызыгында алынган экинчи түрдөгү Фредгольмдун интегралдык теңдемесин чыгарууга алып келинет. Маселени чечүүдө колдонулган Гриндин функциялары айкын түрдө курулган.

Урунттуу сөздөр: чек аралык маселе, чек аралык шарттар, интегралдык теңдемелер, чечимдин жашашы жана жалгыздыгы, Риман жана Грин функциялары, Фредгольмдун экинчи түрдөгү интегралдык теңдемеси.

Доказана теорема существования и единственности решения краевой задачи для смешанного параболо-гиперболического уравнения 4-го порядка с разрывными условиями склеивания. Методом интегральных уравнений, функции Римана и Грина, разрешимость краевой задачи сводится к решению интегрального уравнения Фредгольма второго рода, полученной на линии изменения типа $y=0$. Наглядно построены функции Грина, используемые при решении задачи.

Ключевые слова: краевая задача, краевые условия, интегральные уравнения, единственность и существование решения, функции Римана и Грина, интегральное уравнение Фредгольма второго рода.

Problem statement. In the domain of $D=\{(x,y):0<x<\ell,-h_2<y<h_1\}$ we will consider the equation

$$0=\begin{cases} u_{xxxx}+u_y,(x,y)\in D_1, \\ u_{xxxy}+cu,(x,y)\in D_2, \end{cases} \tag{1}$$

where $D_1=D\cap(y>0),D_2=D\cap(y<0),c=const,h_1,h_2,\ell>0.$

Equation (1) in the domain of D_1 coincides with the equation

$$u_{xxxx}+u_y=0 \tag{2}$$

But in the domain of D_2 – with the equation

$$u_{xxxy}+cu=0. \tag{3}$$

According to the classification of the research paper [1], equation (2) belongs to the parabolic type, since the line $y=0$ is a four-fold real characteristic, and equation (3) belongs to the hyperbolic type, since it has two different real characteristics: a straight line $y=0$ is three-fold, straight $x=0$ is a simple valid characteristic. This implies that the type of equation (1) changes when passing through the lines of $y=0$. Therefore, equation (1) is an equation of mixed parabolic-hyperbolic type.

Problem 1. To find the function of $u(x,y)$, from the class $u\in C(\overline{D_i})\cap C^1(D_i),u_{xxxx}\in C(D_1),\ u_{xxxy}\in C(D_2)$, satisfying equation (1) in the domain of $D\setminus(y=0)$, boundary conditions:

$$u(0,y)=\varphi_1(y),\quad u(\ell,y)=\varphi_2(y),0\le y\le h_1, \tag{4}$$

$$u_x(0,y)=\varphi_3(y),\quad u_x(\ell,y)=\varphi_4(y),0\le y\le h_1, \tag{5}$$

$$u(0,y)=\chi_1(y),\quad u(\ell,y)=\chi_2(y),-h_2\le y\le 0, \tag{6}$$

$$u_x(0,y)=\chi_3(y),\ -h_2\le y\le 0, \tag{7}$$

and pairing conditions

$$u(x,+0)=u(x,-0),0\le x\le \ell, \tag{8}$$

$$u_y(x,+0)=\alpha(x)u_y(x,-0)+\int_0^x \beta(\xi)u_y(\xi,-0)d\xi,\ 0\le x\le \ell, \tag{9}$$

where $\varphi_i(y)\,(i=\overline{1,4}),\ \chi_j(y)\,(j=\overline{1,3}),\ \alpha(x)$ - given functions, and

$$\begin{gathered}\alpha(x),\beta(x)\in C^1[0,\ell],\forall x\in[0,\ell]:\alpha(x)\ne 0,\\ \varphi_i(y)\in C^1[0,h_1]\,(i=\overline{1,4}),\ \chi_j(y)\in C^1[-h_2,0]\,(i=\overline{1,3}),\\ \varphi_1(0)=\chi_1(0),\varphi_2(0)=\chi_2(0),\varphi_3(0)=\chi_3(0),\varphi_1'(0)=\alpha(0)\chi_1'(0).\end{gathered} \tag{10}$$

We will note that the gluing conditions of the form (9) are used by Ya. S. Uflyand [2] when solving the problem of propagation of electrical oscillations in composite lines, when losses are neglected in a limited section of a semi-infinite line, and the rest of the line is considered as a cable without leakage.

Various boundary value problems for equations of mixed and mixed-compound types of the second, third and fourth order have been considered in [3-16].

We will introduce the following notation:

$$u(x,0)=\tau(x),0\le x\le \ell, \tag{11}$$

$$u_y(x,+0)=\nu_1(x),u_y(x,-0)=\nu_2(x),\ 0\le x\le \ell. \tag{12}$$

Then from (8) and (9) we have

$$\nu_1(x)=\alpha(x)\nu_2(x)+\int_0^x \beta(\xi)\nu_2(\xi)d\xi,\ 0\le x\le \ell. \tag{13}$$

Reducing the problem to an integral equation. Passing to the limit at $y\to +0$ from equation (2) we obtain the relation obtained from the domain of D_1:

$$\tau^{(4)}(x)=-\nu_1(x),\ 0<x<\ell. \tag{14}$$

From the boundary conditions (4)-(5) we obtain the following boundary conditions:

$$\tau(0)=\varphi_1(0),\ \tau(\ell)=\varphi_2(0),\ \tau'(0)=\varphi_3(0),\ \tau'(\ell)=\varphi_4(0) \tag{15}$$

First, we find the representation of the solution of the problem (14), (15). To do this, we introduce a new unknown function of $T(x)$:

$$\tau(x)=T(x)+\Phi_1(x), \tag{16}$$

where

$$\Phi_1(x)=\left(\frac{2}{\ell^3}x^3-\frac{3}{\ell^2}x^2+1\right)\varphi_1(0)+\left(-\frac{2}{\ell^3}x^3+\frac{3}{\ell^2}x^2\right)\varphi_2(0)+$$
$$+\left(\frac{x^3}{\ell^2}-\frac{2}{\ell^2}x^2+x\right)\varphi_3(0)+\left(\frac{x^3}{\ell^2}-\frac{2}{\ell^2}\right)\varphi_4(0)$$

Then the problem (14), (15) reduces to the following problem

$$\begin{cases} T^{(4)}(x)=-\nu_1(x),\ 0<x<\ell, \\ T(0)=0,\ T(\ell)=0,\ T'(0)=0,\ T'(\ell)=0. \end{cases}, \tag{17}$$

the solution of which is represented as

$$T(x)=-\int_0^\ell G_1(x,\xi)\nu_1(\xi)d\xi\,, \tag{18}$$

where

$$G_1(x,\xi)=\begin{cases} \dfrac{x^2(\xi-\ell)^2}{6\ell^3}(3\ell\xi-\ell x-2x\xi), & 0\le x<\xi, \\ \dfrac{\xi^2(x-\ell)^2}{6\ell^3}(3\ell x-\ell\xi-2x\xi), & \xi<x\le\ell, \end{cases}$$

– Green’s function of the task (17).

Then from (16) and (18) we will have the relation obtained from the domain of D_1:

$$\tau(x)=\Phi_1(x)-\int_0^\ell G_1(x,\xi)\nu_1(\xi)d\xi\,. \tag{19}$$

On the other hand, from equation (3) at $y\to-0$ we have the relation obtained from the domain of D_2:

$$\nu_2''''(x)=-c\,\tau(x),\ 0<x<\ell\,. \tag{20}$$

To equation (20) we attach the boundary conditions that follow directly from the formulation of problem 1:

$$\nu_2(0)=\chi_1'(0),\ \ \nu_2(\ell)=\chi_2'(0),\nu_2'(0)=\chi_3'(0). \tag{21}$$

Assuming that the right side of equation (20) is known, the solution of problem (20), (21) is represented as:

$$\nu_2(x) = \chi(x) - c\int_0^{\ell} G_2(x,\xi)\tau(\xi)d\xi, \tag{22}$$

where

$$G_2(x,\xi) = \begin{cases} \frac{1}{2\ell^2}\left[x^2(\xi-\ell)^2 - \ell^2(x-\xi)^2\right], 0 \le \xi \le x, \\ \frac{1}{2\ell^2}x^2(\ell-\xi)^2, \, x \le \xi \le \ell, \end{cases}$$ – Green's function,

$$\chi(x) = \chi_1'(0) + \chi_3'(0)x + \frac{x^2}{\ell^2}\left[\chi_2'(0) - \chi_1'(0) - \chi_2'(0)\ell\right].$$

Substituting the value $\nu_2(x)$ from (22) in (13), we have

$$\nu_1(x) = g_1(x) + \int_0^x K_1(x,\xi)\tau(\xi)d\xi, \; 0 \le x \le \ell. \tag{23}$$

where $K_1(x,\xi) = -c\alpha(x)G_2(x,\xi) - c\int_0^x \beta(t)G_2(t,\xi)dt$, $g_1(x) = \alpha(x)\chi(x) + \int_0^x \beta(\xi)\chi(\xi)d\xi$,

Excluding $\nu_1(x)$ from (19) and (23), we come to the Fredholm integral equation of the 2nd kind:

$$\tau(x) = \Phi(x) + \int_0^{\ell} K_2(x,\xi)\tau(\xi)d\xi, \tag{24}$$

where

$$K_2(x,\xi) = \int_{\ell}^{\xi} G_1(x,t)K_1(t,\xi)dt, \; \Phi(x) = \Phi_1(x) - \int_0^{\ell} G_1(x,\xi)g_1(\xi)d\xi.$$

Let $Q = \{(x,\xi): 0 < x < \ell, 0 < \xi < \ell\}$, $N = \max\limits_{\substack{0\le x\le \ell \\ 0\le \xi\le \ell}} |K_2(x,\xi)|$.

If the condition is met

$$N\ell < 1, \tag{25}$$

then equation (24) has a unique solution.

By defining from (24) $\tau(x)$:

$$\tau(x)=\Phi(x)+\int_0^{\ell}R(x,\xi)\Phi(\xi)d\xi\,,$$

where $R(x,\xi)$ – core resolvent $K_2(x,\xi)$. After the definition $\tau(x)$ the solution of Problem 1 is reduced to solving the following tasks.

Problem 2. To find in the domain of D_1 the function of $u(x,y)\in C(\overline{D}_1)\cap C^1(D_1),\ u_{xxxx}\in C(D_1)$, satisfying equation (2), conditions (4), (5) and (11).

Problem 3. To find in the domain of D_2 the function of $u(x,y)\in C(\overline{D}_2)\cap C^1(D_2),\ u_{xxxy}\in C(D_2)$ satisfying equation (3), conditions (6), (7) and (11).

The solution of problem 2 in the domain of D_1 can be represented through Green's functions

$$u(x,y)=\frac{1}{\pi}\int_0^{\ell}G(x,y;\xi,0)\tau(\xi)d\xi+\frac{1}{\pi}\int_0^{y}\Big[G_{\xi\xi\xi}(x,y;0,\eta)\varphi_1(\eta)-$$
$$-G_{\xi\xi\xi}(x,y;\ell,\eta)\varphi_2(\eta)+G_{\xi\xi}(x,y;0,\eta)\varphi_3(\eta)-G_{\xi\xi}(x,y;\ell,\eta)\varphi_4(\eta)\Big]d\eta, \tag{26}$$

where $G(x,y;\xi,\eta)=U(x,y;\xi,\eta)-W(x,y;\xi,\eta)$ – Green's function,

$$U(x,y;\xi,\eta)=\begin{cases}(y-\eta)^{1/4}f(t),\ y>\eta,\ t=\dfrac{x-\xi}{(y-\eta)^{1/4}}\\[2mm] 0,\ y\le\eta,\end{cases}$$ – the fundamental solution of the equation (2), but $f(t)=\int_0^{+\infty}e^{-\lambda^4}\cos\lambda t\,d\lambda$ is the solution of the equation $f'''(t)-\dfrac{t}{4}f(t)=0$, a $W(x,y;\xi,\eta)$ – the regular part of Green's function.

To solve problem 3, we will first consider the following boundary condition:

$$u_{xx}(0,y)=\chi_4(y),-h_2\le y\le 0\,, \tag{27}$$

where $\chi_4(y)$ – unknown function to be further defined. We will consider the following auxiliary task.

Problem 4. To find in the domain of D_2 the function of $u(x,y)\in C(\overline{D}_2)\cap C^1(D_2)$, $u_{xxxy}\in C(D_2)$, satisfying equation (3), the first condition (6), conditions (7), (11) and (27).

By the method of the Riemann function, the solution of problem 4 can be represented as:

$$u(x,y)=\tau(x)+\int_0^y\big[\upsilon(x,y;0,\eta)\chi_4'(\eta)-\upsilon_\xi(x,y;0,\eta)\chi_3'(\eta)+$$
$$+\upsilon_{\xi\xi}(x,y;0,\eta)\chi_1'(\eta)\big]d\eta-\int_0^x\upsilon_{\xi\xi\xi}(x,y;\xi)\tau(\xi)d\xi, \tag{28}$$

where $\upsilon(x,y;\xi,\eta)$ is the Riemann function, defined as the solution of the conjugate problem:

$$\upsilon_{\xi\xi\xi\eta}+c\upsilon=0,$$
$$\upsilon(x,y;\xi,\eta)\big|_{\xi=x}=0,\ \upsilon_\xi(x,y;\xi,\eta)\big|_{\xi=x}=0,$$
$$\upsilon_{\xi\xi}(x,y;\xi,\eta)\big|_{\xi=x}=1,\ \upsilon(x,y;\xi,\eta)\big|_{\eta=y}=\omega(x,y;\xi),$$

and $\omega(x,y;\xi)$ is solving the Cauchy problem

$$\omega_{\xi\xi\xi}(x,y;\xi)=0,$$
$$\omega(x,y;\xi)\big|_{\xi=x}=0,\ \omega_\xi(x,y;\xi)\big|_{\xi=x}=0,\ \omega_{\xi\xi}(x,y;\xi)\big|_{\xi=x}=1,$$

Assuming $x=\ell$ in (28), we have

$$\int_0^y\upsilon(\ell,y;0,\eta)\chi_4'(\eta)d\eta=g_2(y), \tag{29}$$

where

$$g_2(y)=\chi_2(y)-\tau(y)+\int_0^y\big[\upsilon_\xi(\ell,y;0,\eta)\chi_3'(\eta)+$$
$$+\upsilon_{\xi\xi}(\ell,y;0,\eta)\chi_1'(\eta)\big]d\eta+\int_0^\ell\upsilon_{\xi\xi\xi}(\ell,y;\xi,0)\tau(\xi)d\xi.$$

By integrating in parts, in (29) we obtain

$$\chi_4(y)=\int_0^y K_3(y,\eta)\chi_4(\eta)d\eta+g_3(y), \tag{30}$$

where $K_3(y,\eta)=\frac{2}{\ell^2}\upsilon_\xi(\ell,y;0,\eta)$, $g_3(y)=\frac{2}{\ell^2}g_2(y)+\tau''(0)$.

Due to the properties of the Riemann function of $\upsilon(x,y,\xi,\eta)$ equation (30) is a Volterra integral equation of the second kind, admitting a unique solution in the class of $C^1[-h_2,0]$.

By defining from (30) the function of $\chi_4(y)$ and substituting it in (28), we get the solution of Problem 3.

Thus, it is proved

Theorem 1. If conditions (10) and (25) are satisfied, then the solution of Problem 1 exists, unique and representable in D_1 in the form of (26), and in D_2 the form of (28).

REFERENCES

1. Djuraev, T. D., Sopuev A. On the theory of partial differential equations of the fourth order. – Tashkent: Fan, 2000. - 144 p.
2. Uflyand, Ya. S. On the question of the propagation of vibrations in composite electrical lines. – Engineering and Physics Journal, 1964. Vol. VII, No. 1. - pp. 89-92.
3. Bitsadze, A. V. Some classes of partial differential equations. – M.: Nauka, 1981. - 448 p.
4. Salakhitdinov M. S. Mixed-compound type equations. – Tashkent: Fan, 1974. - 156 p.
5. Djuraev, T. D. Boundary value problems for equations of mixed and mixed-composite types. [Text] / T. D. Djuraev – Tashkent: Fan, 1979. – 240 p.
6. Djuraev, T. D. Boundary value problems for parabolic-hyperbolic equations. [Text] / T.D. Dzhuraev, A. Sopuev, M. Mamazhanov – Tashkent: Fan, 1986. – 220 p.
7. Salakhitdinov M. S., Urinov A. K. On the spectral theory of mixed-type equations. – Tashkent: Fan, 2010. - 356 p.

8. Nakhushev, A. M. Displacement problems for partial differential equations. – M.: Nauka, 2006. - 287 p.
9. Kalmenov, T. S. On regular boundary value problems and the spectrum for hyperbolic and mixed type equations. Abstract of the dissertation of the doctor of physical and mathematical sciences. - Moscow: MSU, 1982.
10. Zhegalov, V. I. Mironov, A.N. Differential equations with higher partial derivatives. Kazan Mathematical Society. – 2001. – 226 p.
11. Sabitov, K. B. On the theory of mixed–type equations. – M.: FIZMATLIT, 2014. - 304 p.
12. Sabitov, K. B. Direct and inverse problems for equations of mixed parabolic-hyperbolic type. – M.: Nauka, 2016. – 272 p.
13. Sopuev, A., Zheenbaev, N. A. Boundary value problem for a mixed parabolic-hyperbolic equation of the fourth order with junior terms // Science, new technologies and innovations of Kyrgyzstan. – 2017. – №5. – pp. 116-120. http://www.science-journal.kg/media/Papers/nntiik/2017/5/nntiik-2017-N5-116-120.pdf
14. Abdumitalip uulu, K. Boundary value problems for a fourth-order equation containing a parabolic-hyperbolic operator [Text] / K. Abdumitalip uulu // Bulletin of Osh State University. – No. 1. – Osh, 2022. – pp. 20-28.
15. Apakov, Yu. P. On some non-local problems for a parabolic-hyperbolic equation in three-dimensional space // Direct and inverse boundary value problems of mathematical physics. - Tashkent, Fan, 1987. - pp. 80-95.
16. Apakov, Yu. P., Khamitov, A. A. On solving one boundary value problem for a third-order equation with multiple characteristics in three-dimensional space // Scientific Bulletin of Namangan State University. – Namangan, 2020. - No. 4. – pp. 21-31.

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MSC 34K20, 45J05

ON THE STRUCTURE OF SOLUTIONS OF THE CAUCHY PROBLEM OF PARTIAL DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER

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The solvability of the initial problem and the structure of solutions for nonlinear partial differential equations are studied.

Keywords: solution transformation method, nonlinear partial differential differential equations, Cauchy problem, contraction mapping principle, Lipschitz conditions.

Сызыктуу эмес жекече туундулуу дифференциалдык теңдемелерге коюлган Коши маселесинин чыгарымдуулугу жана чыгарылыштарынын түзүлүшү изилденди.

Урунттуу сөздөр: чыгарылышты өзгөртүп түзүү усулу, сызыктуу эмес жекече туундулуу дифференциалдык теңдемелер, Коши маселеси, кысып чагылтуу принциби, Липшиц шарттары.

Исследованы разрешимость начальной задачи и структура решений для нелинейных дифференциальных уравнений в частных производных.

Ключевые слова: метод преобразования решений, нелинейные дифференциальные уравнения в частных производных, задача Коши, принцип сжатых отображений, условия Липшица.

1. Introduction

Differential equations in partial derivatives, which allow mathematical representation of processes, play an important role in mathematics and its applications. Until now, the problem of finding out the solvability of the Cauchy problem for differential equations in partial derivatives remains a little-studied area. The main fundamental results were obtained in [1-3].

In [4-7], the solvability and structure of solutions of the Cauchy problem for partial integro-differential equations were found, where an analytical method for constructing solutions of the classical Cauchy problem was proposed for them. The essence of the proposed method is to transform the solutions of the original Cauchy problem into an equivalent Volterra integral equation, to which the principle of contracted mappings is applicable.

In [8] researches to problems of existence and uniqueness of solutions to initial problems for linear Volterra integro-differential equations of convolution type in Banach spaces.

In works [9-12], various questions of the solvability of the Cauchy problem for integro-differential, partial differential equations, and systems of equations were studied using the method of transforming solutions. The essence of this approach is to transform the original Cauchy problem into an integral equation equivalent to it, to which one can apply the topological method - the principle of compressed mappings.

In this section, we study the solvability of the Cauchy problem and the structure of solutions for non-linear partial differential equations

$$u_{xxxt}+u_{xxx}+3\beta u_{xxt}+u_{ttx}+3\beta u_{xx}+\left(3\beta^2+2\alpha+1\right)u_{xt}+\beta(\beta^2+2)u_t+\left[\left(3\beta^2+\alpha\right)+\right.$$
$$\left.+(\alpha^2+1)\right]u_x+\beta\left[\ \left(\beta^2+\alpha+1\right)+1\right]u=f(t,x,u(t,x)), \tag{1}$$

with initial condition

$$u(0,x)=\varphi(x), \tag{2}$$

$$u_t(0,x)=\psi(x), \tag{3}$$

where $\alpha,\beta\in R_+$ $\alpha,\ \beta$, $f(t,x,u(t,x))$ are known continuous functions; $\varphi(x),\psi(x)$ is a known continuous function uniformly bounded together with its derivatives included in (1).

The solution of the Cauchy problem (1)-(3) will be sought in the form

$$u(t,x)=c(t,x)+\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-\nu)}\left[(t-s)-\cos(x-\nu)\right]Q(\nu,s)d\nu ds\,, \tag{4}$$

where $c(t,x)\in\overline{C}^{(2,3)}([0,T]\times R)$ is a known continuous function, while

$c(0,x)=\varphi(x), c_t(0,x)=\psi(x)$, $Q(t,x)$ is a new unknown function that needs to be determined [4, 5].

Let

$$H\left(t,c(t,x)\right)=c_{xxxt}\left(t,x\right)+c_{xxx}\left(t,x\right)+c_{ttx}+3\beta c_{xxt}\left(t,x\right)+\beta(\beta^2+2)c_t\left(t,x\right)+$$
$$+\left(3\beta^2+2\alpha+1\right)c_{xt}\left(t,x\right)+3\beta c_{xx}\left(t,x\right)+\left[\left(3\beta^2+\alpha\right)+\alpha^2+1\right]c_x\left(t,x\right)+\left[\beta\left(\beta^2+\alpha+1\right)+1\right]c\left(t,x\right).$$

Assumption (A). Let

$$f(t,x,u)\in\bar{C}\left(\left[0,T\right]\times R\times R\right)\cap Lip\left(L|_u\right),$$

$$H(t,x)\in\bar{C}^{(2,3)}([0,T]\times R),$$

$$\frac{2L}{\alpha\beta}+\frac{\alpha-1}{\alpha}\left(1-e^{-\alpha T_0}\right)+\frac{3}{\beta}<\frac{1}{2}.$$

To determine the function $Q(t,x)$, it is necessary to substitute (4) into (1). For this purpose, taking into account (4), we successively calculate the following relations.

We get

$$u_x(t,x)=c_x(t,x)-\beta\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-\nu)}\left[(t-s)-\cos\left(x-\nu\right)\right]Q(\nu,s)d\nu ds+$$
$$+\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-\nu)}\sin\left(x-\nu\right)Q(\nu,s)d\nu ds,$$

hence, taking into account (4), we have

$$u_x=c_x(t,x)-\beta(u-c)+\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-\nu)}\sin\left(x-\nu\right)Q(\nu,s)d\nu ds. \tag{5}$$

Next, we have

$$u_t(t,x)=c_t(t,x)-\alpha\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-\nu)}\left[(t-s)-\cos\left(x-\nu\right)\right]Q(\nu,s)d\nu ds+$$
$$+\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-\nu)}Q(\nu,s)d\nu ds+\int_{-\infty}^x e^{-\beta(x-\nu)}\left[-\cos\left(x-\nu\right)\right]Q(\nu,t)d\nu.$$

Or

$$u_t(t,x)=c_t(t,x)-\alpha(u-c)+$$
$$+\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-\nu)}Q(\nu,s)d\nu ds+\int_{-\infty}^x e^{-\beta(x-\nu)}\left[-\cos(x-\nu)\right]Q(\nu,t)d\nu,$$
$$u_{tt}+\alpha u_t=c_{tt}+\alpha c_t+\int_{-\infty}^x e^{-\beta(x-\nu)}Q(\nu,t)d\nu,$$
$$u_{ttx}+\alpha u_{tx}=c_{ttx}+\alpha c_{tx}x-\beta\int_{-\infty}^x e^{-\beta(x-\nu)}Q(\nu,t)d\nu+Q(x,t). \tag{6}$$

Differentiating both parts of relation (5) with respect to t, we obtain

$$u_{xt}=c_{xt}-\beta(u_t-c_t)+\int_{-\infty}^x e^{-\beta(x-\nu)}\sin(x-\nu)Q(\nu,t)d\nu-$$
$$-\alpha\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-\nu)}\sin(x-\nu)Q(\nu,s)d\nu ds.$$

Hence, taking into account (5), we have

$$u_{xt}=c_{xt}-\beta(u_t-c_t)-\alpha\left[u_x-c_x+\beta(u-c)\right]+\int_{-\infty}^x e^{-\beta(x-\nu)}\sin(x-\nu)Q(\nu,t)d\nu\ . \tag{7}$$

It is known that there is always $u_{xt}=u_{tx}$. Differentiating both parts of relation (6) with respect to x, we obtain

$$u_{xx}=c_{xx}-\beta(u_x-c_x)-\beta\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-\nu)}\sin(x-\nu)Q(\nu,s)d\nu ds+$$
$$+\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-\nu)}\cos(x-\nu)Q(\nu,s)d\nu ds.$$

Or, taking into account (5), we have

$$u_{xx}=c_{xx}-\beta(u_x-c_x)-\beta\left[u_x-c_x+\beta(u-c)\right]+$$
$$+\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-\nu)}\cos(x-\nu)Q(\nu,s)d\nu ds. \tag{8}$$

Further, differentiating (8) with respect to x, we obtain

$$u_{xxx}=c_{xxx}-\beta(u_{xx}-c_{xx})-\beta\left[u_{xx}-c_{xx}+\beta(u_x-c_x)\right]+$$
$$+\int_0^t e^{-\alpha(t-s)}Q(s,x)ds-\beta\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-\nu)}\cos(x-\nu)Q(\nu,s)d\nu ds-$$
$$-\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-\nu)}\sin(x-\nu)Q(\nu,s)d\nu ds. \tag{9}$$

Taking into account (5), (8) from (9) we have

$$u_{xxx}=c_{xxx}-\beta(u_{xx}-c_{xx})-\beta\left[u_{xx}-c_{xx}+\beta(u_x-c_x)\right]+\int_0^t e^{-\alpha(t-s)}Q(s,x)ds-$$
$$-\beta\left\{u_{xx}-c_{xx}+\beta(u_x-c_x)+\beta\left[u_x-c_x+\beta(u-c)\right]\right\}-\left[u_x-c_x+\beta(u-c)\right]. \quad (10)$$

Differentiating both parts of (10) with respect to t, we obtain

$$u_{xxxt}=c_{xxxt}-\beta(u_{xxt}-c_{xxt})-\beta\left[u_{xxt}-c_{xxt}+\beta(u_{xt}-c_{xt})\right]+$$
$$+Q(t,x)-\alpha\int_0^t e^{-\alpha(t-s)}Q(s,x)ds-$$
$$-\beta\left\{u_{xxt}-c_{xxt}+\beta(u_{xt}-c_{xt})+\beta\left[u_{xt}-c_{xt}+\beta(u_t-c_t)\right]\right\}-\left[u_{xt}-c_{xt}+\beta(u_t-c_t)\right]. \quad (11)$$

From (11), grouping identical terms, we have

$$u_{xxxt}+3\beta u_{xxt}+3\beta^2u_{xt}+\beta(\beta^2+1)u_t=c_{xxxt}+3\beta c_{xxt}+3\beta^2c_{xt}+\beta(\beta^2+1)c_t+$$
$$+Q(t,x)-\alpha\int_0^t e^{-\alpha(t-s)}Q(s,x)ds. \quad (12)$$

Adding term by term (6), (7), (10), (12), we obtain

$$u_{xxxt}+u_{xxx}+3\beta u_{xxt}+u_{ttx}+3\beta u_{xx}+\left(3\beta^2+1\right)u_{xt}+\beta(\beta^2+2)u_t+\left[3\left(\beta^2+\alpha\right)+(\alpha^2+1)u_x+\right.$$
$$+\beta\left[\ \left(\beta^2+\alpha+1\right)+1\right]u=H\left(t,c(t,x)\right)+2Q\left(t,x\right)-\beta\int_{-\infty}^{x} e^{-\beta(x-\nu)}Q(\nu,t)d\nu- \quad (13)$$
$$-(\alpha-1)\int_0^t e^{-\alpha(t-s)}Q(s,x)ds+\int_{-\infty}^{x} e^{-\beta(x-\nu)}\sin\left(x-\nu\right)Q(\nu,t)d\nu.$$

Then from (1), taking into account (4), (13), we have the nonlinear Volterra integral equation

$$Q(t,x)=\frac{1}{2}\left\{\ f\left[t,x,c\left(t,x\right)+\int_0^t\int_{-\infty}^{x} e^{-\alpha(t-s)-\beta(x-\nu)}\left[(t-s)-\cos\left(x-\nu\right)\right]Q(\nu,s)d\nu ds\right]-\right.$$
$$-H(t,c(t,x))+\left(\alpha-1\right)\int_0^t e^{-\alpha(t-s)}Q(s,x)ds-2\int_{-\infty}^{x} e^{-\beta(x-\nu)}\sin\left(x-\nu\right)Q(\nu,t)d\nu+$$
$$\left.+\int_0^t\int_{-\infty}^{x} e^{-\alpha(t-s)-\beta(x-\nu)}Q(\nu,s)d\nu ds+\int_{-\infty}^{x} e^{-\beta(x-\nu)}\left[-\cos\left(x-\nu\right)\right]Q(\nu,t)d\nu\ \right\}\equiv P\left[Q\right].$$

(14)

We will apply the contraction mapping principle [3] to the nonlinear equation (14). Let us consider the right side of (14) as an operator PQ acting on $Q(t,x)$.

Consider the set of continuous functions

$$Q(t,x)=\{Q(t,x): Q(t,x)\in C([0,T_0]\times R)\cap\|Q(t,x)\|\le h\},$$

where T_0 and h will be defined below.

Then from (14) we have

$$\|PQ\|\le\frac{1}{2}\left\|f\left(t,x,c(t,x)+\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-v)}\left[(t-s)-\cos(x-v)\right]Q(v,s)dvds\right)\right\|+$$
$$+\frac{1}{2}\|H(t,x)\|+\frac{\alpha-1}{2\alpha}\left\|\int_0^t e^{-\alpha(t-s)}Q(s,x)ds\right\|+\frac{1}{2}\left\|2\int_{-\infty}^x e^{-\beta(x-v)}\sin(x-v)Q(v,t)dv+\int_{-\infty}^x e^{-\beta(x-v)}\cos(x-v)Q(v,t)dv+\right\|\le$$
$$\le\frac{1}{2}\left\{\ M+N+\left[\frac{\alpha-1}{\alpha}\left(1-e^{-\alpha T_0}\right)+\frac{3}{\beta}\right]h\ \right\},$$

where $M=\max\limits_Q\|f(t,x,u)\|,\quad N=\max\limits_Q\|H(t,x)\|.$

Using assumption (A), we choose α,β,T_0,h so that

$$M+N+\left[\frac{\alpha-1}{\alpha}\left(1-e^{-\alpha T_0}\right)+\frac{3}{\beta}\right]h\le 2h.$$

Then the operator $PQ: Q\to Q$.

Let us now show that the operator PQ is a contraction operator. From (14), using assumption (A), we have

$$\|PQ_1-PQ_2\|\le\frac{1}{2}\left\{2L\int_0^t\int_{-\infty}^x e^{-\alpha(t-s)-\beta(x-v)}dvds+\left[\frac{\alpha-1}{\alpha}\left(1-e^{-\alpha T_0}\right)+\frac{3}{\beta}\right]\right\}\|Q_1-Q_2\|\le$$
$$\le\frac{1}{2}\left[\frac{2L}{\alpha\beta}+\frac{\alpha-1}{\alpha}\left(1-e^{-\alpha T_0}\right)+\frac{3}{\beta}\right]\|Q_1-Q_2\|. \qquad (15)$$

By virtue of assumption (A), it follows from (15) that the operator PQ is a contraction operator on the set Q . By the principle of compressed mappings, it follows that the nonlinear integral equation (14) has a unique continuous solution $Q(t,x)\in Q$. Substituting the found function in (4) we obtain the solution of the original Cauchy problem (1)-(3). Obviously, condition (2)-(3) is satisfied due to the selection of the function $c(t,x)$.

Thus, fair

Theorem. Let assumption (A) hold.

Then the Cauchy problem (1)-(3) has a solution $u(t,x) \in \bar{C}^{(2,3)}([0,T_0] \times R)$, which can be represented in the form (4).

REFERENCES

1. Kolmogorov, AN Elements of the theory of functions and functional analysis. Textbook / A. N. Kolmogorov, S. V. Fomin. - Moscow: Fizmatlit, 2012. - 572 p. – ISBN 978-5-9221-0266-7. – EDN RBBEND.
2. Tikhonov, A. N. Equations of Mathematical Physics: A Textbook for Students of Specialized Physics and Mathematics Universities / A. N. Tikhonov; Moscow State University M.V. Lomonosov. – 7th edition. - Moscow: Moscow State University. M.V. Lomonosov, 2004. - 798 p. – (Classic university textbook). – ISBN 5-211-04843-1. – EDN QJNNLR.
3. Tsalyuk Z.B. A remark about the Lyapunov method for integro-differential equations // Mat. Analysis. - Kazan: From Kazansk. un-ta, 1978. S. 103-107.
4. Smirnov M.M. Second order partial differential equations. Moscow: Nauka, 1964
5. Oleinik, OA Lectures on partial differential equations / OA Oleinik; O. A. Oleinik. - 2nd ed., Rev. and additional .. - Moscow: Binom. Lab. knowledge, 2005. - 260 p. - (Classical university textbook / Moscow State University named after M. V. Lomonosov). – ISBN 5-94774-208-X. – EDN QJOBJF.
6. Tikhonov A.N., Samarsky A.A. Equations of mathematical physics. - Moscow: Nauka, 1972. - 540 p.
7. Magnitskii, N. A. New methods for chaotic dynamics / N. A. Magnitskii, S. V. Sidorov. – Singapure : World Scientific Publishing Co., 2006. – 384 p. – EDN UCKROL.
8. Petrovsky, I. G. Lectures on partial differential equations / I. G. Petrovsky. - Moscow: FIZMATLIT, 2009. - 401 p. – ISBN 978-5-9221-1090-7. – EDN UGLIPJ.
9. Lavrent'ev, M. M. Theory of operators and ill-posed problems / M. M. Lavrent'ev, L. Ya. Saveliev; M. M. Lavrentiev, L. Ya. Saveliev. – [2nd ed.,

revised. and add.]. - Novosibirsk: Publishing House of the Institute of Mathematics, 2010. - ISBN 978-5-86134-162-2. – EDN QJXUUB.

10. Godunov S.K., Equations of mathematical physics: textbook. allowance for high fur boots Godunov S. K. - 2nd ed., corrected. and additional - M. : Nauka, 1979. - 391 p. - Bibliography: p. 389.
11. Imanaliev M.I. , Imanaliev T.M. , Kakishev K. On Cauchy problems for nonlinear partial differential equations of the sixth order // Issled. by integro-differential. equations. - Bishkek: Ilim, 2007. - Issue. 36. - S. 19-28.
12. Imanaliev, M. I. On the theory of integro-differential equations in partial derivatives / M. I. Imanaliev, A. B. Baizakov // Proceedings of the National Academy of Sciences of the Kyrgyz Republic. - 2009. - No. 4. - P. 115-120. – EDN XAZNIP.
13. Imanaliev M.I. , Baizakov A.B. On Cauchy problems for third-order nonlinear partial differential equations jj Issled by integro-differential equations. - Bishkek: Ilim, 2008. - Issue. 38. - S. 3-8.
14. Orlov, S. S. Generalized solutions of high-order integro-differential equations in Banach spaces / S. S. Orlov. - Irkutsk: Irkutsk State University, 2014. - 149 p. – ISBN 978-5-9624-1030-2. – EDN SZRIVF.
15. Dzheenbaeva, G. A. On the solvability of the Cauchy problem for systems of nonlinear integro-differential partial differential equations with a parameter / G. A. Dzheenbaeva // International Scientific Research Journal. - 2018. - No. 9-1(75). - S. 17-24. – DOI 10.23670/IRJ.2018.75.9.003. – EDN UZFQFI.
16. Voroshilov, A. A. Cauchy problem for the diffusion-wave equation with Caputo partial derivative / A. A. Voroshilov, A. A. Kilbas // Differential Equations. - 2006. - T. 42. - No. 5. - S. 599-609. – EDN HUIEFF.
17. Voroshilov, A. A. Existence conditions for a classical solution of the cauchy problem for the diffusion-wave equation with a partial Caputo derivative / A. A. Voroshilov, A. A. Kilbas // Doklady Mathematics. – 2007. – Vol. 75. – No 3. – P. 407-410. – DOI 10.1134/S1064562407030209. – EDN XLVQPD.

18. Baizakov, A. B. Sufficient conditions of the Cauchy problem for nonlinear differential equations in private derivatives / A. B. Baizakov, G. A. Jeenbaeva, Yu. N. Ananyeva // Herald of Institute Mathematics of the National Academy of Sciences of the Kyrgyz Republic. – 2021. – No 1. – P. 69-74. – DOI 10.52448/16948173_2021_1_69. – EDN VDAOJQ.
19. Baizakov, A. B. On the solvability of the initial problem of integro-differential equations in partial derivatives of the third order / A. B. Baizakov, G. A. Dzheenbaeva // Science, new technologies and innovations of Kyrgyzstan. - 2016. - No. 6. - P. 8-11. – EDN WISNOJ.
20. Baizakov, A. B. On the structure of solutions of the initial problem of nonlinear integro-differential equations in partial derivatives of the fourth order / A. B. Baizakov, G. A. Dzheenbaeva, M. M. Sharshenbekov // Herald of Institute Mathematics of the National Academy of Sciences of the Kyrgyz Republic. – 2022. – No 1. – P. 72-76. – DOI 10.52448/16948173_2022_1_72. – EDN ORNBAG.
21. Baizakov, A. B. On eigenvalues and eigenfunctions of Volterra integral and integro-differential equations with a singularity / A. B. Baizakov, K. A. Aitbaev, M. M. Sharshenbekov // Herald of Institute Mathematics of the National Academy of Sciences of the Kyrgyz Republic. – 2020. – No 2. – P. 40-46. – DOI 10.52448/16948173_2020_2_40. – EDN IVOXGG.
22. Baizakov, A. B. On the solvability of the Cauchy problem for integro-differential equations in partial derivatives of the fourth order / A. B. Baizakov, G. A. Dzheenbaeva, A. K. Aitbaev // Herald of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic. - 2019. - No. 1. - P. 123-127. – DOI 10.52448/16948173_2019_1_123. – EDN YBNOBU.
23. A. B. Baizakov, K. A. Aitbaev, G. A. Dzheenbaeva // On sufficient conditions for the initial problem for nonlinear integro-differential equations in partial derivatives // Herald of Institute Mathematics of the National Academy of Sciences of the Kyrgyz Republic. – 2022. – No 2. – P. 47-52. – DOI 10.52448/16948173_2020_2_40. – EDN IVOXGG.

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ON THE ASYMPTOTIC SMALLESS OF SOLUTION COMPONENTS OF A SYSTEM OF LINEAR IMPLICIT VOLTERRA-TYPE INTEGRO-DIFFERENTIAL EQUATIONS ON THE HALF-AXIS

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Sufficient conditions are established for the corresponding components of all solutions to tend to zero as the argument grows unbounded, i.e., asymptotic smallness of the corresponding components of all solutions of the system of linear implicit integro-differential equations of the first order of the Volterra type. For this, a non-standard method of reducing to a system, the method of transforming the equations of V.Volterra, the method of V.R. Vinokurov to connect the kernel and the free term, the Cauchy formula and the L'Hopital rule in the form of Stolz are used.

Keywords: first-order integro-differential equation, implicit Volterra-type system, tend convergence of corresponding components to zero, asymptotic smallness, non-standard reduction to a system.

Сызыктуу айкын эмес биринчи тартиптеги Вольтерра тибиндеги интегро-дифференциалдык теңдемелер системасынын бардык чыгарылыштарынынын компоненттеринин аргумент чексиз өскөндө нөлгө умтулуусунун, б.а. асимптотикалык кичинекейлигинин жетиштүү шарттары табылат. Бул үчүн теңдемелерди системага стандарттык эмес келтирүү методу, В.Вольтерранын теңдемелерди өзгөртүп түзүү методу, В.Р. Винокуровдун ядро менен бош мүчөнү байланыштыруу методу өнүктүрүлөт, Кошинин формуласы жана Штольц формасындагы Лопиталь эрежеси колдонулат.

Урунтуу сөздөр: биринчи тартиптеги интегро-дифференциалдык теңдеме, Вольтерра тибиндеги айкын эмес система, тиешелүү компоненттердин нөлгө умтулгандыгы, асимптотикалык кичинекейлик, системага стандарттык эмес келтирүү.

Устанавливаются достаточные условия стремления к нулю соответствующих компонент всех решений при неограниченном росте аргумента, т.е. асимптотической малости соответствующих компонент всех решений системы линейных неявных интегро-дифференциальных уравнений первого порядка типа Вольтерра. Для этого разиваются нестандартный метод сведения к системе, метод преобразования уравнений В.Вольтерра, метод В.Р. Винокурова для связи ядра и свободного члена, применяются формула Коши и правило Лопиталя в форме Штольца.

Ключевые слова: интегро-дифференциальное уравнение первого порядка, неявня система типа Вольтерра, стремление соответствующих компонент к нулю, асимптотическая малость, нестандартное сведение к системе.

All functions appearing are continuous and the relations hold for $t \geq t_0$, $t \geq \tau \geq t_0$; I=[t_0,∞); norms $\|x\|$ and $\|A\|$ for n×1 vector $x = \{x_i\}$ and n×n matrix $A = (a_{ij})$ mean (see, for example, [1, c.142; 2, c.10]): $\|x\|^2 = x_1^2 + \cdots + x_n^2$ and $\|A\|^2 = \gamma$, where γ is the largest eigenvalue of the symmetric matrix $A^T A$; A^T is the transposed for $A = (a_{ij})$, i.e. $A^T = (a_{ji})$; relations for matrices are understood as relations for their quadratic forms with any non-zero vector; $< x, y >$ is the scalar product of the vectors $x = \{x_i\}, y = \{y_i\}$, i.e. $< x, y >= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$; IDE is integro-differential equation.

Definition. The component $x_i(t)$ $(1 \leq i \leq n)$ n×1 of the vector $x(t) = \{x_i(t)\}$ is called asymptotically small if the following relation holds:

$\lim_{t\to\infty} x_i(t) = 0 \quad (1 \leq i \leq n)$.

The problem. Establish sufficient conditions for the asymptotic smallness of the components $x_i(t)$ $(1 \leq i \leq n)$ of all solutions of the n-dimensional system of linear implicit Volterra-type IDEs of the form:

$$x'(t) + A(t)x(t) + \int_{t_0}^{t} [Q_0(t,\tau)x(\tau) + Q_1(t,\tau)x'(\tau)]d\tau = f(t), t \geq t_0. \qquad (1)$$

The solution $x(t)$ of the IDE system (1) is understood as $x(t) \epsilon C^1(I, R^n)$ with any initial Cauchy vectors $x(t_0)$. As is known, each such solution of the IDE system (1) exists and is unique.

Note that such a problem appears in the mathematical modeling of some processes, for example, from cosmodynamics [3, p. 274-275] for dynamic stabilization of the plasma column, from aeroautoelasticity [4] for flight control of aircraft at high speeds. In the article [5], a similar problem was solved for another class of a system of the form (1) by developing a modified method of weight and cut functions [6, p. 28-29;7] and applying the lemma [5].

In this paper, to solve the formulated problem, we develop a non-standard method of reduction to a system [8,9], a method for transforming the equations of V. Volterra [10, p.194-217], method of V.R. Vinokurov for connection of the kernel and the free member [11], apply the Cauchy formula [12, c.393-394] and the L'Hopital rule in the form of Stolz [13, p.115; 14].

Let us proceed to obtain the main result.

In the IDE system (1), we make the following nonstandard substitution

$$x'(t) + \Lambda x(t) = W(t)y(t), \tag{2}$$

where $\Lambda = (\lambda_i)$ is some $n\times n$ diagonal constant matrix with auxiliary matrices λ_i $(i = 1..n), W(t)$ is some $n\times n$ nondegenerate matrix weight function, $y(t)$ is a new unknown $n\times 1$ vector function. Taking into account (2), from the IDE system (1) we have

$$[A(t) - \Lambda]x(t) + W(t)y(t) + \int_{t_0}^{t}\{[Q_0(t,\tau) - Q_1(t,\tau)\Lambda]x(\tau) +$$
$$+Q_1(t,\tau)W(\tau)y(\tau)\}d\tau = f(t), t \geq t_0. \tag{3}$$

Multiplying (3) from the left by the matrix $W^{-1}(t)$, we obtain

$$y(t) + \int_{t_0}^{t} W^{-1}(t)\, Q_1(t,\tau)W(\tau)y(\tau)d\tau$$
$$= W^{-1}(t)f(t) - W^{-1}(t)[A(t) - \Lambda]x(t) -$$
$$- \int_{t_0}^{t} W^{-1}(t)\,[Q_0(t,\tau) - Q_1(t,\tau)\Lambda]x(\tau)d\tau. \tag{4}$$

Let us introduce the notation:

$K(t,\tau) \equiv W^{-1}(t)Q_1(t,\tau)W(\tau), F(t) \equiv W^{-1}(t)f(t),$

$B(t) \equiv W^{-1}(t)[A(t) - \Lambda], P(t,\tau) \equiv W^{-1}(t)[Q_0(t,\tau) - Q_1(t,\tau)\Lambda].$

Then from (4) we have

$$y(t) + \int_{t_0}^{t} K(t,\tau)y(\tau)d\tau = F(t) - B(t)x(t) - \int_{t_0}^{t} P(t,\tau)x(\tau)d\tau. \tag{5}$$

Further, combining (2), (5), we obtain the following system, which is equivalent to the given implicit first-order IDE system (1)

$$\begin{cases} x'(t) + \Lambda x(t) = W(t)y(t), \\ y(t) + \int_{t_0}^{t} K(t,\tau)y(\tau)d\tau = F(t) - B(t)x(t) - \int_{t_0}^{t} P(t,\tau)x(\tau)d\tau. \end{cases} \tag{6}$$

For any solution $(x(t), y(t))$ of the IDE system (6), similarly to [10, c.194-217], its first equation is a system of differential equations of the first order multiplied scalarly by $x(t)$, the second - a system of integral equations - on $y(t)$, add the obtained relations, then integrate within the limits of t_0 to t and we will have the following identity

$$\|x(t)\|^2 + 2\int_{t_0}^{t} < \Lambda x(s), x(s) > ds + 2\int_{t_0}^{t} \|y(s)\|^2 ds +$$

$$+2\int_{t_0}^{t}\int_{t_0}^{s} < K(s,\tau)y(\tau), y(s) > ds \equiv \|x(t)\|^2 + 2\int_{t_0}^{t} < W(s)y(s), x(s) > ds +$$

$$+2\int_{t_0}^{t} < F(s), y(s) > ds - 2\int_{t_0}^{t} < B(s)x(s) + \int_{t_0}^{s} P(s,\tau)x(\tau)d\tau, y(s) > ds. \quad (7)$$

With the introduction of some scalar function $c(t)$ and using transformations from [11,15], we obtain the following relationship between the double integral with $K(t,\tau)$ and the integral with F(t) in the form

$$2\int_{t_0}^{t} < \int_{t_0}^{s} K(s,\tau)y(\tau)d\tau - F(s), y(s) > ds =< K(t,t_0)Y(t,t_0), Y(t,t_0) > -$$

$$-2 < F(t), Y(t,t_0) > +c(t) - \int_{t_0}^{t} < K_s'(s,t_0)Y(s,t_0)Y(s,t_0) > -$$

$$-2 < F'(s), Y(s,t_0) > +c'(s)]ds + \int_{t_0}^{t} < K_\tau'(t,\tau)Y(t,\tau)Y(t,\tau) > d\tau -$$

$$-\int_{t_0}^{t}\int_{t_0}^{s} < K_{s\tau}''(s,\tau)Y(s,\tau), Y(s,\tau) > d\tau ds - c(t_0), \quad (8)$$

where

$$Y(t,\tau) \equiv \int_{\tau}^{t} y(\eta)d\eta.$$

The following inequalities also hold

$$2\int_{t_0}^{t} < W(s)y(s), x(s) > ds \leq 2\int_{t_0}^{t} \|W(s)y(s), x(s)\| ds \leq$$

$$\leq \int_{t_0}^{t} \|W(s))\|^2 \|x(s))\|^2 ds + \int_{t_0}^{t} \|y(s))\|^2 ds, \quad (9)$$

$\exists$ is a number $\varepsilon \in (0,1)$ such that

$$-2\int_{t_0}^{t} < B(s)x(s) + \int_{t_0}^{s} P(s,\tau)x(\tau), y(s) > ds \le$$

$$\le 2\int_{t_0}^{t} \left\| B(s)x(s) + \int_{t_0}^{s} P(s,\tau)x(\tau)d\tau \right\| \|y(s)\| ds \le$$

$$\le \varepsilon^{-1}\int_{t_0}^{t} \left\| B(s)x(s) + \int_{t_0}^{s} P(s,\tau)x(\tau)d\tau \right\|^2 + \varepsilon\int_{t_0}^{t} \|y(s)\|^2 ds. \qquad (10)$$

Based on identity (7), taking into account relations (8)-(10), similarly to [16], we prove

Theorem 1. Let 1) $\Lambda \ge 0$; 2) $K(t,t_0) \ge 0$, $K'_t(t,t_0) \le$, there is a scalar function $c(t)$ such that for any$n\times 1$ vector u the following relations hold: $< K(t,t_0)u,u > -2 < F(t),u > +c(t) \ge 0$, $< K'_t(t,t_0)u,u) > -2 < F'(t),u >$ $+c'(t) \le 0$; 3) $K'_\tau(t,\tau) \ge 0$, $K''_{t\tau}(t,\tau) \le 0$; 4) $\|W(t))\|^2 + [\|B(t)\| +$ $+\int_{t_0}^{s} ||P(t,\tau)||d\tau]^2 \in L^1(I, R_+ \setminus \ \{0\})$. Then for any solution $(x(t),y(t))$ of the IDE system (6) the following properties are true:

$$\|x(t)\| = O(1), \int_{t_0}^{t} y(\eta)d\eta = O(1). \qquad (11)$$

Next, for any $x(t_0)$, based on a non-standard replacement, we write the Cauchy formula for the components $x_i(t)$ of the solutions $x(t) = \{x_i(t)\}$:

$$x_i(t) = e^{-\lambda_i(t-t_0)}[x_i(t_0) + \int_{t_0}^{t} e^{-\lambda_i(s-t_0)}\sum_{j=1}^{n} W_{ij}(s)y_j(s)ds] \ (i = 1..n), \quad (12)$$

where $W(t) = (W_{ij}(t))$.

Taking into account the relation $y_j(s)ds = (\int_{t_0}^{s} y_j(\eta)d\eta)'ds$ $(j = 1..n)$ and integration by parts from (12) we get:

$$x_i(t) = e^{-\lambda_i(t-t_0)}\{x_i(t_0) + \sum_{j=1}^{n} W_{ij}(t)\,(\int_{t_0}^{t} y_j(\eta)d\eta) -$$

$$-\int_{t_0}^{t} e^{\lambda_i(s-t_0)}\sum_{j=1}^{n}[\lambda_i W_{ij}(s) + W'_{ij}(s)](\int_{t_0}^{s} y(\eta)d\eta)\,ds\} \ (i = 1..n). \qquad (13)$$

Theorem 2. Let 1) all conditions of the theorem 1 be satisfied; 2) $\lambda_i > 0$, $W_{ij}(t) = O(1)$, $\lambda_i W_{ij}(t) + W'_{ij}(t) \to 0$ $(1 \le i \le n, j = 1..n)$. Then the component x_i $(i \le i \le n)$ of any solution $x(t) = \{x_i(t)\}$ tends to zero at $t \to \infty$.

To prove Theorem 2, the L'Hopital rule in the form of Stolz is applied to formula (13) [13, p.115; 14].

Note that the idea of replacement (2) is taken from [17].

REFERENCES

1. Barbashin E.A. *Introduction to the theory of stability.* Moscow, Nauka Publ., 1967, 224 p. (in Russ.)

2. Tyshkevich V.A. *Some questions of the theory of stability of functional-differential equations.* Kyiv, Nauk. Dumka Publ., 1980, 80 p. (in Russ.)

3. Bellman R., Cook K.L. *Differential-difference equations*. Moscow, Mir Publ, 1967, 548 p. (in Russ.)

4. Belotserkovsky S.M., Kochetkov Yu.A., Krasovsky A.A., Novitsky V.V. *Introduction to aeroautoelasticity*. Moscow, Nauka Publ.,1980,384 p. (in Russ.)

5. Iskandarov S., Satyeva G.G. Asymptotic representations and properties of solutions to a linear implicit system of Volterra-type integro-differential equations. *Investigations on integro-differential equations,* 2001, issue 30, pp.45-51. (in Russ.)

6. Iskandarov S. *Method of weighting and cutting functions and asymptotic properties of solutions of integro-differential and integral equations of Volterra type*. Bishkek, Ilim Publ., 2002, 216 p. (in Russ.)

7. Iskandarov S. Asymptotic representation of solutions of a weakly nonlinear second-order integro-differential equation of the second order of the Volterra type. *Izv. NAS of the Kyrgyz Republic*, 1998, no. 2-3, pp.9-13. (in Russ.)

8. Iskandarov S. Method for Studying the Asymptotic Properties of Solutions to a System of Second-Order Linear Differential Equations on the Half-Line. *Differential Equations*, 2020, vol.56, no.4, pp.533-537.

9. Iskandarov S. Estimate and Asymptotic Smallness of Solutions of a Weakly Nonlinear Implicit Volterra Integro-Differential Equation of the First Order on the Semiaxis. *Lobachevskii Journal of Mathematics*, 2022, vol.43, no.3, pp.-585-591.

10. Volterra V. *Mathematical struggle for existence*. Moscow, Nauka Publ., 1976, 288 p. (in Russ.)

11. Vinokurov V.R. Asymptotic behavior of solutions of a class of Volterra integro-differential equations. *Differtial equations,* 1967, vol.3, no. 10, pp.1732-1744. (in Russ.)

12. Matveev N.M. *Methods for integrating ordinary differential equations*. Moscow, Higher School Publ., 1967, 564 p. (in Russ.)

13. Demidovich B.P. *Lectures on the mathematical theory of stability*. Moscow, Nauka Publ., 1967, 472 p. (in Russ.)

14. Landau E. *Introduction to differential and integral calculus*. Moscow, IL Publ., 1948, 464 p. (in Russ.)

15. Kiffe T.R. On Nonlinear Volterra Equations of Nonconvolution Type. J.different.equat, 1976, vol.22, pp.349-367.

16. Iskandarov S. On boundedness and stability of solutions of Volterra-type integro-differential equations. *Investigations on integro-differential equations,* 1979, pp.85-102. (in Russ.)

17. Iskandarov S., Bokobaeva Z.B. On estimates of solutions and their first derivatives of a linear Volterra implicit second-order integro-differential equation. *Vestnik IM NAS KR*, 2018, no.1, pp. 49-55. (in Russ.)

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REGULARIZATION AND STABILITY OF SOLUTIONS OF FREDHOLM LINEAR INTEGRAL EQUATIONS OF THE FIRST KIND ON A SEMI-AXIS

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In this article, regularizations and stability of solutions of Fredholm linear integral equations of the first kind are obtained by the methods of functional analysis.

Key words: linear, inteqral equations, first kind, regularization.

Бул макалада функционалдык анализдин усулдарынын жардамы менен Фредгольмдун биринчи түрдөгү сызыктуу интегралдык тендемелердин чечимдеринин регуляризациясы жана туруктуулук баалары алынды.

Урунттуу сөздөр: сызыктуу, интегралдык тендемелер, биринчи түрдөгү, регуляризация.

В данной работе, с помощью методам функционального анализа получены регуляризации и оценки устойчивости решений линейных интегральных уравнений Фредгольма первого рода.

Ключевые слова: линейные, интегральные уравнения, первого рода, регуляризация.

1. Introduction

Inverse and ill-posed problems are currently attracting great interest. The theory and numerical methods for solving inverse and ill-posed problems were studied in [1-18].The notion of correctness in the works of A.N. Tikhonov [1], M.M. Lavrent'ev [2] and V.K. Ivanov [3], different from the classical one, provided a means for studying ill-posed problems and stimulated interest in integral equations with large applied value.

The fundamental results for Fredholm integral equations of the first kind were obtained by M.M. Lavrentiev in [4], [5], where the regularizing operators in the sense of M.M. Lavrentiev.

In [10]-[12], uniqueness theorems were proved and regularization operators in the sense of Lavrent'ev were constructed for systems of linear Volterra and Fredholm integral equations of the third kind.

In [14]-[16] problems of uniqueness and stability of solutions for linear Fredholm integral equations of the first kind were investigated.

In this work, we apply the method of integral transformation to prove regularizations and stability of solutions of Fredholm linear integral equations of the first kind in the semi-axis.

Consider of the Fredholm linear integral equations

$$Ku \equiv \int_{-\infty}^{a} K(t,s)u(s)ds = f(t), \quad t \in (-\infty, a] \tag{1}$$

where

$$\int_{-\infty}^{a}\int_{-\infty}^{a} |K(t,s)|^2 dsdt < \infty,$$

$$K(t,s) = \begin{cases} A(t,s), & -\infty < s \le t \le a, \\ B(t,s), & -\infty < t \le s \le a. \end{cases} \tag{2}$$

$B(t,s)$ and $A(t,s)$ are the known functions, respectively on the domains

$$\{(t,s): -\infty < s \le t \le a\}, \ \{(t,s): -\infty < t \le s \le a\}.$$

$u(t) \in L_2(-\infty, a]$, where $L_2(-\infty, a]$- the space of square-summable functions in $(-\infty, a]$.

Equation (1) by virtue of relation (2) can be expressed as

$$\int_{-\infty}^{t} A(t,s)u(s)ds + \int_{t}^{a} B(t,s)u(s)ds = f(t). \tag{3}$$

Taking the multiplication of both sides of the equation (1) with $u(t)$, integrating the results on $-\infty < t \le a$, we obtain

$$\int_{-\infty}^{a}\int_{-\infty}^{t} A(t,s)u(s)\,u(t)\,dsdt + \int_{-\infty}^{a}\int_{t}^{a} B(t,s)u(s)\,u(t)\,dsdt = \int_{-\infty}^{a} f(t)\,u(t)\,dt,$$

$$\int_{-\infty}^{a}\int_{-\infty}^{t} A(t,s)u(s)\,u(t)\,dsdt + \int_{-\infty}^{a}\int_{s}^{a} B(s,t)u(t)\,u(s)\,dtds = \int_{-\infty}^{a} f(t)\,u(t)\,dt. \tag{4}$$

Integrating by parts and using the Dirichlet formula we have

$$\int_{-\infty}^{a}\int_{s}^{a} B(s,t)u(t)\,u(s)\,dtds = \int_{-\infty}^{a}\left[\int_{-\infty}^{t} B(s,t)u(s)dt\right]u(t)dt.$$

Then from (4) we obtain

$$2\int_{-\infty}^{a}\int_{-\infty}^{t} H(t,s)u(s)ds\,u(t)dt = \int_{-\infty}^{a} f(t)u(t)dt \tag{5}$$

where

$$H(t,s) = \frac{1}{2}\left[A(t,s)+B(s,t)\right], (t,s)\in\{(t,s): -\infty < s \le t \le a\} \tag{6}$$

$$\int_{-\infty}^{a}\int_{-\infty}^{t} H^2(t,s)dsdt < +\infty.$$

We introduce a new function $M(t,s)$ as follows

$$M(t,s) = \begin{cases} H(t,s), & -\infty < s \le t \le a, \\ H(s,t), & -\infty < t \le s \le a. \end{cases} \tag{7}$$

It's clear that $\quad M(t,s) = M(s,t), \quad (t,s)\in(-\infty,a]\times(-\infty,a].$

It is easy to verify the validity of the equality

$$\int_{-\infty}^{a}\int_{-\infty}^{a} |M(t,s)|^2 dsdt < +\infty.$$

Then, it is known that

$$M(t,s) = \sum_{i=1}^{\infty} \frac{\varphi_i(t)\varphi_i(s)}{\lambda_i}, \tag{8}$$

where, λ_i - the characteristic numbers of the matrix kernel $M(t,s)$, which are arranged in ascending order of their modules, $|\lambda_1| \le |\lambda_2| \le \dots$ and $\varphi_1(t), \varphi_2(t)\dots$ the corresponding orthonormal eigen functions.

It is assumed that $M(t,s)$ is a complete kernel and $0<\lambda_1\le\lambda_2\le\ldots$. In this case, the solution of equation (1) will be unique in $L_2(-\infty,a]$.

In what follows, we will assume that all characteristic numbers of the matrix kernel are positive.

We distinguish the family of correctness sets depending on the parameter as follows

$$M_\alpha=\left\{u(t)\in L_2(-\infty,a]:\sum_{v=1}^{\infty}\lambda_v^{\alpha}\left|u^{(v)}\right|^2\le c\right\}, \tag{9}$$

where $c>0,\quad 0<\alpha<\infty,$

$$u^{(v)}=\int_{-\infty}^{a}u(t)\varphi_v(t)\,dt, \tag{10}$$

It is clear that if $u(t)\in M_\alpha$, then

$\|u(t)\|^2\le c\lambda_1^{-\alpha}$, where

$$\|u(t)\|=\left(\int_{-\infty}^{a}|u(t)|^2\,dt\right)^{\frac{1}{2}}=\left(\sum_{v=1}^{\infty}\left|u^{(v)}\right|^2\right)^{\frac{1}{2}}$$

We will assume that $f(t)\in K(M_\alpha)$. Then equation (1) has a solution $u(t)\in M_\alpha$ and is valid

$$\sum_{i=1}^{\infty}\frac{1}{\lambda_i}\left|\int_{-\infty}^{a}u(t)\varphi_i(t)\,dt\right|^2=\int_{-\infty}^{a}f(t)u(t)\,dt.$$

Hence, using Hölder's inequalities, we have

$$\sum_{v=1}^{\infty}\lambda_v^{-1}\left|u^{(v)}\right|^2\le\|f(t)\|\cdot\|u(t)\|. \tag{11}$$

On the other side

$$\|u(t)\|^2 = \sum_{i=1}^{\infty} \frac{|u_i|^{\frac{2\alpha}{1+\alpha}}}{\lambda_i^{\frac{\alpha}{1+\alpha}}} \lambda_i^{\frac{\alpha}{1+\alpha}} |u_i|^{\frac{2}{1+\alpha}} \le \left(\sum_{i=1}^{\infty} \frac{|u_i|^2}{\lambda_i} \right)^{\frac{\alpha}{1+\alpha}} \left(\sum_{i=1}^{\infty} \lambda_i^{\alpha} |u_i|^2 \right)^{\frac{1}{1+\alpha}} .$$

Here we have applied Hölder's inequality for $p = 1 + a, q = \frac{(1+\alpha)}{\alpha}$. Taking into account $u(t) \in M_\alpha$ and (11), from the last inequality we have

$$\|u(t)\|^2 \le c^{\frac{1}{1+\alpha}} \left(\|f(t)\| \|u(t)\| \right)^{\frac{\alpha}{1+\alpha}} .$$

Hence we obtain the following stability estimate

$$\|u(t)\| \le c^{\frac{1}{2+\alpha}} \|f(t)\|^{\frac{\alpha}{2+\alpha}} \text{ , } 0 < \alpha < \infty \tag{12}$$

Thus, the following theorem has been proved.

Theorem 1. Let the operator M generated by the matrix kernel $M(t,s)$ be positive, where it is defined $M(t,s)$ by formula (7). Then on the set $K(M_\alpha)$ the operator K^{-1}, the inverse of K, is uniformly continuous with the Hölder exponent $\frac{\alpha}{2+\alpha}$, i.e. estimate (12) is valid.

Let us show that the solution of the system of equations

$$\varepsilon u(t,\varepsilon) + \int_{-\infty}^{a} K(t,s) u(s,\varepsilon) ds = f(t), \quad t \in (-\infty, a], \varepsilon > 0 \tag{13}$$

will be regularizing for equation (1) on the set M_α .

Indeed, by making the following substitution into the equation (13)

$$u(t,\varepsilon) = u(t) + \xi(t,\varepsilon),$$

where $u(t) \in M_\alpha$ - solution of equation (1), we obtain

$$\varepsilon\xi(t,\varepsilon)+\int_{-\infty}^{a}K(t,s)\xi(s,\varepsilon)ds=-\varepsilon u(t).$$

Multiplying the last equation by $\xi(t,\varepsilon)$ and integrating, from $-\infty$ to a, taking into account (2) and (8), we have:

$$\varepsilon\|\xi(t,\varepsilon)\|^2+\sum_{\nu=1}^{\infty}\lambda_\nu^{-1}|\xi_\nu(\varepsilon)|^2\le\varepsilon\sum_{\nu=1}^{\infty}|u^{(\nu)}||\xi_\nu(\varepsilon)|, \tag{14}$$

where $\xi_i(\varepsilon)$-are the Fourier coefficients for the function $\xi(t,\varepsilon)$, according to the orthonormal system $\{\varphi_\nu(t)\}$ т.е. $\xi_\nu(\varepsilon)=\int_{-\infty}^{a}\xi(t,\varepsilon)\varphi_\nu(t)dt$.

Applying the Hölder inequality for $p=q=\frac{1}{2}$, from (14) we obtain

$$\|\xi(t,\varepsilon)\|\le\|u(t)\|, \tag{15}$$

$$\sum_{\nu=1}^{\infty}\lambda_\nu^{-1}|\xi_\nu(\varepsilon)|^2\le\varepsilon\|u(t)\|^2\le\varepsilon c\lambda_1^{-\alpha},\ \varepsilon>0, \tag{16}$$

on the other side

$$\sum_{\nu=1}^{\infty}|u^{(\nu)}||\xi_\nu(\varepsilon)|=\sum_{\nu=1}^{\infty}\frac{|\xi_\nu(\varepsilon)|^{\frac{\alpha}{1+\alpha}}}{\lambda_\nu^{\frac{\alpha}{2(1+\alpha)}}}\cdot\lambda_\nu^{\frac{\alpha}{2(1+\alpha)}}|u^{(\nu)}|^{\frac{1}{1+\alpha}}|\xi_\nu(t,\varepsilon)|^{\frac{1}{1+\alpha}}|u^{(\nu)}|^{\frac{\alpha}{1+\alpha}}.$$

Therefore, after applying the generalized Hölder inequality to the right-hand side for

$p=\frac{2(1+\alpha)}{\alpha},\ q=2(1+\alpha),\ m=2(1+\alpha),\ n=\frac{2(1+\alpha)}{\alpha}$ we have

$$\sum_{\nu=1}^{\infty}|u^{(\nu)}||\xi_\nu(\varepsilon)|\le\left(\sum_{\nu=1}^{\infty}|\xi_\nu(\varepsilon)|^2\lambda_\nu^{-1}\right)^{\frac{a}{(1+\alpha)2}}\left(\sum_{\nu=1}^{\infty}\lambda_\nu^{\alpha}|u^{(\nu)}|^2\right)^{\frac{1}{(1+\alpha)2}}\|\xi(t,\varepsilon)\|^{\frac{1}{1+\alpha}}\|u(t)\|^{\frac{\alpha}{1+\alpha}},$$

$$\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|\left|\xi_\nu(\varepsilon)\right| \le \left(\sum_{\nu=1}^{\infty}\left|\xi_\nu(\varepsilon)\right|^2 \lambda_\nu^{-1}\right)^{\frac{1}{p}}\left(\sum_{\nu=1}^{\infty}\lambda_\nu^{\alpha}\left|u^{(\nu)}\right|^2\right)^{\frac{1}{q}}\left\|\xi(t,\varepsilon)\right\|^{\frac{2}{q}}\left\|u(t)\right\|^{\frac{2}{p}}.$$

Further in force

$u(t)\in M_\alpha$, (15) and (16) from the last inequality we have

$$\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|\left|\xi_\nu(\varepsilon)\right| \le \left(\varepsilon c\lambda_1^{-\alpha}\right)^{\frac{1}{p}} c^{\frac{1}{q}}\left\|u(t)\right\|^{\frac{2}{q}+\frac{2}{p}},$$

$$\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|\left|\xi_\nu(\varepsilon)\right| \le \left(\varepsilon c\lambda_1^{-\alpha}\right)^{\frac{1}{p}} c^{\frac{1}{q}}(c\lambda_1^{-\alpha})^{\frac{p+q}{pq}},$$

Hence, substituting $p=\dfrac{2(1+\alpha)}{\alpha}$, $q=2(1+\alpha)$, we get

$$\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|\left|\xi_\nu(\varepsilon)\right| \le c^{\frac{1}{2(1+\alpha)}}(c\lambda_1^{-\alpha})^{\frac{1}{2}}(\varepsilon c\lambda_1^{-\alpha})^{\frac{\alpha}{2(1+\alpha)}}. \tag{17}$$

$$\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|\left|\xi_\nu(\varepsilon)\right| \le c^{\frac{1}{2(1+\alpha)}}c^{\frac{1}{2}}c^{\frac{\alpha}{2(1+\alpha)}}\lambda_1^{\frac{-\alpha}{2}}\lambda_1^{-\frac{\alpha^2}{2(1+\alpha)}}\varepsilon^{\frac{\alpha}{2(1+\alpha)}}$$

$$\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|\left|\xi_\nu(\varepsilon)\right| \le c\lambda_1^{\frac{-\alpha(2\alpha+1)}{2(1+\alpha)}}\varepsilon^{\frac{\alpha}{2(1+\alpha)}}. \tag{18}$$

Taking into account (18), from (14) we have

$$\left\|u(t,\varepsilon)-u(t)\right\|_{L_2} \le c^{\frac{1}{2}}\lambda_1^{\frac{-\alpha(2\alpha+1)}{4(1+\alpha)}}\varepsilon^{\frac{\alpha}{4(1+\alpha)}},\quad 0<\alpha<\infty. \tag{19}$$

Thus proven.

Theorem 2. Let the operator M generated by the matrix kernel $M(t,s)$ be positive and $f(t)\in K(M_\alpha)$. Then estimate (19) is valid, where $u(t,\varepsilon)$ - is a solution to equation (13) u(t) is a solution to equation (1) $M(t,s)$ and is determined by formula (8).

REFERENCES

1. Tikhonov A.N. On methods for solving ill-posed problems. // In the book: Abstracts of reports, International Congress of Mathematicians. - M., 1966.(russ).
2. Lavrent'ev M. M. On some ill-posed problems of mathematical physics. - Novosibirsk: Publishing House of Sib. Department of the Academy of Sciences of the USSR, 1962. - 92 p.(russ).
3. Ivanov V.K. About ill-posed tasks. Differential Equations.-1968.- No. 2.-P.61.
4. Lavrent'ev M. M. On integral equations of the first kind Doklady of the Academy of Sciences of the USSR. 1959. V. 127. No. 1. S. 31.(russ).
5. Lavrent'ev M. M., Romanov V. G., Shishatskii S. P. Ill-Posed Problems of Mathematical Physics and Analysis (Providence, RI: American Mathematical Society). – 1986.
6. Apartsyn, A. S. Nonclassical Linear Volterra Equations of the First Kind / A. S. Apartsyn. – Utrecht-Boston: VSP, 2003. – 168 p. – ISBN 90-6764-375-0. – EDN RUCKLT.
7. Magnitskii N. A. Linear VolterrA Integral Equations of the First and Third Kind // Doklady Akademii Nauk. - 1991. - Vol. 317, No 2.-Pp. 330-333.
8. Bukhgeim A.L. Volterra Equations and Inverse Problems, VSP, Utrecht, The Netherlands, 1999.
9. A.Asanov, Regularization, Uniqueness and Existence of Solutions of Volterra Equations of the First Kind. VSP, Utrecht, p.272. (1998).
10. Imanaliev, M. I., Asanov A., Solutions of system of nonlinear Volterra integral equations of the first kind, Soviet Math. Dokl. 309, (1989) 1052–1055.
11. Imanaliev, M. I., Asanov A., Asanov R. A. On a class of systems of linear and nonlinear Fredholm integral equations of the third kind with multipoint

singularities // Differential Equations. - 2018. - T. 54. - No. 3. - P. 387. – DOI 10.1134/S037406411803010X. – EDN YQYHAP. (Russ).

12. Imanaliev, M. I., Asanov A. Regularization and uniqueness of solutions to systems of nonlinear Volterra integral equations of the third kind // Doklady Mathematics. – 2007. – Vol. 415. – No 1. – P. 14-17. – DOI 10.1134/S1064562407040035. – EDN XLWHPB.(Russ)
13. Imanaliev M. I., Asanov A., Asanov R. A. A class of systems of linear Fredholm integral equations of the third kind // Doklady Mathematics. – 2011. – Vol. 83. – No 2. – P. 227-231. – DOI 10.1134/S1064562411020293. – EDN XKKSFX.
14. Asanov A., Kadenova Z. A. Regularization and Stability of Systems of Linear Integral Fredholm Equations of the First Kind, Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 38, Samara State Technical University, Samara, 2005, Pp.11–14. (In Russ.), http://mi.mathnet.ru/eng/vsgtu/v38/p11, DOI: https://doi.org/10.14498/vsgtu363.(Russ).
15. Imanaliev M. I., Asanov A., Kadenova Z. A., A class of linear integral equations of the first kind with two independent variables.// Doklady Mathematics. – 2014. – Vol. 89. – No 1. – P. 98-102. – DOI 10.1134/S1064562414010281. – EDN XLBERD.
16. Asanov A., M. H. Chelik, Kadenova Z. A., Uniqueness and stability of solutions of linear intergral equations of the first kind with two variables. // International Journal of Mathematical Analysis. – 2013. – Vol. 7. – No 17-20. – P. 907-914. – DOI 10.12988/ijma.2013.13088. – EDN XKWECX.
17. Asanov A., Kadenova Z.A., Bekeshova D. On the uniqueness of solutions of Fredholm linear integral equations of the first kind on a semi-axis, // Herald of Institute Mathematics of the National Academy of Sciences of the Kyrgyz Republic. – 2022. – No 1. – P. 82-87. – DOI 10.52448/16948173_2022_1_82. – EDN FXALAA.

18. Asanov A., Kadenova Z.A., Bekeshova D. On the uniqueness of solutions of Volterra linear equations of the first kind on the semi-axis// Unstable problems of computational mathematics: Abstracts of the seminar with international participation, Irkutsk, August 15–19, 2022. - Irkutsk: Irkutsk State University, 2022. - P. 36-40. – EDN KFMOZP.

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SQUARE INTEGRABLE SOLUTION TO THE LINEAR TWO-DIMENSIONAL VOLTERRA-STIELTES EQUATION OF THE SECOND KIND IN THE INFINITE DOMAIN

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This article explores the question of the square integrability and uniqueness of solutions of linear integral equations of the Volterra-Stieltjes with two independent variables.

Keywords: integration by parts, the Dirichlet formula.

Бул макалада эки көз карандысыз өзгөрмөлүү Вольтерра-Стиельтьес тибиндеги сызыктуу интегралдык теңдемелерди чыгаруунун квадраттык интегралдуулугу жана уникалдуулугу маселеси каралат.

Урунттуу сөздөр:: бөлүктөр боюнча интеграция, Дирихле формуласы

В этой статье рассматривается вопрос о квадратичной интегрируемости и единственности решения линейных интегральных уравнений типа Вольтерра-Стилтьеса с двумя независимыми переменными.

Ключевые слова: интегрирование по частям, формулой Дирихле.

Introduction

The article [2] introduced the concepts of derivative and differential of a function with respect to an increasing function $\varphi(x)$. Here, on the basis of this concept, questions of quadratic integrability of solutions of the Volterra-Stieltjes integral equations in an infinite domain are studied. Questions of boundedness of quadratic integrability and stability of solutions of differential, integral, and integro-differential equations have been studied in many works, for example, in [1,3]. In [1-3] considered the question of limiting solutions for an infinite domain of integral and integro-differential equations on the half. Questions of boundedness of quadratic integrability and stability of solutions for differential, integral, and integro-differential equations have been studied in many works, for example, in [4-

6]. In this article, applying the method in [7], we prove a theorem of uniqueness and boundedness of solutions of integro-differential equations in partial derivatives of the second order in infinite domains.

In [2], the concept of partial derivatives of a function $a(t,x,s)$ with respect to an increasing function is defined as follows:

$$a'_{\varphi(t)}(t,x,s)=\lim_{\Delta t\to 0}\frac{a(t+\Delta t,x,s)-a(t,x,s)}{\varphi(t+\Delta t)-\varphi(t)}.$$

Finding the derivative of a function $a(t,x,s)$with $\varphi(t)$respect to will be called differentiation with$\varphi(t)$ respect to this function. If a function $a(t,x,s)$ at a point(t,x,s) has a finite derivative with respect to$\varphi(t)$, then the function $a(t,x,s)$ is said to be differentiable with respect to at that point$\varphi(t)$. A function $a(t,x,s)$ that is differentiable with respect to all points $\varphi(t)$ of the domainG_1 is called differentiable with $\varphi(t)$ respect to $G_1=\{(t,x,s)/0\le s\le t<\infty,\quad 0\le x<\infty\}$.

Consider the equation:

$$u(t,x)+\int_0^t a(t,x,s)u(s,x)d\varphi(s)+\int_0^x b(t,x,y)u(t,y)d\psi(y)+$$

$$+\int_0^t\int_0^x c(t,x,s,y)u(s,y)d\psi(y)d\varphi(s)=f(t,x),\ (t,x)\in G, \tag{1}$$

where $a(t,x,s), b(t,x,y), c(t,x,s,y), f(t,x)$ - known continuous functions, $u(t,x)$ - unknown function , $\varphi(t)$ и $\psi(x)$ - increasing continuous functions in $[0,\infty)$, here all integrals are understood according to Stieltjes.

Denote by $C(G)$ -the space of all continuous functions on -

$G=\{(t,x)/0\le t<\infty,\quad 0\le x<\infty\}$. Denote $L^2_{\varphi,\psi}(G)$ by the space of all functions $u(t,x)$ satisfying the condition,

$$\int_0^\infty\int_0^\infty |u(t,x)|^2 d\varphi(t)d\psi(x)<\infty$$

THEOREM. If the conditions are met: $f(t,x)\in L^2_{\varphi,\psi}(G)\cap C(G)$,

a) functions $a(t,x,s),a'_{\varphi(t)}(t,x,s),a'_{\varphi(s)}(t,x,s),a''_{\varphi(t)\varphi(s)}(t,x,s)\in C(G_1)$,

$G_1=\{(t,x,s)/0\le s\le t<\infty,\quad 0\le x<\infty\}$, $a(t,x,0)\ge 0$ and $a'_{\varphi(t)}(t,x,0)\le 0$ at $(t,x)\in G$;

$a'_{\varphi(s)}(t,x,s)\ge 0$ and $a''_{\varphi(t)\varphi(s)}(t,x,s)\le 0$ at $(t,x,s)\in G_1$;

б) functions $b(t,x,y)$, $b'_{\psi(x)}(t,x,y)$, $b'_{\psi(y)}(t,x,y)$, $b''_{\psi(x)\psi(y)}(t,x,y)$ $\in C(G_2)$,

$G_2=\{(t,x,y)/0\le t<\infty,\quad 0\le y\le x<\infty\}$, $b(t,x,0)\ge 0$ and $b'_{\psi(x)}(t,x,y)\le 0$ at $(t,x)\in G$;

$b'_{\psi(y)}(t,x,y)\ge 0$ and $b''_{\psi(x)\psi(y)}(t,x,y)\le 0$ at $(t,x,y)\in G_2$;

в) functions $c(t,x,s,y)$, $c'_{\varphi(t)}(t,x,s,y)$, $c'_{\psi(x)}(t,x,s,y)$, $c''_{\varphi(t)\psi(x)}(t,x,s,y)$, $c'_{\varphi(s)}(t,x,s,y)$, $c'_{\psi(y)}(t,x,s,y)$, $c''_{\varphi(t)\psi(y)}(t,x,s,y)$, $c''_{\varphi(t)\varphi(s)}(t,x,s,y)$, $c''_{\varphi(s)\psi(y)}(t,x,s,y)$, $c'''_{\varphi(t)\varphi(s)\psi(y)}(t,x,s,y)$, $c'''_{\varphi(s)\psi(x)\psi(y)}(t,x,s,y)$, $c^{(IV)}_{\varphi(t)\varphi(s)\psi(x)\psi(y)}(t,x,s,y)$ $\in C(G_3)$, $G_3=\{(t,x,s,y)/0\le s\le t<\infty,\ 0\le y\le x<\infty\}$

$c'_{\psi(y)}(t,x,0,y)\ge 0$ at $(t,x,y)\in G_2$, $\quad c'_{\varphi(s)}(t,x,s,0)\ge 0$ at $(t,x,s)\in G_1$,

$c(t,x,0,0)\ge 0$, $c'_{\varphi(t)}(t,x,0,0)\le 0$, $c'_{\psi(x)}(t,x,0,0)\le 0$, $c''_{\varphi(t)\phi(s)}(t,x,s,0)\le 0$, $c''_{\varphi(t)\psi(x)}(t,x,0,0)\ge 0$, $c''_{\varphi(t)\psi(y)}(t,x,0,y)\le 0$ at $(t,x)\in G$ $\quad c''_{\varphi(s)\psi(x)}(t,y,s,0)\le 0$, $c''_{\varphi(s)\psi(y)}(t,x,s,y)\ge 0$, $c''_{\psi(x)\psi(y)}(t,x,0,y)\le 0$, $c'''_{\varphi(t)\varphi(s)\psi(x)}(t,x,s,0)\ge 0$, $c'''_{\varphi(t)\varphi(s)\psi(y)}(t,x,s,y)\le 0$, $c'''_{\varphi(t)\psi(x)\psi(y)}(t,x,0,y)\ge 0$, $c'''_{\varphi(t)\varphi(s)\psi(y)}(t,x,s,y)\le 0$, $c^{(IV)}_{\varphi(t)\varphi(s)\psi(x)\psi(y)}(t,x,s,y)\ge 0$ at $(t,x,s,y)\in G_3$;

г) $c^2(t,x,0,0)-a'_{\varphi(t)}(t,x,0)b'_{\psi(x)}(t,x,0)\le 0$ at $(t,x)\in G$,

$$\varphi(s)\psi(y)\left(c''_{\varphi(s)\psi(y)}(t,x,s,y)\right)^2-a''_{\varphi(t)\varphi(s)}(t,x,s)b''_{\psi(x)\psi(y)}(t,x,y)\le 0$$

at $(t,x,s,y)\in G_4$,

then equation (1) has a unique solution in $L^2_{\varphi,\psi}(G)\cap C(G)$.

PROOF. From the continuity of all known functions in (1) it follows that this equation has a continuous solution $u(t,x)$.

We multiply both parts of identity (1) by $u(t,x)$ and integrate $G_{t,x}\{0\le s\le t;\,0\le y\le x\}$.

Then

$$\int_0^t\int_0^x u^2(s,y)\,d\psi(y)d\varphi(s)+\int_0^t\int_0^x\int_0^s a(s,y,\tau)u(\tau,y)u(s,y)\,d\varphi(\tau)d\psi(y)d\varphi(s)+$$

$$+\int_0^t\int_0^x\int_0^y b(s,y,z)u(s,z)u(s,y)\,d\psi(z)d\psi(y)d\varphi(s)+\int_0^t\int_0^x\int_0^s\int_0^y c(s,y,\tau,z)\times$$

$$\times u(\tau,z)u(s,y)\,d\varphi(\tau)d\psi(z)d\psi(y)d\varphi(s)=\int_0^t\int_0^x f(s,y)u(s,y)\,d\psi(y)d\varphi(s). \qquad (2)$$

We transform each of the integrals of the left side of equation (2) according to the integration by parts formula, the generalized Dirichlet formula and use the following formula:

$$\mathrm{KWW}_{\varphi(s)}{}'=\frac{1}{2}(KW^2)_{\varphi(s)}{}'-\frac{1}{2}K_{\varphi(s)}{}'W^2.$$

$$\int_0^t\int_0^x\int_0^s a(s,y,\tau)u(\tau,y)u(s,y)\,d\varphi(\tau)d\psi(y)d\varphi(s)=-\int_0^t\int_0^x\int_0^s a(s,y,\tau)\frac{\partial}{\partial\varphi(\tau)}\left(\int_\tau^s u(\xi,y)\,d\varphi(\xi)\right)\times$$

$$\times d\varphi(\tau)u(s,y)\,d\psi(y)d\varphi(s)=\int_0^t\int_0^x a(s,y,0)\left(\int_0^s u(\xi,y)\,d\varphi(\xi)\right)u(s,y)\,d\psi(y)d\varphi(s)+$$

$$+\int_0^t\int_0^x\int_0^s a'_{\varphi(\tau)}(s,y,\tau)\left(\int_\tau^s u(\xi,y)\,d\varphi(\xi)\right)u(s,y)\,d\varphi(\tau)d\psi(y)d\varphi(s)=\frac{1}{2}\int_0^x a(t,y,0)\times$$

$$\left(\int_0^t u(\xi,y)\,d\varphi(\xi)\right)^2 d\psi(y)-\frac{1}{2}\int_0^t\int_0^x a'_{\varphi(s)}(s,y,o)\left(\int_0^s u(\xi,y)\,d\varphi(\xi)\right)^2 d\psi(y)d\varphi(s)+$$

$$+\frac{1}{2}\int_0^t\int_0^x a'_{\varphi(\tau)}(t,y,\tau)\left(\int_\tau^t u(\xi,y)\,d\varphi(\xi)\right)^2 d\psi(y)d\varphi(\tau)-$$

$$-\frac{1}{2}\int_0^t\int_0^x\int_\tau^t a''_{\varphi(s)\varphi(\tau)}(s,y,\tau)\left(\int_\tau^s u(\xi,y)\,d\varphi(\xi)\right)^2 d\varphi(s)d\psi(y)d\varphi(\tau).$$

Applying the generalized Dirichlet formula, we have:

$$\int_0^t\int_0^x\int_0^s a(s,y,\tau)u(\tau,y)u(s,y)\,d\varphi(\tau)d\psi(y)d\varphi(s)=\frac{1}{2}\int_0^x a(t,y,0)\left(\int_0^t u(\xi,y)\,d\varphi(\xi)\right)^2 d\psi(y)-$$

$$-\frac{1}{2}\int_0^t\int_0^x a'_{\varphi(s)}(s,y,0)\left(\int_0^s u(\xi,y)d\varphi(\xi)\right)^2 d\psi(y)d\varphi(s)+\frac{1}{2}\int_0^t\int_0^x a'_{\varphi(\tau)}(t,y,\tau)\left(\int_\tau^t u(\xi,y)d\varphi(\xi)\right)^2\times$$

$$\times d\psi(y)d\varphi(\tau)-\frac{1}{2}\int_0^t\int_0^x\int_0^s a''_{\varphi(s)\varphi(\tau)}(s,y,\tau)\left(\int_\tau^s u(\xi,y)d\varphi(\xi)\right)^2 d\varphi(\tau)d\psi(y)d\varphi(s). \qquad (3)$$

Similarly, we obtain for the second term:

$$\int_0^t\int_0^x\int_0^y b(s,y,z)u(s,z)u(s,y)d\psi(z)d\psi(y)d\varphi(s)=\frac{1}{2}\int_0^t b(s,x,0)\left(\int_0^x u(s,v)d\psi(v)\right)^2 d\varphi(s)-$$

$$-\frac{1}{2}\int_0^t\int_0^x b'_{\psi(y)}(s,y,0)\left(\int_0^y u(s.v)d\psi(v)\right)^2 d\psi(y)d\varphi(s)+\frac{1}{2}\int_0^t\int_0^x b'_{\psi(z)}(s,x,z)\left(\int_z^x u(s,v)d\psi(v)\right)^2 d\psi(z)\times$$

$$\times d\varphi(s)-\frac{1}{2}\int_0^t\int_0^x\int_0^y b''_{\psi(y)\psi(z)}(s,y,z)\left(\int_z^y u(s,v)d\psi(v)\right)^2 d\psi(z)d\psi(y)d\varphi(s). \qquad (4)$$

To transform the third integral, we use the following formula:

$$Cv''_{\varphi(\tau)\psi(z)}=(Cv)''_{\varphi(\tau)\psi(z)}-\left(C'_{\varphi(\tau)}v\right)'_{\psi(z)}-\left(C'_{\psi(z)}v\right)'_{\varphi(\tau)}+C''_{\varphi(\tau)\psi(z)}v$$

at $v(s,y,\tau,z)=\int_\tau^s\int_z^y u(\xi,v)d\psi(v)d\varphi(\xi)$.

Then, integrating by parts, we have:

$$\int_0^t\int_0^x\int_0^s\int_0^y c(s,y,\tau,z)u(\tau,z)u(s,y)d\psi(z)d\varphi(\tau)d\psi(y)d\varphi(s)==\int_0^t\int_0^x\int_0^s\int_0^y c(s,y,\tau,z)\frac{\partial^2}{\partial\varphi(\tau)\partial\psi(z)}\times$$

$$\times\left(\int_\tau^s\int_z^y u(\xi,v)d\psi(v)d\varphi(\xi)\right)d\psi(z)d\varphi(\tau)u(s,y)d\psi(y)d\varphi(s)=\int_0^t\int_0^x\left[\int_0^s\int_0^y\frac{\partial^2}{\partial\varphi(\tau)\partial\psi(z)}\times\right.$$

$$\times\left(c(s,y,\tau,z)\int_\tau^s\int_z^y u(\xi,v)d\psi(v)d\varphi(\xi)\right)d\psi(z)d\varphi(\tau)-\int_0^s\int_0^y\frac{\partial}{\partial\psi(z)}\left(c'_{\varphi(\tau)}(s,y,\tau,z)\int_\tau^s\int_z^y u(\xi,v)\times\right.$$

$$\left.\times d\psi(v)d\varphi(\xi)\right)d\psi(z)d\varphi(\tau)-\int_0^s\int_0^y\frac{\partial}{\partial\varphi(\tau)}\left(c'_{\psi(z)}(s,y,\tau,z)\int_\tau^s\int_z^y u(\xi,v)d\psi(v)d\varphi(\tau)\right)\times$$

$$\left.\times d\psi(z)d\varphi(\tau)+\int_0^s\int_0^y c''_{\varphi(\tau)\psi(z)}(s,y,\tau,z)\int_\tau^s\int_z^y u(\xi,v)d\psi(v)d\varphi(\xi)d\psi(z)d\varphi(\tau)\right]u(s,y)d\psi(y)\times$$

$$\times d\varphi(s)=\int_0^t\int_0^x c(s,y,0,0)\left(\int_0^s\int_0^y u(\xi,v)d\psi(v)d\varphi(\xi)\right)u(s,y)d\psi(y)d\varphi(s)+\int_0^t\int_0^x\int_0^s c'_{\varphi(\tau)}(s,y,\tau,0)\times$$

$$\times\int_\tau^s\int_0^y u(\xi,v)d\psi(v)d\varphi(\xi)u(s,y)d\varphi(\tau)d\psi(y)d\varphi(s)+\int_0^t\int_0^x\int_0^y c'_{\psi(z)}(s,y,0,z)\int_0^s\int_z^y u(\xi,v)d\psi(v)\times$$

$$\times d\varphi(\xi)u(s,y)d\psi(z)d\psi(y)d\varphi(s)+\int_0^t\int_0^x\int_0^s\int_0^y c''_{\varphi(\tau)\psi(z)}(s,y,\tau,z)\int_\tau^s\int_z^y u(\xi,v)d\psi(v)d\varphi(\xi)\times$$

$$\times u(s,y)d\psi(z)d\varphi(\tau)d\psi(y)d\varphi(s). \qquad (5)$$

Hence, using the formula

$$Cvv''_{\varphi(s)\psi(y)}=\frac{1}{2}(Cv^2)''_{\varphi(s)\psi(y)}-\frac{1}{2}(C'_{\varphi(s)}v^2)'_{\psi(y)}-\frac{1}{2}(C'_{\psi(y)}v^2)'_{\varphi(s)}+\frac{1}{2}C''_{\varphi(s)\psi(y)}v^2-Cv'_{\psi(y)}v'_{\varphi(s)}$$

and the generalized Dirichlet formula, we get:

$$\int_0^t\int_0^x\int_0^s\int_0^y c(s,y,\tau,z)u(\tau,z)u(s,y)d\psi(z)d\varphi(\tau)d\psi(y)d\varphi(s)=\frac{1}{2}c(t,x,0,0)\left(\int_0^t\int_0^x u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2-$$

$$-\frac{1}{2}\int_0^t c'_{\varphi(s)}(s,x,0,0)\left(\int_0^s\int_0^x u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2 d\varphi(s)-\frac{1}{2}\int_0^x c'_{\psi(y)}(t,y,0,0)\times$$

$$\times\left(\int_0^t\int_0^y u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2 d\psi(y)+\frac{1}{2}\int_0^t\int_0^x c''_{\varphi(s)\psi(y)}(s,y,0,0)\left(\int_0^s\int_0^y u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2\times$$

$$\times d\psi(y)d\varphi(s)-\int_0^t\int_0^x c(s,y,0,0)\left(\int_0^s u(\xi,y)d\varphi(\xi)\right)\left(\int_0^y u(s,v)d\psi(v)\right)d\psi(y)d\varphi(s)+$$

$$+\frac{1}{2}\int_0^t c'_{\varphi(\tau)}(t,x,\tau,0)\left(\int_\tau^t\int_0^x u(\xi,v)d\varphi(\xi)d\psi(v)\right)^2 d\varphi(\tau)-\frac{1}{2}\int_0^t\int_0^s c''_{\varphi(s)\varphi(\tau)}(s,x,\tau,0)\times$$

$$\times\left(\int_\tau^s\int_0^x u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2 d\varphi(\tau)d\varphi(s)-\frac{1}{2}\int_0^t\int_0^x c''_{\varphi(\tau)\psi(y)}(t,y,\tau,0)\left(\int_\tau^t\int_0^y u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2\times$$

$$\times d\psi(y)d\varphi(\tau)+\frac{1}{2}\int_0^t\int_0^x\int_0^s c'''_{\varphi(s)\varphi(\tau)\psi(y)}(s,y,\tau,0)\left(\int_\tau^s\int_0^y u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2 d\varphi(\tau)d\psi(y)d\varphi(s)-$$

$$-\int_0^t\int_0^x\int_0^s c'_{\varphi(\tau)}(s,y,\tau,0)\left(\int_\tau^s u(\xi,y)d\varphi(\xi)\right)\left(\int_0^y u(s,v)d\psi(v)\right)d\varphi(\tau)d\psi(y)d\varphi(s)+\frac{1}{2}\int_0^x c'_{\psi(z)}(t,x,0,z)\times$$

$$\times\left(\int_0^t\int_z^x u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2 d\psi(z)-\frac{1}{2}\int_0^t\int_0^x c''_{\varphi(s)\psi(z)}(s,x,0,z)\left(\int_0^s\int_z^x u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2 d\psi(z)\times$$

$$\times d\varphi(s)-\frac{1}{2}\int_0^x\int_0^y c''_{\psi(y)\psi(z)}(t,y,0,z)\left(\int_0^t\int_z^y u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2 d\psi(z)d\psi(y)+$$

$$+\frac{1}{2}\int_0^t\int_0^x\int_0^y c'''_{\varphi(s)\psi(y)\psi(z)}(s,y,0,z)\left(\int_0^s\int_z^y u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2 d\psi(z)d\psi(y)d\varphi(s)-$$

$$-\int_0^t\int_0^x\int_0^y c'_{\psi(z)}(s,y,0,z)\left(\int_0^s u(\xi,y)d\varphi(\xi)\right)\left(\int_z^y u(s,v)d\psi(v)\right)d\psi(z)d\psi(y)d\varphi(s)+$$

$$+\frac{1}{2}\int_0^t\int_0^x c''_{\varphi(\tau)\psi(z)}(t,x,\tau,z)\left(\int_\tau^t\int_z^x u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2 d\psi(z)d\varphi(\tau)-\frac{1}{2}\int_0^t\int_0^x\int_0^y c'''_{\varphi(\tau)\psi(y)\psi(z)}(t,y,\tau,z)\times$$

$$\times\left(\int_\tau^t\int_z^y u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2 d\psi(z)d\psi(y)d\varphi(\tau)-\frac{1}{2}\int_0^t\int_0^x\int_0^s c'''_{\varphi(s)\varphi(\tau)\psi(z)}(s,x,\tau,z)\times$$

$$\times\left(\int_\tau^s\int_z^x u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2 d\varphi(\tau)d\psi(z)d\varphi(s)+\frac{1}{2}\int_0^t\int_0^x\int_0^s\int_0^y c^{(IV)}_{\varphi(s)\varphi(\tau)\psi(y)\psi(z)}(s,y,\tau,z)\times$$

$$\times\left(\int_\tau^s\int_z^y u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2 d\psi(z)d\varphi(\tau)d\psi(y)d\varphi(s)-\int_0^t\int_0^x\int_0^s\int_0^y c''_{\varphi(\tau)\psi(z)}(s,y,\tau,z)\times$$

$$\times\left(\int_\tau^s u(\xi,y)d\varphi(\xi)\right)\left(\int_z^y u(s,v)d\psi(v)\right)d\psi(z)d\varphi(\tau)d\psi(y)d\varphi(s). \qquad (6)$$

Taking into account formulas (3), (4), (5), (6) and condition c), from (2) we have

$$\int_0^t\int_0^x u^2(s,y)d\psi(y)d\varphi(s)+\frac{1}{2}\int_0^t\int_0^x\left[-a'_{\varphi(s)}(s,y,0)\left(\int_0^s u(\xi,y)d\varphi(\xi)\right)^2-2c(s,y,0,0)\times\right.$$

$$\left.\times\left(\int_0^s u(\xi,y)d\varphi(\xi)\right)\left(\int_0^y u(s,v)d\psi(v)\right)-b'_{\psi(y)}(s,y,0)\left(\int_0^y u(s,v)d\psi(v)\right)^2\right]d\psi(y)d\varphi(s)+$$

$$+\frac{1}{2}\int_0^t\int_0^x\int_0^s\int_0^y\left\{-\frac{a''_{\varphi(\tau)\varphi(s)}(s,y,\tau)}{\psi(y)-\psi(0)}\left(\int_\tau^s u(\xi,y)d\varphi(\xi)\right)^2-2c''_{\varphi(\tau)\psi(z)}(s,y,\tau,z)\left(\int_\tau^s u(\xi,y)d\varphi(\xi)\right)\times\right.$$

$$\left.\times\left(\int_z^y u(s,v)d\psi(v)\right)-\frac{b''_{\psi(y)\psi(z)}(s,y,\tau)}{\varphi(s)-\varphi(0)}\left(\int_z^y u(s,v)d\psi(v)\right)^2\right\}d\psi(z)d\varphi(\tau)d\psi(y)d\varphi(s)+$$

$$+\frac{1}{2}\int_0^t\int_0^x\left[a'_{\varphi(\tau)}(t,y,\tau)\left(\int_\tau^t u(\xi,y)d\varphi(\xi)\right)^2 d\psi(y)d\varphi(\tau)+\int_0^t\int_0^x b'_{\psi(z)}(s,x,z)\left(\int_z^x u(s,v)d\psi(v)\right)^2 d\psi(z)\times\right.$$

$$\times d\varphi(s)+\frac{1}{2}\int_0^x a(t,y,0)\left(\int_0^t u(\xi,y)d\varphi(\xi)\right)^2 d\psi(y)+$$

$$\frac{1}{2}\int_0^t b(s,x,0)\left(\int_0^x u(s,v)d\psi(v)\right)^2 d\varphi(s)+$$

$$+\frac{1}{2}c(t,x,0,0)\left(\int_0^t\int_0^x u(\xi,v)d\psi(v)d\varphi(\xi)\right)^2-$$

$$-\frac{1}{2}\int_0^t c'_{\varphi(s)}(s,x,0,0)\left(\int_0^s\int_0^x u(\xi,v)\,d\psi(v)\,d\varphi(\xi)\right)^2 d\varphi(s)-$$

$$-\frac{1}{2}\int_0^x c'_{\psi(y)}(t,y,0,0)\left(\int_0^t\int_0^y u(\xi,v)\,d\psi(v)\,d\varphi(\xi)\right)^2 d\psi(y)+\frac{1}{2}\int_0^t\int_0^x c''_{\varphi(s)\psi(y)}(s,y,0,0)\times$$

$$\times\left(\int_0^s\int_0^y u(\xi,v)\,d\psi(v)\,d\varphi(\xi)\right)^2 d\psi(y)\,d\varphi(s)+$$

$$+\frac{1}{2}\int_0^t\int_0^x c''_{\varphi(\tau)\psi(z)}(t,x,\tau,z)\left(\int_\tau^t\int_z^x u(\xi,v)\,d\psi(v)\,d\varphi(\xi)\right)^2 d\psi(z)\,d\varphi(\tau)$$

$$\frac{1}{2}\int_0^t\int_0^x\int_0^y c'''_{\varphi(\tau)\psi(y)\psi(z)}(t,y,\tau,z)\left(\int_\tau^t\int_z^y u(\xi,v)\,d\psi(v)\,d\varphi(\xi)\right)^2 d\psi(z)\,d\psi(y)\,d\varphi(\tau)-$$

$$-\frac{1}{2}\int_0^t\int_0^x\int_0^s c'''_{\varphi(s)\varphi(\tau)\psi(z)}(s,x,\tau,z)\left(\int_\tau^s\int_z^x u(\xi,v)\,d\psi(v)\,d\varphi(\xi)\right)^2 d\varphi(\tau)\,d\psi(z)\,d\varphi(s)+$$

$$+\frac{1}{2}\int_0^t\int_0^x\int_0^s\int_0^y c^{(IV)}_{\varphi(s)\varphi(\tau)\psi(y)\psi(z)}(s,y,\tau,z)\left(\int_\tau^s\int_z^y u(\xi,v)\,d\psi(v)\,d\varphi(\xi)\right)^2 d\psi(z)\,d\varphi(\tau)\,d\psi(y)\,d\varphi(s)=$$

$$=\int_0^t\int_0^x f(s,y)u(s,y)\,d\psi(y)\,d\varphi(s)\,. \tag{7}$$

By virtue of conditions a), b), c) and d), from (7) we have

$$\int_0^t\int_0^x u^2(s,y)\,d\psi(y)\,d\varphi(s)\le\left|\int_0^t\int_0^x f(s,y)u(s,y)\,d\psi(y)\,d\varphi(s)\right|. \tag{8}$$

On the right side of inequality (8), applying the Cauchy-Bunyakovsky inequality, we obtain

$$\int_0^t\int_0^x u^2(s,y)\,d\psi(y)\,d\varphi(s)\le\left(\int_0^t\int_0^x f^2(s,y)\,d\psi(y)\,d\varphi(s)\right)^{\frac{1}{2}}\left(\int_0^t\int_0^x u^2(s,y)\,d\psi(y)\,d\varphi(s)\right)^{\frac{1}{2}}.$$

Hence it follows that

$$\left(\int_0^t\int_0^x u^2(s,y)\,d\psi(y)\,d\varphi(s)\right)^{\frac{1}{2}}\le\left(\int_0^t\int_0^x f^2(s,y)\,d\psi(y)\,d\varphi(s)\right)^{\frac{1}{2}} \text{ at } (t,x)\in G\,.$$

From the last inequality, passing to the limit at $t\to\infty$ and $x\to\infty$ we get

$$\|u(t,x)\|_{L^2_{\varphi,\psi}(G)} = \left(\int_0^\infty\int_0^\infty u^2(s,y)d\psi(y)d\varphi(s)\right)^{\frac{1}{2}} \le \left(\int_0^\infty\int_0^\infty f^2(s,y)d\psi(y)d\varphi(s)\right)^{\frac{1}{2}} =$$
$$= \| f(t,x)\|_{L^2_{\varphi,\psi}(G)}. \quad (9)$$

Thus, the theorem is proved.

REFERENCES

1. Imanaliev M.I. Oscillations and stability of solutions of singularly perturbed systems. - Frunze: Ilim, 1974. - 352 p.

2. Asanov A. The denivative of a function by means of a increasing function. Fen Bilimleri Dergisi, Kyrgyz-Turkish Manas University, 1. 18-64 (2001).
– C. 18-64.

3. Iskandarov S. On belonging to the space of the solution of the Volterra integral equation //Proc. report All-Union. conf. by the asymptotic method in the theory of singularly perturbed equations. - Alma-Ata: Science, 1979. - Part 1. - S. 150-151.

4.Abdukarimov, A. M. On the quadratic integrability of solutions of linear integral equations of the Volterra-Stieltjes type on infinite domains / A. M. Abdukarimov // Bulletin of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic. - 2018. - No. 1. - P. 105-111. – DOI 10.52448/16948173_2018_105. – EDN SGEKVU.

5.Asanov A., Abdukarimov A.M. On the quadratic integrability of solutions of partial differential equations on infinite domains //Vestn. KGNU. - Bishkek, 2001. - Issue. 6. – P. 80–84.

6.Asanov A., Abdukarimov A.M. On quadratic integrability of solutions of linear two-dimensional integro-differential equations with partial derivatives on infinite domains //Vestn. Osh State University. Ser.phys.-math. Sciences. - Osh, 2003. - Issue. 7. – P. 35–40.

7.Asanov A. On a class of systems of Volterra integral equations of the first kind. // Functional analysis and its applications., 1983.- T. 17, Issue 4.- P.73-74.

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QUASI-LINEAR PARABOLIC EQUATION WITH A DISCONTINUOUS INITIAL FUNCTION

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In this section we construct asymptotic representations for a solution of quasi-linear parabolic equation, when the solution of the corresponding degenerate problem is a discontinuous function.

Key words: quasilinear parabolic equation, excited problem, solution, multiple lines, asymptotic expansion, function, discontinuous function.

Бул макалада тиешелүү козголгон маселенин чечими үзгүлтүктүү функция болгондо, квазисызыктуу параболалык теңдемени чечүүдө асимптотикалык ажыроолор тургузулат.

Урунттуу сөздөр: квази сызыктуу параболалык теңдеме, козголгон маселе, чечим, бир нече сызык, асимптотикалык ажыроо, функция, үзгүлтүктүү функция.

В данной статье строятся асимптотические представления для решения квазилинейного параболического уравнения, когда решение соответствующей вырожденной задачи является разрывной функцией.

Ключевые слова: квазилинейное параболическое уравнение, возбужденная задача, решение, кратные прямые, асимптотическое разложение, функция, разрывная функция.

1. In the strip $\Pi_T\{(t,x)|0 < t \le T, \quad -\infty < x < \infty\}$ let us consider the problem

$$L_\epsilon u \equiv \epsilon^2 \frac{\partial^2 u}{\partial x^2} - \phi'_u(u)\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0, \quad u(0,x) = f(x). \qquad (1)$$

The function ϕ will be assumed to be infinitely differentiable, uniformly bounded for bounded values of the argument, $\phi(0) = \phi'(0) = 0$, $\phi''(0) = 1$, and $\phi''(u) \ge \phi_0 > 0$ for any u. Let the function $f(x)$ be continuous and uniformly bounded for

$x \neq 0$ and possess uniformly bounded derivatives of any order for $x \neq 0$, which have finite limiting values as $x \to 0$ and $x \to +0$.

A long with Problems A_ϵ, we consider in the strip Π_T Problem A_0,

$$A_0 \equiv \frac{\partial u}{\partial t} + \phi_u'(u)\frac{\partial u}{\partial x} = 0, \qquad u(0,x) = f(x) \tag{2}$$

As is known, Problem A_0 may have no solution in the class of differentiable functions even for an arbitrarily smooth initial function. In order for the problem (2) to be solvable for a continuous or piecewise continuous initial function, it is necessary to seek its solution in the class of discontinuous functions.

As it follows from O. A. Oleinik's work [1], for an arbitrary bounded measurable initial function $f(x)$ a solution of Problem A_ϵ is infinitely dif ferentiable for $t > 0$. At the same time, a solution of Problem A_ϵ may appear to be a discontinuous function even for an infinitely smooth initial function, and for a discontinuous initial function a solution of the problem may be continuous for $t > 0$. In connection with these specific peculiarities, an asymptotic, as $\epsilon \to 0$, representation of a solution of Problem A_ϵ may or may not possess terms of boundary layer character; moreover, boundary layer terms of expansion can describe both the ordinary differential equations and equations of parabolic type.

Hence, the structure of an asymptotic, as 0, representation of a solution of Problem A, depends essentially on the properties of the solution of Problem A_0. It is not difficult to prove the following

Theorem 1. If $f(x)$ is a function, infinitely differentiable and bounded along with its derivatives of any order, and a solution of Problem A_0 is infinitely differentiable in the strip Π_T, then for the solution of Problem A_ϵ we can construct a uniform, asymptotic as $\epsilon \to 0$ expansion of the type

$$U(t,x,\epsilon) \sim \sum_{k=0}^{\infty} \epsilon^{2k} u_{2k}(t,x),$$

and for that expansion in the strip Π_T the estimate

$$\left\| u(t,x) - \sum_{k=0}^{N} \epsilon^{2k} u_k\,(t,x) \right\|_{C^2} \le C\epsilon^{2N+2}$$

is valid.

Thus, the occurrence of terms of boundary layer character in an asymptotic expansion of the solution of Problem A_ϵ is due to the occurrence of sin- gularities in the solution of Problem A_0. In the present section we describe an asymptotic representation of the solution of Problem A_ϵ, in the case, where a solution of Problem A_0, is a discontinuous function for $0 \le\ t \le T$.

Thus, let the initial function satisfy the above-formulated conditions, pro- vided $f(-0) > f(+0)$. Let the solution of the problem (3), (4) be an infinitely differentiable function everywhere in $\overline{\Pi}_T$, except for an infinitely smooth line $x = x_p\,(t)$, $x_p(0) =\ 0$ at the points of which the solution of Problem A_0 and any its derivatives have finite limiting values as $x \to\ x_p(t) - 0$ and $x \to\ x_p(t) + 0$. As is known (see, e.g., [51]), on the line $x = x_p\,(t)$ the relation

$$x_p'(t) = \frac{\phi\left(u_0\big(t, x_p(t)+0\big)\right) - \phi\left(u_0\big(t, x_p(t)-0\big)\right)}{u_0\big(t, x_p(t)+0\big) -\ u_0\big(t, x_p(t)-0\big)} \qquad (3)$$

is fulfilled, where $u_0(t,x)$ is a solution of Problem A_0.

In what follows, for the sake of simplicity we will assume that the identity $x_p(t) = 0$ holds; the consideration of the general case somewhat complicates technical details of the construction and is not a matter of principle.

2. An asymptotic expansion of the solution of Problem A_ϵ will be sought in the form

$$U(t,x,\epsilon) \sim \sum_{k=0}^{\infty} \epsilon^{2k} u_{2k}\,(t,x) + \sum_{k=0}^{\infty} \epsilon^{2k} u_k\,(t, x/\epsilon^2) \qquad (4)$$

The function $u_0(t,x)$ is assumed to be known, and the functions $u_{2k}(t,x)$ for $k \ge\ 1,\ \ \vartheta_k(t,\xi)$ are to be defined.

The functions $u_{2k}(t,x)$ must ensure the closeness of the asymptotic ex- pansion and the solution of Problem A_ϵ, everywhere outside the neighbor- hood of the line of discontinuity of the solution of Problem A_0.

The functions $\vartheta_k(t,\xi)$ will be constructed under the assumption that they are of boundary layer character of variation as $|\xi| \to \infty$ and guarantee along with their derivatives of the first order the continuity of the asymptotic expansion (4).

A solution of Problem A_0 under the condition (3) is defined uniquely. Characteristica of the equation (2) are straight lines intersecting on the line $x = 0$. The function $u_0(t,x)$ is a solution of the functional equation

$$u_0 = f(x - \phi'(u_0)t), \qquad (5)$$

which shows that the derivative of the function $u_0(t,x)$, a) with respect to the variable x for $x \neq 0$ can be found by the formula

$$u'_{0x}(t,x) = \frac{f'(x - \phi'(u_0)t)}{1 + t\phi''(u_0)f'\big(x - t\phi'(u_0)\big)} = \frac{f'(x^0)}{1 + t\phi''(f(x^0))f'(x^0)}$$

In particular, from the condition of solvability of the equation **(5)** it follows that in case, the initial function is smooth, the smoothness of the solution of Problem A_0 cannot be violated for $t \in [0, T_0)$, where

$$T_0 = \min_{-\infty<x<\infty} [-\phi''(f(x))f'(x)]^{-1}$$

Using the standard techniques, we can see that the functions $u_{2k}(t,x)$ for $k \geq 1$ are defined as solutions of the equations

$$L_0 u_{2k} \equiv \frac{\partial u_{2k}}{\partial t} + \frac{\partial}{\partial x}[\phi'(u_0)u_{2k}] = \frac{\partial^2 u_{2k-2}}{\partial x^2} - \sum_{s=0}^{2k} \frac{\phi^{(s+1)}(u_0)}{s!}$$

$$\sum_{|\alpha|+\beta=2k} u_{\alpha_1} u_{\alpha_2} \ldots u_{\alpha_s} \frac{\partial u_\beta}{\partial x} \equiv F_{2k}(t,x), \qquad (6)$$

satisfying the zero-initial condition. Here, each of the indices $\beta, \alpha_1, \alpha_2, \ldots, \alpha_s$ may be any even integer from 0 to $2k-2$. $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_s$. It can be easily seen that solutions of the equations (6) are, generally speaking, discontinuous for $x = 0$.

Let us now pass to the construction of the functions $\vartheta_{2k}(t,\xi)$ which are defined separately for $\xi < 0$ and $\xi > 0$. Using again the well-known proce- dure of constructing the asymptotic expansions, we find that the functions $\vartheta_{2k}(t,\xi)$ are obtained as solutions of the following differential equations:

$$\frac{\partial^2\vartheta_0}{\partial\xi^2} - \phi'\left(u_0^{\pm} + \vartheta_0\right)\frac{\partial\vartheta_0}{\partial\xi} = 0 \qquad (7)$$

$$L_0 u_{2k} \equiv \frac{\partial^2\vartheta_{2k}}{\partial\xi^2} - \frac{\partial}{\partial\xi}\left[\phi'\left(u_0^{\pm} + \vartheta_0\right)\vartheta_{2k}\right] = u_{2k}^{\pm}\phi''\left(u_i^{\pm} + \vartheta_0\right)\frac{\partial\vartheta_0}{\partial\xi}\phi_{2k}(t,\xi); \quad (8)$$

here $u_{2k}^{-} = \ lim_{x\to-0}\, u_{2k}(t,x)$, $u_{2k}^{+} = \ lim_{x\to+0}\, u_{2k}(t,x)$, $u_{2k}^{\pm} = u_{2k}^{-}$ for $\xi <\ 0$, $u_{2k}^{\pm} = u_{2k}^{+}$ for $\xi > 0$, and the function $\phi_{2k}(t,\xi)$ is represented as a sum of a finite number of summands, each containing as a multiplier at least one of the functions $\vartheta_{2i}(t,\xi)$ or ${\vartheta'}_{2i\xi}(t,\xi)$ for $i < k$. The equations (7) and (8) are solved separately for $\xi < 0$ and $\xi >\ 0$; for $\xi = 0$ we impose the following conditions of continuity of:

- the asymptotic expansion

$$u_{2k}(t,-0) + \vartheta_{2k}(t,-0) =\ u_{2k}(t,+0) + \vartheta_{2k}(t,+0); \qquad (9)$$

- the first derivatives of the asymptotic representations:

$$[\vartheta_0(t,-0)]'_\xi -\ [\vartheta_0(t,+0)]'_\xi = 0,$$

$$[\vartheta_{2k}(t,-0)]'_\xi -\ [\vartheta_0(t,+0)]'_\xi = [u_{2k-2}(t,+0)]'_x -\ [u_{2k-2}(t,+0)]'_x, k \geq 1. \quad (10)$$

Lemma 1. The equation (7) possesses a solution of boundary layer type, which satisfies the conditions (9), (10) and tends exponentially to zero as $|\xi| \to 0$.

Proof. In equation (7), for $\xi < 0$ and $\xi >\ 0$ we change the unknown functions $z^- = u_0^- + \vartheta_0$, *and* $z^+ = u_0^+ + \vartheta_0$, respectively. It is readily seen that both equations take the same form, and their coefficients coincide for $\xi = 0$ by virtue of the condition (9).

For $\infty\ < \xi < \infty$, let us consider the equation

$$z''_{\xi\xi} - \phi'(z)z'_\xi = 0 \qquad (11)$$

and show that it has a solution, tending to u^- as $\xi \to -\infty$ and to u^+ as $\xi \to +\infty$. Bearing this in mind, we integrate both parts of equation (11) with respect to the variable ξ and obtain $z'_\xi - \phi(z) + \mathrm{C(t)} = 0$. Then we - choose a constant $C(t)$ of integration in such a way that the derivative of the function $z(t,\xi)$ would tend to zero as $z \to u_0^-: C(t) = \phi(u_0^-)$. By virtue of the condition **(3)**, $\partial z/\partial\xi \to 0$ as $z \to u_0^+$. Thus, the function $z(t,\xi)$ satisfies the equation $z'_\xi = \phi(z) - \phi(u_0^-) =$

$= \phi(z) - \phi\,(u_0^+)$. The function $\psi(z) = \phi(z) - \phi\,(u_0^+)$ is convex and vanishing for $z = u_0^-$ and $z = u_0^+$. Every solution of the equation under consideration satisfies the equality

$$\xi = \int_{z(t,0)}^{z(t,\xi)} \frac{\partial\omega}{\psi(\omega)} = \int_{z(t,0)}^{z(t,\xi)} \frac{\partial\omega}{\phi(\omega) - \phi(u_0^-)}.$$

The values $z = u_0^-$ and $z = u_0^+$ are first order zeros of the function **(2).** Hence the integral on the right-hand side of the last equality diverges as $z \to u_0^-$ and $z \to u_0^+$, that is, the function $z(t,\xi)$ takes the values u_0^- and u_0^+ for none of the finite value ξ, if $z(t,0) \neq u_0^\pm$. This means that a solution of equation (11) having the above-mentioned limiting values as $\xi \to \pm\infty$ exists $\xi \in (-\infty,\infty)$ and is defined uniquely by its valuation for $\xi = 0$. Because function $\psi(z)$ for $u_0^+ < z < u_0^-$ is of constant sign, every such solution is given in terms of the monotonically decreasing function of the variable ξ; note that $\partial z/\partial\xi$ vanishes for none of the value ξ. Since $z''_{\xi\xi} = \phi'(z)\psi(z)$, $\phi'(u_0^-) > 0$, $\phi'(u_0^+) < 0$, the function $z(t,\xi)$ has only one point of inflection $\xi = C_0(t)$, and therefore $\phi'(z(t, C_0(t))) = 0$. We fix the integral curve with inflection for $\xi = 0$, and let $z = \bar{z}(t,\xi)$ of that curve. Then the function $z = \bar{z}(t, \xi - C_0(t))$, where $C_0(t)$ is an arbitrary smooth function, defined for all $t \in [0,T]$, satisfies equation (11) and has an inflection for $\xi = C_0(t)$. Therefore $C_0(t)$ can be interpreted as the constant of integration distinguishing a unique solution of equation (7):

$$\vartheta_0(t,\xi) = \begin{cases} \bar{\mathrm{z}}(\mathrm{t}, \xi - \mathrm{C}_0(\mathrm{t})) - \mathrm{u}_0^-, \xi < 0 \\ \bar{\mathrm{z}}(\mathrm{t}, \xi - \mathrm{C}_0(\mathrm{t})) + \mathrm{u}_0^+,\ \xi > 0 \end{cases} \tag{12}$$

Obviously, the condition (10) is fulfilled for any choice of the function $C_0(t)$; thus, in constructing a zero approximate asymptotic expansion $u^{(0)} = u_0(t,x) +$ $+ \vartheta_0(t, x/\epsilon)$ one constant of integration (being a function of the parameter t)

remains still undetermined. The proof presents no difficulties when solutions of equation (7), satisfying the conditions $0 < \vartheta_0(t, 0) < u_0^- - u_0^+$, tend exponentially to zero.

Lemma 2. Solutions of the boundary layer type equation (8) exponentially tending to zero as $|\xi| \to 0$ and satisfying the conditions (9), (10**)** exist. For every fixed k the set of functions $\{\vartheta_s(t,\xi)\}$, $s = 0, 1, \ldots, k$ is defined to within one integration constant $C_k(t)$ which is a function of the parameter t; moreover, each of the integration constants $C_s(t)$, $s = 0,1, \ldots, k$ is defined as a solution of some linear, for $s \geq 1$, first order ordinary differential equation. The initial value $C_s(0)$ for the solution of each of these equations is arbitrary.

Proof. The existence and exponential tending to zero as $|\xi| \to 0$ of solutions of equations **(7)** is obvious. By virtue of the above-described properties of the function $\vartheta_0(t,\xi) = \overline{\vartheta_0}(t, \xi - C_0(t))$, the solution of either equation (8) which tends to zero as $|\xi| \to 0$, exists and depends on the two constants of integration being the functions of the variable t. Let us consider equation (8) for $k = 1$. Obviously, we shall have

$$\frac{\partial \vartheta_2}{\partial \xi} - \phi'\left(u_0^\pm + \vartheta_0\right)\vartheta_2 = \Phi_2^\pm(t,\xi), \tag{13}$$

$$\Phi_2^\pm(t,\xi) = \int_{\pm\infty}^{\xi} \left\{\frac{\partial \vartheta_0}{\partial t} + [\,\phi'\left(u_0^\pm + \vartheta_0\right) - \phi'\left(u_0^\pm\right)] \left(\frac{\partial u_0}{\partial x}\right)^\pm \right.$$

$$\left. + \phi''\left(u_0^\pm + \vartheta_0\right)\left[u_2^\pm + \eta\left(\frac{\partial u_0}{\partial x}\right)^\pm\right]\frac{\partial \vartheta_0}{\partial \eta}\right\} \partial\eta,$$

where the indices " $-$ " and " $+$ " are chosen for $\xi < 0$ and $\xi > 0$, respectively. A solution of equation (13) is defined by means of two constants of integration. Condition (9) can be satisfied by choosing only one of the constants. As is follows from (13), no choice of the second constant of integration can satisfy that condition for the function $\vartheta_2(t,\xi)$; only the remaining, for the time being unknown constant of integration $C_0(t)$ can satisfy the condition (9). From equation (13) and condition (10) we have

$$\frac{\partial u_0(t,+0)}{\partial x} - \frac{\partial u_0(t,-0)}{\partial x} = \int_{-\infty}^{0} \frac{\partial \vartheta_0}{\partial t} d\xi + \int_{0}^{\infty} \frac{\partial \vartheta_0}{\partial t} d\xi + \bar{\Phi}_2(t), \qquad (14)$$

where the function $\bar{\Phi}_2(\mathrm{t})$ is defined by means of limiting, as $x \to \pm 0$, values of the function $u_0(t,x)$ and its derivative with respect to the variable x. Using the representation (12) for the function $\vartheta_0(t,\xi)$, form the last equation it is not difficult to get for the function $C_0(t)$ the differential equation $C_0'(t) = q_0(t, C_0)$, whose right-hand side is the continuous function and satisfies the Lipschitz condition in the second argument for all $t \in [0, T_0)$. It follows from the relations (8) and (13) that the conditions (9) and (10) will be fulfilled for any choice of the initial condition for $\xi = 0$ for the solution of equation (13). The lemma is proved for the case $k = 1$.

To prove the lemma for the general case $k \geq 2$, we write the relation for the jump of first derivatives of the function $\vartheta_{2k}(t,\xi)$ with respect to the variable ξ for $\xi = 0$ in the form

$$\frac{\partial \vartheta_{2k}(t,-0)}{\partial \xi} - \frac{\partial u_{2k}(t,+0)}{\partial \xi} = \int_{-\infty}^{\infty} \frac{\partial \vartheta_{2k-2}}{\partial t} d\xi + q_k(t), \qquad (15)$$

where $q_k(t)$ is the given function? Using the general type of the solution of equation **(8)** for $k \geq 2$ and performing integration with respect to the variable ξ, we can get a differential equation for the integration constant $C_{2k-2}(t)$:

$$a(t)C_{2k-2}'(t) = \rho_k(t)C_{2k-2}(t) + q_k(t), \qquad (16)$$

where

$$a(t) = \int_{-\infty}^{\infty} exp\left[\int_{0}^{\xi} \phi'(u_0^{\pm} + \vartheta_0)\partial\eta\right] d\xi \geq a_0 > 0,$$

$p_k(t), q_k(t)$ are the known smooth functions.

Note that the conditions (9) and (10) are fulfilled for any choice of the initial value for the solution of equation (16). Moreover, in the approximation $u^{(2k)}(t,x,\epsilon)$, the integration constant. $C_{2k}(t)$ and all the initial values $C_{2s}(0)$, $\mathrm{s} \leq \mathrm{k} - 1$, remain still undetermined.

Remark 1. Note that we have completely constructed a formal asymptotic expansion of the solution which, in fact, is the asymptotic residual expansion of the equation. By no choice of values of the constants $C_{2s}(0)$ can one achieve that the given expansion would satisfy the initial condition uniformly for all $|x| < \infty$. At the same time, it is obvious that in order that the constructed by us formal expansion approximate the exact solution for $t > 0$, it is necessary to determine these constants uniquely, as long as the choice of constants enables us to determine the structure of the solution in the neighborhood of the line of discontinuity of the solution of the degenerate problem.

Lemma 3. Let $\alpha = const, 0 < \alpha < 1/2$. *For* $|x| \geq \delta_0 = \epsilon^{1-2\alpha}$ the inequality $\left|u(t,x) - u^{(N)}(t,x)\right| \leq M\epsilon^{\alpha}$ holds.

Lemma 4. For $|x| \geq M\delta_0$ the estimate $\left|u(t,x) - u^{(N)}(t,x)\right| \leq M\epsilon^{2N+2}$ is valid.

Proof. Let $p > 0$ be some number. Let us construct a function $f_1(t,x)$ possessing the following properties: $f_1(t,-\rho) = f_1(t,\rho) = 0$; $f_1(t,x) \geq 1$ for $|x| \geq m_1 p, m_1 = const$,

$$\left|\rho^s \frac{\partial^1 f_1}{\partial x^s \partial t^{l-s}}\right| \leq M; \quad \frac{\partial f_1}{\partial t}\phi'\left(u^{(N)}\right)\frac{\partial f_1}{\partial x} = q_1(t,x) \leq 0 \; for \; |x| > \rho.$$

It is not difficult to see that if $\rho \gg \epsilon ln\epsilon$, then the function exists. Consider an auxiliary function $z_1(t,x,\epsilon) = e^{-\alpha_2 t} f_1(t,x) z(t,x,\epsilon)$ which for $x > \rho$ satisfies the equation

$$L_1 z_1 \equiv \epsilon^2 \frac{\partial^2 z_1}{\partial x^2} - \phi'(u)\frac{\partial z_1}{\partial x} - \frac{\partial z_1}{\partial t} - \alpha_2(t,x,\epsilon) z_1 = \Phi_1(t,x,\epsilon)$$

where $\alpha_2(t,x,\epsilon) \geq 1$ for $|x| \geq \rho$, and the function $\Phi_1(t,x,\epsilon)$ is written in an obvious manner. Note that by virtue of the properties of coefficients of the asymptotic expansion the constant α_2 can be chosen independent of ϵ. For $t = 0, |x| \geq \rho$ the condition $|z_1(0,x,\epsilon)| \leq M\epsilon^{2N+2}$ is fulfilled for the function $z_1(t,x,\epsilon)$.

Suppose that the function $z_1(t,x,\epsilon)$ at some point $P_1(t_1,x_1)$ of the half- strip $x \geq \rho, 0 < t \leq T$ reaches its largest (for that half-strip) positive value. Since the equality $zf'_{1x} + f_1 z'_x = 0$ is fulfilled at the point P_1, that is $f'_{1x} z'_x = -z(f'_{1x})^2/f_1$, the function $\bar{z}_1 = z_1 - \beta_0$, where β_0 is some constant, satisfies at that point P_1 the relation

$$L_1\bar{z}_1 \equiv \epsilon^2\frac{\partial^2\bar{z}_1}{\partial x^2} - \phi'(u)\frac{\partial\bar{z}_1}{\partial x} - \alpha_2\bar{z}_1 - \frac{\partial\bar{z}_1}{\partial t} = \beta_0\alpha_2(t,x,\epsilon) +$$

$$+e^{-\alpha_2 t}z[\epsilon_{1x}^2 - q_1(t,x,\epsilon)] - 2\epsilon^2 e^{-\alpha_2 t}(f'_{1x})^2/f_1 - e^{-\alpha_2 t}f_1F_n(t,x,\epsilon) -$$

$$- e^{-\alpha_2 t}z_{1x}^2\int_0^1 \phi''\left(u\theta + u^{(N)}\right)(1-\theta))\theta. \qquad (17)$$

Consider the value of the right-hand side of (17) at the point P_1. Denote

$$8e^{-\alpha_2 t_1}[\rho f'_{1x}] = d_1, \quad 4e^{-\alpha_2 t_1}\rho f'_{1x}\int_0^1 \phi''\left(u\theta + u^{(N)}\right)(1-\theta))d\theta = d_2,$$

$4e^{-\alpha_2 t_1}\rho^2 f''_{1xx} = d_3, \quad 4e^{-\alpha_2 t_1}\epsilon^{-(2N+2)}f_1F_n = d_4$

and choose a constant β_0 as follows:

$$\beta_0 = max\{\epsilon^2\rho^{-2}d_1f_1^{-1}z; \ \ \rho^{-1}d_2z^2; \ \ \epsilon^2\rho^{-2}d_3z; \ \ \epsilon^{2N+2}d_4\}.$$

For such a choice of the constant β_0, the expression $L_1\bar{z}_1$, is nonnegative at the point P_1, since $\alpha_2(P_1) \geq 1, q_1(P_1,\epsilon) \leq 0$. As it follows from equation (17), the largest value the function $z_1(t,x,\epsilon)$ reaches at the point P_1 is nonnegative, i.e.,

$$f_1(P_1)z(P_1,\epsilon) \leq e^{-\alpha_2 t_1}\beta_0. \qquad (18)$$

Suppose first that $\beta_0 = \epsilon^2\rho^{-2}d_1f_1^{-1}(P_1)z(P_1,\epsilon)$. Then inequality (18)
implies $f_1^2(\mathrm{P}_1) \leq \epsilon^2\rho^{-2}d_1e^{-\alpha_2 t_1}$. Therefore, by virtue of Lemma 3, $f_1(P_1)z(P_1,\epsilon) \leq M\epsilon^{1+\alpha}\rho^{-1}$.

If $\beta_0 = \rho^{-1}d_2z^2(P_1,\epsilon)$, then $f_1(P_1)z(P_1,\epsilon) \leq M\epsilon^{2\alpha}\rho^{-1}$. However, if $\beta_0 = \ \epsilon^2\rho^{-2}d_3z(P_1,\epsilon)$, then $f_1(P_1)z(P_1,\epsilon) \leq M\epsilon^{2+\alpha}\rho^{-2}$.

Finally, for $\beta_0 = \epsilon^{2N+2}d_4$ we obtain the inequality $f_1(P_1)z(P_1,\epsilon) \leq M\epsilon^{2N+2}$. Comparing these estimates and supposing $\alpha = (m_0+3)/(3m_0)$.

$m_0 \geq 6, \rho = \delta_0$, we can state that the estimate $z(t,x,\epsilon) \leq M\epsilon^{2+\alpha} = M\epsilon^{\alpha+3/m_0}$ holds for all $x \geq m_1\rho, \ \ 0 \leq t \leq T$.

Let now $z_2(t,x,\epsilon) = f_2(t,x)z(t,x,\epsilon)$, where the function $f_2(t,x)$ is con- structed in the half-strip $x \geq m_1\rho, \ \ 0 \leq t \leq T$ just in the same way as the function $f_1(t,x)$ in the half-strip $x \geq \rho, \ \ 0 \leq t \leq T$. Repeating for the function $z_2(t,x,\epsilon)$ the same arguments as for the function $z_1(t,x,\epsilon)$, we shall arrive at the conclusion that for $x \geq 2m_1\rho, \ \ 0 \leq t \leq T$, when the function $z_2(t,x,\epsilon)$ reaches at the the point P_2 its positive maximum, we have the estimate

$u(P_2) - u^{(N)}(P_2) \leq M\epsilon^{\alpha+9/m_0}$.

Repeating our reasoning successively in the half-strips $x \geq 3m_1\,\rho,\ x \geq 4m_1\,\rho$ and so on, we shall step by step raise the exactness of the estimate until we obtain the inequality $z(P_r, \epsilon) \leq M\epsilon^{2N+2}$. It is easily seen that a number of steps r, which we have to do for getting a final estimate, does not exceed the value $[(2N+2)]/3+1$.

The lower bound for the asymptotic expansion error can be found analogously.

Lemma 5. In the rectangle D, the estimate

$\omega_N(t,x,\epsilon) \leq M\epsilon^{2N+\alpha} + M\epsilon^2 exp\{-\gamma_0 t/[\epsilon^2\, In\epsilon^{-1}] - \gamma_0 x^2/[\epsilon^4\, In\epsilon^{-1}]\}$ (19) is valid, where γ_0 is a constant not depending on ϵ.

Theorem 2. Outside of the neighborhood $\Omega = \left\{(t,x)|0 \leq t \leq M\epsilon^2\, In\epsilon^{-1},\ |x| \geq M\epsilon^2 \sqrt{In\epsilon^{-1}}\right\}$ of the point where the line of discontinuity of the solution of the degenerate equation and the initial straight line meet, the inequality $\left|u(t,x,\epsilon) - u^{(N)}(t,x,\epsilon)\right| \leq M\epsilon^{2N+2}$ is valid.

Proof. Let $t_1 \geq M\epsilon^2\, In\epsilon^{-1}$. It follows from (19) that for any $a, b, a < b$ the inequality

$$\left|\int_a^b [u(t_1,x) - u^{(2N+3)}(t_1,x)]dx\right| \leq M\epsilon^{4N+6}$$

is fulfilled. Let x_1 be a point of the line $t = t_1$ at which the difference $u(t_1,x) - u^{(2N+3)}(t_1,x)$ reaches its largest positive value m_0. Let a, b be chosen in such a way that $b = x_1$, and on the interval $[a,b]$ the function $z(t,x,\epsilon) = u(t_1,x) - u^{(2N+3)}(t_1,x)$ is nonnegative. Since $\partial z/\partial x < m_1\epsilon^{-2}$ in the strip Π_T, the length of the interval $[a,b]$ is not less than $m_0\epsilon^2/m_1$ Therefore

$$\frac{m_0^2\epsilon^2}{2m_0} \leq \int_a^b z(t_1,x,\epsilon)dx \leq M\epsilon^{4N+6}$$

whence $m_0 \leq M\epsilon^{2N+6}$. Analogously one can prove the assertions of the lemma in the case of negative minimum of the function $z(t,x,\epsilon)$ for an arbitrary point lying outside the neighborhood of Ω.

REFERENCES

1.Oleinik O.A. and Kruzhkov S.N., Quasilinear parabolic equations of second order with several indepe№ndent variables. Uspekin Mat/Nauk 16(1961), №5, 115-155. (in Russian)

2.Moiseev N.N. Asymptotic methods in nonlinear mechanics. Nauka, Moskow, 1969. (in Russian)

3.Tikhonov A.N. Systems of differential equations containing small parameters //Math. Sat. - 1952. - V.31, No. 3. - P.575-586.

4.Tikhonov A.N. On systems of differential equations containing parameters// Mat. Sat. - 1950. - T.27 (69), No. 1. - P.147-156.

5.Turkmanov J.K. Asymptotics of the Solution of a Perturbed Equation with an Irregular Singular Point // Proceedings of the International Scientific and Practical Conference: "Problems of Mechanics and Applied Mathematics", dedicated to the memory of prof. Frankl F.I. T.2. Applied Mathematics - Bishkek: Republican Higher College, 1995. - V.2.: Applied Mathematics. -p.87-88.

6.Turkmanov J.K. On a class of perturbed differential equations with a weak singularity // Issled. by integro-differential. equations. - Bishkek: Ilim, 1987. - Issue 26. – P.143-147.

7.Turkmanov J.K., Agybaev A.S. On the asymptotic behavior of solutions of perturbed first-order ordinary differential equations with a regular singular point. //News of KSTU im. I. Razzakova. Series #2(62). Theoretical and applied scientific and technical journal, 2022. - P.174-181.

8.Turkmanov J.K., Agybaev A.S., Karynbaeva M.M. Some a priori estimates for a system of quasi-linear parabolic equations. // Herald of Institute of Mathematics of NAS of KR. - Bishkek, 2022. - No. 1. - P.108-120.

9.Turkmanov J.K., Karynbaeva M.M., Agybaev A.S., Solution of a perturbed nonlinear differential equation with a weak singular point. // Herald of Institute of Mathematics of NAS of KR. - Bishkek, 2022. - No. 2. - P. 81-92.

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MSC 70F99

SYSTEM OF DIFFERENTIAL EQUATIONS DESCRIBING REPELLING POINTS ON A SQUARE AND EMPTY CORNERS PHENOMENON

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Considered the system of ordinary differential equations with discontinuous right hand parts presenting mutual repelling of points charges in very viscous media within a square. The empty corners phenomenon is found for numbers of points exceeded 400 (constant of numerosity).

Keywords: mathematical model, repelling, ordinary differential equation, system of equations, phenomenon, constant of numerosity

Абдан жабышкак чөйрөдө чекиттердин түртүшүүнү чагылдыруучу, оң жактары үзгүл болгон кадимки дифференциалдык теңдемелер системасы каралат. Чекиттердин саны 400төн (көпчө туруктуусу) ашканда бош бурч кубулушу табылды.

Урунттуу сөздөр: математикалык модел, түртүшүү, кадимки дифференциалдык теңдеме, теңдемелер системасы, кубулуш, көпчө туруктуусу.

Рассматривается система дифференциальных уравнений с разрывными правыми частями, представляющая взаимное отталкивание точек в очень вязкой среде внутри квадрата. Обнаружено явление пустых углов для количеств точек, превышающих 400 (константа множественности).

*Ключевые слова:*математическая модель, отталкивание, обыкновенное дифференциальное уравнение, система уравнений, явление, константа множественности

Introduction

Distributions of continuous electrical (similar, repelling) charges on conductors were considered in many papers. Also, theorems on uniqueness of such distributions were proven. For instance, continuous charges on surfaces were considered in [1], ones on lines were done in [2]. We considered distribution of discrete electrical charges of same sign. We found that such distributions have other properties, including absence of uniqueness. For example, two charges have three stable arrangements within a regular triangle.

In [3] we put a general problem to investigate arrangements of discrete electrical charges of same sign, considered motion of a charge under sum of all repelling forces from other charges within the interior of a domain and along its boundary. In [4] we proved existence of solution of task for two moving charges on an axis; in [5] we also did for one immovable charge and one moving charge on a half-axis. We also proved existence of solution of evident task for two immovable charges and one moving charge on a segment. In [6] we considered an arbitrary number of charges on a segment. Here we consider repelling points within a square.

1. Stating of the problem

Denote $\boldsymbol{R}:=(-\infty,\infty)$, $\boldsymbol{R}_+:=[0,\infty)$, $\boldsymbol{S_2}:=[0,1]\times[0,1]$. Let the time $t\in\boldsymbol{R}_+$.

We will consider $n\geq 2$ points moving on the square $\boldsymbol{S_2}$. Denote their coordinates as $z_i(t)=\{x_i(t), y_i(t)\}$, $i=0..n$. Their initial positions (all are distinct) are given:

$$\{x_i(0), y_i(0)\}=\{x_{i0}, y_{i0}\},\ i=0..n. \tag{1}$$

They are mutually repelling by the Coulomb law.

Suppose that the media of $\boldsymbol{S_2}$ is very viscous. Then forces of inertia can be neglected and a force pushes a body in the media as well as it is immobile each moment. Also, suppose that the velocity of a body in the media is proportional to the force. Thus we obtain the following system of ordinary differential equations of the first order (the coefficient is 1)

$$F_k=\sum_{i\neq k}\{x_k(t)-x_i(t), y_k(t)-y_i(t)\}\left(\left(x_k(t)-x_i(t)\right)^2+\left(y_k(t)-y_i(t)\right)^2\right)^{-3/2};$$

$$\{x_k{}'(t), y_k{}'(t)\}=\begin{cases} F_k\ if\ \{x_k(t), y_k(t)\}\ is\ in\ the\ interior\ of\ S_2; \\ \quad projection\ of\ F_k\ on\ the\ side \\ \quad if\ \{x_k(t), y_k(t)\}\ is\ in\ the\ side\ of\ S_2; \\ 0\ if\ \{x_k(t), y_k(t)\}\ is\ in\ the\ corner\ of\ S_2; \end{cases}\quad t\in R_+ \tag{2}$$

We hope to prove existence of a solution of the initial value problem (1)-(2) by the method used in [4].

2. System of difference equations

We wrote the following program in pascal: (sides of S_2 are 700 pixels; #N means number of points; initial values are chosen randomly):

```
PROGRAM sab_023;
USES CRT, graph;
var hxy,vx,vy,dx,dy,dxy,dxy1,hxy1,z,z2,xj,yj,dxy2,dxyd: double;
i,j,nxy,it,nt,np,ihand,n_time,ik: longint;
var drv, mode,f,n: integer;
x,y:array[1..700] of double;
xn,yn:array[1..700] of integer;
begin {main}
drv:=0; mode:=VgaHi;
InitGraph(drv,mode,'c:\tp\bgi');
randomize;
SetTextStyle(0,0,2);
OutTextXY(30,20,'Tagaeva, 2023. Repelling #N charges on square');
OutTextXY(100,40,'(Wait a little)');
z:=700.; z2:=z/2.0;
np:=10; {nt:=500*n_time;} hxy:=1.0; hxy1:=hxy; nt:=1000; nxy:=#N;
for ik:=1 to nxy do begin  x[ik]:=z*random; y[ik]:=z*random;
xn[ik]:=round(x[ik]); yn[ik]:=round(y[ik]);
Setcolor(green); circle(xn[ik]+80,yn[ik]+70,2);
          end;
 for it:=0 to nt do
     begin {it} if it>np then hxy:=2.0*hxy1;
 if it>2*np then hxy:=4.0*hxy1;
     for i:=1 to nxy do
   begin {i=ix} vx:=0.; vy:=0.; for j:=1 to nxy do
```

```
begin if j<>i then begin
xj:=x[j]; {if xj>x[i]+z2 then xj:=xj-z; if xj<x[i]-z2 then xj:=xj+z;}
yj:=y[j]; {if yj>y[i]+z2 then yj:=yj-z; if yj<y[i]-z2 then yj:=yj+z;}
dxy2:=sqr(x[i]-xj)+sqr(y[i]-yj)+1.;
dxy1:=z/(dxy2*sqrt(dxy2)); if dxy1<sqr(z)/nxy{*0.5} then begin
dx:=(x[i]-xj)*dxy1; dy:=(y[i]-yj)*dxy1;
vx:=vx+dx; vy:=vy+dy; end;
end; end;
 x[i]:=x[i]+vx*hxy; if x[i]>z then x[i]:=z; if x[i]<0. then x[i]:=0.;
 y[i]:=y[i]+vy*hxy; if y[i]>z then y[i]:=z; if y[i]<0. then y[i]:=0.;
      end {i=ix};
for ik:=1 to nxy do begin xn[ik]:=round(x[ik]);
yn[ik]:=round(y[ik]) end;
{for ik:=1 to nxy do write(xn[ik]:5,yn[ik]:5); writeln;
      readln;}
         end {it};
Setcolor(white);
repeat
for ik:=1 to nxy do begin {circle(xn[ik]+80,yn[ik]+70,8);}
circle(xn[ik]+80,yn[ik]+70,6); circle(xn[ik]+80,yn[ik]+70,4);
circle(xn[ik]+80,yn[ik]+70,2) end;
delay(100);
until keypressed;
END.
```

3. Experiments with the system of difference equations

The more is #N, the more points are situated on sides of $\boldsymbol{S_2}$. It corresponds to the known fact that all charges are situated at the surface of the charged conductor. But since #N>400 there arise empty sectors in corners of $\boldsymbol{S_2}$, with the only point in the vertex. It contradicts the known fact that the density of surficial charges is proportional to the surface curvature. Hence, the found phenomena is specific for discrete charges.

Conclusion

The third example of implementation of effect of numerosity in irgöö-type processes [7] is found in the paper. It also is an example of the law of the transformation of quantity into quality.

REFERENCES

1. Kolahal Bhattacharya. On the Dependence of Charge Density on Surface Curvature of an Isolated Conductor // Physica Scripta, 2016, Volume 91, Number 3, pp.1-5.

2. http://teacher.pas.rochester.edu/PHY217/LectureNotes/Chapter4 /LectureNotes Chapter4.html\sharp318

3. Pankov P., Tagaeva S. Mathematical modeling of distribution of discrete electrical charges // Abstracts of the V International Scientific Conference “Asymptotical, To-pological and Computer Methods in Mathematics” devoted to the 85 anniversary of Academician M. Imanaliev / Ed. by Academician A.Borubaev. - Bishkek, 2016. – P. 58.

4. Тагаева С.Б. Исследование систем обыкновенных дифференциальных уравнений с разрывными правыми частями, описывающих отталкивание частиц // Вестник Жалал-Абадского государственного университета, 2016, № 1(32). – С.78-82.

5. Тагаева С.Б. Исследование систем обыкновенных дифференциальных уравнений с разрывными правыми частями, описывающих отталкивание частиц с различными зарядами // Естественные и математические науки в современном мире / Сб. статей по материалам XLIX междунар. научно-практ. конф. № 12 (47). Новосибирск: Изд. АНС «СибАК», 2016. 136 с. – С. 85-91.

6. Tagaeva S.B. Existence and stabilization of solution of system of differential equations describing arrangement of many discrete electrical charges on a segment //Интернет-журнал ВАК КР, 2017, № 1. - 6 с.

7. Kenenbaeva G.M., Tagaeva S.B. On constants related to effect of "numerosity" // Herald of Institute of Mathematics of NAS of KR, 2021, No. 1. - Pp. 10-16.

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ALGORITHM OF CONTROLLED MOTION OF STRETCHED OBJECT ON TOPOLOGICAL TORUS

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Supra algorithms of controlled motion of points in kinematical spaces were implemented. We proposed controlled motion of stretched objects in kinematical spaces. An algorithm of controlled motion of a square on topological torus is proposed in the paper.

Key words: axiomatization, topological space, kinematical space, velocity, motion, topological torus.

Мурда кинематикалык мейкиндиктердеги чекиттердин башкарылуучу кыймылынын алгоритмдери ишке ашырылган. Биз кинематикалык мейкиндиктерде созулган объектилердин башкарылуучу кыймылын сунуш кылдык. Макалада топологиялык шакек боюнча чарчынын башкарылуучу кыймылынын алгоритми сунуш кылынган.

Урунттуу сөздөр: аксиомалаштыруу, топологиялык мейкиндик, кинематикалык мейкиндик, ылдамдык, кыймылдоо, топологиялык шакек.

Ранее были реализованы алгоритмы управляемого движения точек в кинематических пространствах. Мы предложили управляемое движение растянутых объектов в кинематических пространствах. В статье предложен алгоритм управляемого движения квадрата по топологическому тору.

Ключевые слова: аксиоматизация, топологическое пространство, кинематическое пространство, скорость, движение, топологический тор.

1. Introduction

Definition of a kinematical space M (the metric ρ_K is the minimal time of passing between points) with controlled motion of points was introduced in [1], see Section 2 below.

We proposed controlled motion of stretched sets [2], see Section 3 below.

Section 3 contains an algorithm to implement such motion.

2. Definitions for motion of points with bounded velocity

As a codifying the ancient idea of controlled motion with bounded velocity, the notion of kinematical space was introduced [1].

D e f i n i t i o n 1. A computer program is said to be a **presentation** of a computer kinematical space if:

P1) there is an (infinite) metrical space X of points and a set X_1 of display-presentable points being sufficiently dense in X;

P2) the user can pass from any point x_1 in X_1 to any other point x_2 by a sequence of adjacent points in X_1 by their will;

P3) the minimal time to reach x_2 from x_1 is (approximately) equal of the minimal time to reach x_2 from x_1.

The space X is said to be a **kinematic space**; the space X_1 is said to be a **computer kinematic space**; this minimal time is said to be the **kinematical distance** ρ_X between x_1 and x_2; a sequence of adjacent points is said to be a **route**. Passing to a limit as X_1 tends to X we obtain the following.

There is a set K of **routes**; each route M, in turn, consists of the positive real number T_M (**time** of route) and the function $m_M : [0, T_M] \to X$ (**trajectory** of route);

(K1) For $x_1 \neq x_2 \in X$ there exists such $M \in K$ that $m_M(0) = x_1$ and $m_M(T_M) = x_2$, and the set of values of such T_M is bounded with a positive number below;

(K2) If $M = \{T_M , m_M(t)\} \in K$ then the pair $\{T_M , m_M (T_M - t)\}$ is also a route of K (the reverse motion with same speed is possible); (cf. P3).

(K3) If $M = \{T_M, m_M(t)\} \in K$ and $T^* \in (0, T_M)$ then the pair: T^* and function $m^*(t) = m_M(t)$ $(0 \leq t \leq T^*)$ is also a route of K (one can stop at any desired moment);

(K4) concatenation of routes for three distinct points x_1, x_2, x_3.

If there exists a kinematic consistent with the given metric then the metric space is said to be **kinematizable**.

Program [1] in *pascal* of controlled motion along Riemann surface of the function Log(z).

```
program Ln_z;
uses crt, graph; var drv, mode, x, y, x1, y1, xr, yr, d,
```

```
n0, n1, n2, n3: integer;  r, r1: real;  r_k: char;
procedure read_key;
{'NumLock' must be on} begin
r_k:=readkey; x1:=x; y1:=y;
Case r_k of  '6': x1:=x+d;   {right} '8': y1:=y+d;   {up}
'4': x1:=x-d;   {left} '2': y1:=y-d;   {down} end;
r1:=sqrt(x1*x1+y1*y1);
if (r1<100) and (r1>10) then begin
if (y>0) and (x*x1<0) then n3:=n3-(x1-x);
if (y<0) and (x*x1<0) then n1:=n1+(x1-x);
if (x>0) and (y*y1<0) then n2:=n2+(y1-y);
if (x<0) and (y*y1<0) then n0:=n0-(y1-y);
x:=x1; y:=y1; r:=r1
{passing to new point} end; end;
begin drv:=0; mode:=VgaHi; InitGraph(drv,mode,'');
x:=30; y:=30; r:=sqrt(x*x+y*y);
n0:=10; n1:=12; n2:=14; n3:=16; d:=4;
repeat
xr:=round(200*x/r); yr:=round(200*y/r);
cleardevice;
Circle(300+x,200-y,3); {Moving point}
line(300,200,300-xr,200+yr);
line(300+150,200,300+150+n0,200);
line(300,200-150,300,200-150-n1);
line(300-150,200,300-150-n2,200);
line(300,200+150,300,200+150+n3);
read_key; until (r_k='q') or (r_k='Q'); {Quit}  end.
```

3. Axiomatization of motion of stretched sets with bounded velocity

Consider the practical task. Let there be a thing and obstacles. It is necessary to move the thing to another place. Is possible? If yes then in what minimal time it can be done?

D e f i n i t i o n 2. There is a family P of connected subsets of the set X (we will call them **passes**); each pass has the positive **length (time)** and a family Q of connected

subsets of the set X (we will call them **things**).

(It means that a thing moves along a pass).

The space X is said to be a **generalized kinematic space**.

(G1) For each $x\in p\in P$ there exists such $q\in Q$ that $x\in q$ [a thing can be in each place of a pass].

(G2) For each $x_1\neq x_2\in X$ there exists such pass $p\in P$ that $x_1, x_2 \in P$ and the set of lengths of such p is bounded with a positive number below; this infimum is said to be the **generalized kinematical distance** ρ_X between points x_1 and x_2.

(G3) For each $q_1\neq q_2\in Q$ there exists such pass $p\in P$ that $q_1, q_2 \in P$ and the set of lengths of such p is bounded with a positive number below; this infimum is said to be the **generalized kinematical distance** ρ_X between things p_1 and p_2.

(G4) If $x_1, x_2 \in p_1$ and $x_2, x_3 \in p_2$ then there exists such pass $p_3\in P$ that $x_1, x_2, x_3 \in p_3$ and $length(p_3) \leq length(p_1)+ length(p_2)$.

If

(G5) For each $x_1\neq x_2\in X$ there exists such pass $p_{12}\in P$ that $length(p_{12}) =\rho_X (x_1, x_2)$

then the generalized kinematical space X is said to be **flat** (with respect to P).

If there exists a generalized kinematic (families P and Q) consistent with the given Hausdoff metric for Q then the metric space is said to be K-kinematizable.

4. Algorithm of motion of square on topological torus

All numbers are integer.

Given a topological torus obtained by gluing sides of $N\times N$-square: $(N+1)\equiv 0$; a list of obstacles $X[M]$, $Y[M]$, $M=1..K$. The initial position of the controlled square Q is $[0;L]\times[0;L]$, $L<M$. The goal is (U,V) for the point of the controlled

square *Q* with initial position *(0,0)*. All obstacles are out of the initial position of *Q*.

The commands of control are *C={N, E, S, W}*.

Algorithm.

Define function [gluing]

F(Z):= Z; If Z<0 then F(Z):= Z+N+1; If Z>N then F(Z):= Z-N-1;

Cycle { Cycle { XQ[P,T]:=P; YQ[P,T]:=T | T=1..N}| P=1..N};

[initial position of Q]

Repeat {Input C; DX:=0; DY:=0;

If C=N then DY:= 1; If C=E then DX:=1;

If C=S then {DY:=-1}; If C=W then {DX:=-1};

Cycle { Cycle { XQ1[P,T]:= F(XQ[P,T]+DX);

YQ1[P,T]:=F(YQ[P,T]+DY) | T=1..N}| P=1..N};

[attempt to shift to direction C];

OB:=true;

Cycle { Cycle { Cycle { If (XQ1[P,T]= X[M] and

YQ1[P,T]:= Y[M]) then OB:=false | T=1..N}| P=1..N} | M=1..K};

If OB then Cycle { Cycle { XQ[P,T]:= XQ1[P,T]);

YQ[P,T]:= YQ1[P,T] | T=1..N}| P=1..N} else

Output ("Obstacle!");

If (X[0,0]=U and Y[0,0]=V) then (Output ("Goal!"); Exit) }

5. Conclusion

We hope that the idea of this algorithm can be extended to other cases of controlled motion in kinematical spaces with obstacles.

REFERENCES

1. Borubaev A.A., Pankov P. S. Computer presentation of kinematic topological spaces. The Kyrgyz State National University, Bishkek, 1999, 131 p.

2. Zhoraev A.H. Axiomatization of kinematical spaces. Herald of Institute of Mathematics of NAS of KR, 2021, No. 1. - Pp. 16-21.

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OPTIMIZATION OF PRODUCTION AND THE SIZE OF IRRIGATED AND CULTIVATED SOWN AREAS AT THE LOWEST TOTAL COST

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The article develops an economic-mathematical model of the problem of determining the size of irrigated and cultivated sown areas and the volume of crops grown on the basis of a contractual condition, with minimal total costs. The performance of the model is demonstrated by a numerical example.

Key words: mathematical model, irrigated and cultivated sown areas, volume, agricultural crop, contractual terms, total costs.

Макалада сугат жана айдоо аянттарынын өлчөмүн жана келишимдик шарттын негизинде өстүрүлгөн айыл чарба өсүмдүктөрүнүн көлөмүн минималдуу жалпы чыгымдар менен аныктоо маселесинин экономикалык-математикалык модели иштелип чыккан. Моделдин иштеши сандык мисал менен көрсөтүлөт.

Урунттуу сөздөр: математикалык модель, сугат жана айдоо аянттары, көлөм, айыл чарба өсүмдүктөрү, келишимдик шарттар, жалпы чыгымдар.

В статье разработана экономико-математическая модель задачи определения размера орошаемых и освоенных посевных площадей и объема, выращенных сельхоз культур на основе договорного условия, при минимальных суммарных затратах. Работоспособность модели продемонстрирована на числовом примере.

Ключевые слова: математическая модель, орошаемые и освоенные посевные площади, объем, сельхоз культура, договорные условия, суммарные затраты.

The climatic features of Kyrgyzstan determine the development of predominantly irrigated agriculture, which is the most productive in the conditions of the Republic.

Many landowners have changed their attitude to agriculture. A significant step in this direction is the transfer of their land plots to irrigated land [1]. In accordance with this, land owners transfer less valuable agricultural land to more valuable types of land or to equivalent types of agricultural land. In addition, they expanded land plots, at the expense of waste land plots. In the course of solving this problem, waste land plots were turned into suitable ones for use in agriculture, through radical land improvement. The latter are reduced to hydrotechnical (drainage, irrigation) or agro-reclamation (uprooting, deep plowing, liming) and other measures.

Modeling these and other processes of activity of private owners, cooperatives and farms is associated with solving various optimization problems [2].

In the future, the activities of private owners, cooperatives and farms cannot be imagined without the use of economic and mathematical models and digital information technologies [3]. The use of economic and mathematical models and digital information technologies allows us to complete the task, to explore individual elements of agricultural activity.

In particular, using economic and mathematical models and methods [4,5] it is possible to achieve significant results in improving the organization of agricultural production and increasing the efficiency of the use of agricultural land, improving the quality of cultivated crops.

In this paper, we formulate an economic-mathematical model for determining the optimal size of irrigated sown areas and developed land plots and the volume of grown products, each type, with minimal total costs.

<u>Formulation of the problem</u>. Let the cooperative have a crop area of s (ha), which includes developed plots that can be used for growing k - types of agricultural products, $k \in K = \{1,2,...,p\}$. The cooperative has an agreement with the enterprise on the supply of agricultural products in the amount q_k, $k \in K$.

To fulfill the terms of the agreement, the cooperative used the sown areas and developed plots, as well as additional funds in the amount of D (conventional unit). We assume that the yield of each type of agricultural product grown on the sown area is known. Also known are the costs of the cooperative for the cultivation of products of each type in the unit size of the sown area c_k, $k \in K$, and in the unit size of the cultivated sown area $\bar{c}_k$, $k \in K$.

It is required to determine the optimal size of the sown and developed sown areas of the cooperative and the volume of grown products of each type so that the total costs for growing products in the sown and developed sown areas would be minimal and at the same time the terms of the contract would be fully satisfied.

Let us formulate a mathematical model of the problem.

Let us introduce the following notation:

k – is the index of the type of cooperative product, $k \in K$;

K – is a set of types of agricultural products, $K = \{1,2,\ldots,p\}$.

Known options:

s – the size of the sown area of the cooperative, together with the developed areas;

q_k – is the volume of agricultural products of each type (tons) specified in the contract with the enterprise, $k \in K$;

a_k – is the volume of the k-th agricultural product (in tons) obtained per unit of sown area (yield), $k \in K$;

b_k – is the volume of the k-th agricultural product (in tons) obtained per unit of cultivated sown area (yield), $k \in K$;

d_k – is the volume of the k-th agricultural product grown in the sown areas of the cooperative, $k \in K$;

$\bar{d}_k$ – the volume of the k-th agricultural product grown in the developed areas of the cooperative, $k \in K$;

c_k – the cost of growing products of the k-th type per unit size of the sown area of the cooperative, $k \in K$;

$\bar{c}_k$ – the cost of growing products of the k-th type per unit size of the developed sown area of the cooperative, $k \in K$, where $\bar{c}_k = c_k + c_0 + c$, $k \in K$,

c_0 – liming the unit size of the developed sown area of the cooperative,

c – deep plowing of a unit size of the cultivated sown area of the cooperative.

Required variables:

x_k – is the size of the sown area for the k-th type of agricultural products, $k \in K$;

z_k – is the size of the developed area for the k-th type of agricultural products, $k \in K$.

According to the accepted notation, the mathematical model of the problem of determining the optimal size of sown and developed areas and the volume of cultivation of each type of product can be represented as

Find the minimum

$$L(x, z) = \sum_{k\in K} c_k x_k + \sum_{k\in K} \bar{c}_k z_k \tag{1}$$

under conditions

$$\sum_{k\in K} x_k + \sum_{k\in K} z_k = s, \tag{2}$$

$$\sum_{k\in K} \bar{c}_k z_k = D, \tag{3}$$

$$a_k x_k = d_k, \quad k \in K, \tag{4}$$

$$b_k z_k = \bar{d}_k, \quad k \in K, \tag{5}$$

$$d_k + \bar{d}_k = q_k, \quad k \in K, \tag{6}$$

$$0 \le d_k \le q_k, \quad k \in K, \tag{7}$$

$$x_k \ge 0, \tag{8}$$

$$z_k \ge 0, \tag{9}$$

where $x = \left|x_k\right|_{|1||K|}$, $z = \left|z_k\right|_{|1||K|}$.

The conditions of problem (1)-(9) can be presented in the form of Table. 1.

Table. 1.

x_1	x_2	…	x_p	z_1	z_2	…	z_p	d_1	d_2	….	d_p	$\bar{d}_1$	$\bar{d}_2$	…	$\bar{d}_p$		
1	1	…	1	1	1	…	1									=	S_0
a_1								-1								=	0
	a_2								-1							=	0
		…								…						…	…
			a_p								-1					=	0
				b_1								-1				=	0
					b_2								-1			=	0

						…				…				…		…	…
							b_p								-1	=	0
								1		…		1				=	q_1
									1				1			=	q_2
		…				…				…				…		…	…
											1			…	1	=	q_p
									1							≤	q_1
										1						≤	q_2
		…				…				…				…		…	…
											1					≤	q_p
				$\bar{c}_1$	$\bar{c}_2$	…	$\bar{c}_p$			…				…		=	D
c_1	c_2	…	c_p	$-\bar{c}_1$	$-\bar{c}_2$	…	$-\bar{c}_p$	0	0	…	0	0	0	…	0	→	min

Let us give an example to test the performance of the formulated model and methods for solving it.

Consider a cooperative with a cultivated area along with a cultivated area of 75 thousand hectares, which is used to grow four types of crops, k=(1,2,3,4). For the development of the site of the cooperative, additional costs in the amount of 1000000 (conventional units) are required.

The cooperative has an agreement with the enterprise on the supply of agricultural products in the amount of q_k,, (tons) k=(1,2,3,4), i.e. q_k ={ q_1, q_2, q_3, q_4}={180000, 110000, 200000, 210000}.

To fulfill the terms of the agreement, the cooperative used sown areas and developed plots, as well as additional funds in the amount of 1,000,000 (conventional units). We assume that the yield of each type of agricultural product grown on the sown area is known, i.e. $a_k=(a_1, a_2, a_3, a_4)=(9,8,10,10)$ (c) and in developed areas, i.e. $b_k= (b_1,b_2,b_3,b_4)$=(9,8,10,10) (c). Also known are the costs of the cooperative (in conventional units) for the cultivation of products of each type per unit size of the sown area c_k, $k=(1,2,3,4)$, i.e. c_k, ={ c_1, c_2, c_3, c_4 }={*2400, 2430, 2561, 2074*}; expenses for payment of land reclamation measures i.e. for deep plowing, the unit size of the cultivated sown area $c=2000$ (in conventional units); liming of the size unit of the cultivated sown area of the cooperative c_0=300 (in standard units).

Let us calculate the costs of the cooperative for growing products of each type on the developed sown area, taking into account payment for deep plowing and liming of the developed sown area of the cooperative according to the formula $\bar{c}_k = c_k + c_0 + c$, $k=(1,2,3,4)$.

We have = $\bar{c}_k = \{\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4\} = \{4700, 4760, 4861, 4374\}$.

It is required to determine the optimal size of the sown and developed areas of the cooperative and the volume of cultivated products of each type so that the total costs of sown and developed sown areas would be minimal and at the same time the terms of the contract would be fully satisfied.

In accordance with the data above, the mathematical model of the problem will be written in the form

Find the minimum

$$L(x,z) = 2400x_1 + 2439x_2 + 2561x_3 + 2074x_4 + \\ + 4700z_1 + 4760z_2 + 4861z_3 + 4374z_4 \qquad (10)$$

under conditions

$$\sum_{k=1}^{4} x_k + \sum_{k=1}^{4} z_k = 75000,$$

$$4700z_1 + 4760z_2 + 4861z_3 + 4374z_4 = 1000000, \qquad (11)$$

$$9x_1 = d_1, \quad 8x_2 = d_2, \quad 10x_3 = d_3, \quad 10x_4 = d_4, \qquad (12)$$

$$9z_1 = \bar{d}_1, \quad 8z_2 = \bar{d}_2, \quad 10x_3 = \bar{d}_3, \quad 10z_4 = \bar{d}_4, \qquad (13)$$

$$d_1 + \bar{d}_1 = 180000, \quad d_2 + \bar{d}_2 = 110000, \quad d_3 + \bar{d}_3 = 200000, \quad d_4 + \bar{d}_4 = 210000, \qquad (14)$$

$$0 \le d_1 \le 180000, \; 0 \le d_2 \le 110000, \quad 0 \le d_3 \le 200000, \quad 0 \le d_4 \le 210000, \qquad (15)$$

$$x_k \ge 0, \quad k=1,2,3,4, \qquad (16)$$

$$z_k \ge 0, \quad k=1,2,3,4, \qquad (17)$$

We write the condition of problem (10)-(17) in the form of the following table.2.

Table 2.

x_1	x_2	x_3	x_4	z_1	z_2	z_3	z_4	d_1	d_2	d_3	d_4	$\bar{d}_1$	$\bar{d}_2$	$\bar{d}_3$	$\bar{d}_4$		
1	1	1	1	1	1	1	1									=	75000
9								-1								=	0
	8								-1							=	0
		10								-1						=	0
			10								-1				1	=	0
				9								-1				=	0
					8								-1			=	0
						10								-1		=	0
							10								-1	=	0
								1				1				=	180000
									1				1			=	110000
										1				1		=	200000
											1				1	=	210000
								1								≤	180000
									1							≤	110000
										1						≤	200000
											1					≤	210000
				5867	5954	4861	4374									=	1000000
2400	2430	2561	2074	4700	4760	4861	4374	0	0	0	0	0	0	0	0	→	*min*

Having solved the problem (10)-(17) using the methods given in [5,6], we determine the size of the sown and developed area and the volume of agricultural products of each type grown by the cooperative to fulfill the terms of the contract with the enterprise, i.e. we have

$X=\{x_1=20000;\quad x_2=13750;\quad x_3=19794;\quad x_4=21000;\quad z_3=206;$ $d_1=180000;\ d_2=110000;\quad d_3=197943;\quad d_4=210000;\quad \bar{d}_3=2057\}$ at which the value

$L(x,z)$ = 176659654 conv. units.

Therefore, it follows from the optimal solution that the cooperative has fulfilled the terms of the contract for agricultural products of the 1st type in the amount of 180,000 tons, the 2nd type in the amount of 110,000 tons, the 3rd type in the amount of 197,943 tons and the 4th type in the amount of 210,000 tons. On the developed site in the amount of 206 hectares received agricultural products of the 3rd type in the amount of 2057 tons which used 1,000,000 conventional units.

Thus, the total costs of the cooperative amounted to 176659654 conventional units.

REFERENCES

1. Kolpakova O.P., Mamontova S.A., Kovaleva Yu.P., Ivanova O.I. Optimization of the structure of sown areas based on the use of environmental criteria // Astrakhan Bulletin of Environmental Education. - 2020. No. 1 (55). –P.97-101.

2. Karachevskaya E.V. , Rogachev A.F. Modeling and evaluation of the economic efficiency of the functioning of the agro-pharmaceutical cluster of the Republic of Belarus // Modern Economy Success. - 2016. - N1. –C.87-110. –EDNWWYIUD..

3. Asankulova M., Zhusupbaev A. Mathematical model of optimization of production and import of agricultural products // Successes of modern science. - 2016, V.3, No. 5. - P.121-123.

4. Zhusupbaev, A. The task of determining the optimal volume of coal purchase and its transportation / A. Zhusupbaev, M. Asankulova, S. Iskandarov // Science, new technologies and innovations of Kyrgyzstan. - 2017. - No. 5. - P. 163-165. – EDN ZEZQXL.

5. Asankulova, M. Determination of the optimal area and project of construction of a housing building / M. Asankulova, A. J. Jolborsova, G. S. Iskandarova // Herald of Institute Mathematics of the National Academy of Sciences of the Kyrgyz Republic. - 2022. - No 1. - P. 152-156. – DOI 10.52448/16948173_2022_1_152. – EDN IZHJTH. ORCID iD: https://orcid.org/0000-0002-4209-6894

6. Jusupbaev, A. The problem of determining the size of the loan and the selection of productive breeds of animals / A. Jusupbaev, M. Asankulova, K. Maatov // Herald of Institute Mathematics of the National Academy of Sciences of the Kyrgyz republic. - 2021. - No 1. - P. 124-133.
– DOI 10.52448/16948173_2021_1_124. – EDN UKWNXM.

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APPLICATION OF THE δ-HOMOGENEOUS VES-PRODUCTION FUNCTION FOR THE ANALYSIS OF THE DEVELOPMENT OF ECONOMIC SYSTEMS

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Structural changes in the national economy can be determined on the basis of quantitative analysis, redistribution of labor and investment resources between the material, fund-forming and consumer sectors of the national economy. Observing the change in the proportions (the share of labor and investment from their total volume directed to the development of each sector) between sectors, we can talk about the transition processes taking place in the economy, as well as judge the direct contribution of each sector to the formation of the gross domestic product of the Kyrgyz Republic. The relevance of the study is due to the fact that the structure of the economy, acting as an independent factor, directly affects not only economic growth, but also determines the nature of economic development. The mathematical model of structural changes can be applied in the study of transients in the main branches of economic activity of the Kyrgyz Republic. The reforms carried out so far have not made it possible to overcome the complex structural imbalances in the economy inherited from the administrative and command system of the past. The recommendations made will be quantitative rather than qualitative, i.e. they will tell you how much and in what proportions labor and investment should be invested in the development of a particular sector of the economy.

Keywords: production function, variable elasticity, replacement of labor with capital, elasticity of labor replacement with capital.

Улуттук экономикадагы структуралык өзгөрүүлөрдү сандык талдоонун, Эмгек жана инвестициялык ресурстарды улуттук экономиканын материалдык, фонд түзүүчү жана керектөө секторлорунун ортосунда кайра бөлүштүрүүнүн негизинде аныктоого болот. Секторлор ортосундагы пропорциялардын (ар бир секторду өнүктүрүүгө багытталган эмгектин жана инвестициялардын жалпы көлөмүнөн) өзгөрүшүнө байкоо жүргүзүү менен экономикада болуп жаткан өткөөл процесстер жөнүндө айтууга болот, ошондой эле ар бир сектордун КР ички дүң продуктусун түзүүгө түздөн-түз кошкон салымы жөнүндө айтууга болот. Изилдөөнүн актуалдуулугу экономиканын структурасы көз карандысыз фактор

катары кароо менен байланышкан, түздөн-түз экономикалык өсүшкө гана эмес, ошондой эле экономикалык өнүгүүнүн мүнөзүн аныктайт. Структуралык өзгөрүүлөрдүн математикалык модели КР экономикалык ишмердигинин негизги тармактарындагы өткөөл процесстерди изилдөөдө колдонулушу мүмкүн. Буга чейин жүргүзүлгөн реформалар мурунку административдик-командалык системадан мураска калган экономикадагы татаал структуралык диспропорцияларды жеңүүгө мүмкүндүк берген эмес. Алар экономиканын тигил же бул секторун өнүктүрүүгө эмгектин жана инвестициялардын канча жана кандай пропорцияларында жумшалышы керектиги жөнүндө айтып беришет.

Урунттуу сөздөр: өндүрүштүк функция, өзгөрүлмө ийкемдүүлүк, эмгек капиталын алмаштыруу, эмгек капиталын алмаштыруу ийкемдүүлүгү.

Структурные изменения в национальной экономике можно определить на основе количественного анализа, перераспределения трудовых и инвестиционных ресурсов между материальным, фондообразующим и потребительским секторами национальной экономики. Наблюдая за изменением пропорций (долей труда и инвестиций от их общего объема, направляемых на развитие каждого сектора) между секторами, можно говорить о переходных процессах, происходящих в экономике, а также судить о непосредственном вкладе каждого сектора в формирование валового внутреннего продукта КР. Актуальность исследования связано с тем, что структура экономики, выступая как самостоятельный фактор, непосредственно воздействует не только на экономический рост, но и предопределяет характер экономического развития. Математическая модель структурных изменений может быть применена при исследовании переходных процессов в основных отраслях экономической деятельности КР. До сих пор проведенные реформы не позволили преодолеть сложные структурные диспропорции в экономике, унаследованные от административно-командной системы прошлого. Вынесенные рекомендации будут носить скорее количественный, чем качественный характер т.е. они расскажут о том, сколько и в каких пропорциях труда и инвестиций следует вкладывать в развитие того или иного сектора экономики.

Ключевые слова: производственная функция, переменная эластичность, замещение труда капиталом, эластичность замещения труда капиталом.

Introduction

The use of econometric models using the production function is due to the fact that the model under consideration can be applied in the study of transients in the main sectors of economic activity of the country. To date, the reforms carried out in the Kyrgyz Republic have not made it possible to overcome the complex structural imbalances in the economy inherited from the administrative command system of the past.

Currently, in many studies, the problem of the impact of structural shifts on sustainable economic development is carried out using a production function with a constant elasticity of replacement of the labor factor by the capital factor (production function – CES). However, this requires preliminary verification of the fulfillment of the assumptions of the constancy of the elasticity of the replacement of factors with retrospective data on the functioning of the economic system in question. The constancy of the elasticity of replacement, in our opinion, is a simplified approach, and it is necessary to consider a short period of the process. On the other hand, to a certain extent, the possibilities of constructing a model with production functions of the VES-function type (VES- variable elasticity substitution production function) are limited.

Statistical data on the functioning of economic systems published in the open press were used in the modeling [1].

Modeling methods

Production functions are used to analyze the influence of a number of main factors (inputs) on the results of the production process (outputs), since PF as a whole reflect fairly stable quantitative ratios between its inputs and outputs.

The complexity of economic systems, for the description of the functioning of which neoclassical production functions of the type CES- production function are used, does not always allow us to assert that the values of the elasticity of labor substitution with capital σ in the systems under consideration are constant, since this situation is not so common for real economic systems. It is also necessary to make additional assumptions justifying the possibility of using production functions of these types, describing particular cases of the functioning of these systems, for modeling their functioning, which ultimately reduces the accuracy of the obtained forecast estimates.

VES-functions (VES- variable elasticity substitution production function) are a type of neoclassical production functions that take into account changes in the values of elasticity of labor substitution by capital σ in economic systems. Currently, various variants of the analytical representation of the production function of the form VES- function are known.

Revankar [3] and a number of other authors accepted that the marginal rate of replacement of labor with capital γ is characterized by the following dependence on the capital ratio k of the economic system under consideration:

$$\gamma = \alpha + \beta k, \qquad \begin{cases} \beta > 0, \\ -{}^{\alpha}/_{\beta} < k. \end{cases}$$

Then σ is determined by the dependence:

$$\sigma(k) = 1 + \left({}^{\alpha}/_{\beta}\right) k^{-1} \begin{Bmatrix} \sigma(k) < 1, & \sigma(k) > 1 \\ \dfrac{d\sigma(k)}{dk} < 1, & \dfrac{d\sigma(k)}{dk} > 1 \end{Bmatrix}, \qquad \sigma < 0, \sigma > 0,$$

and the VES function has the form:

$$Y = Ae^{\lambda t}\left[(1+\beta)KL^{\beta} + \alpha L^{1+\beta}\right]^{1/_{1+\beta}}.$$

The value of γ can be estimated by the expression:

$$\gamma = k\left(\frac{1}{\alpha + \beta k} - 1\right), \qquad \begin{cases} 0 < \alpha < 1 \\ 0 < \alpha + \beta k < 1 \end{cases}$$

Where α, β are the parameters of the function.

Based on this, he obtained the following dependencies for the σ-function:

$$\sigma(k) = 1 - \frac{\beta k}{(\alpha + \beta k)^2 - \alpha}, \begin{Bmatrix} \sigma(k) < 1, & \sigma(k) > 1 \\ \dfrac{d\sigma(k)}{dk} < 1, & \dfrac{d\sigma(k)}{dk} > 1 \end{Bmatrix}, \beta < 0, \beta > 0, \quad (1)$$

$$Y = Ae^{\lambda t}K^{\alpha}L^{1-\alpha}e^{\beta k} \qquad (2)$$

The assumptions made by the authors mentioned above regarding the nature of the relationships between γ, σ and k ensure changes in the values of σ depending on the magnitude of k, as well as the fulfillment of the requirements imposed on neoclassical production functions in general.

The use of the above variants of VES-functions suggests an additional justification for the possibility of describing changes in the values of γ and σ by the accepted dependencies.

δ- homogeneous VES-production function

To construct δ-homogeneous production functions of the VES-function type, we identify the structure of the production function by solving the following system of differential equations:

$$\begin{cases} \dfrac{g'(k)}{g(k)} = \dfrac{\delta}{\gamma(k) + k} \\ \dfrac{\gamma'(k)}{\gamma(k)} = \dfrac{1}{k\sigma(k)} \end{cases}$$

Here δ is an indicator of the homogeneity of the production function; k is the stock ratio: $k = K/L$; $g(k)$ is a modified production function in the form:

$$Y = f(K,L) = L^{\delta} f(1,k) \to \frac{Y}{L^{\delta}} = f(1,k) = g(k)$$

$\gamma(k)$ is the marginal rate of labor replacement by capital:

$$\gamma(k) = \frac{\delta g(k) - kg'(k)}{g'(k)}$$

$\sigma(k)$ is the elasticity of labor replacement by capital for a δ-homogeneous production function:

$$\gamma(k) = \frac{\delta g(k) - kg'(k)}{g'(k)}$$

For a given value δ (it is assumed that the choice of the value $\delta \epsilon [0,1]$ is carried out according to a previously formulated optimization criterion), it is sufficient to construct a function $g(k)$, which can be defined by the following expressions, taking into account the structure of the function *$\sigma(k)$*:

$$\gamma(k) = b \cdot exp\left(\int_a^k \frac{dt}{\sigma(t)t}\right); \; g(k) = c \cdot exp\left(\int_a^k \frac{\delta dt}{\gamma(t)+t}\right),$$

where *a*,*b*,*c* are some positive constants.

As *$\sigma(k)$*, one can choose, for example, some continuous, including piecewise linear function.

The initial data for constructing a neoclassical δ-homogeneous production function of the VES-function type are the sets of values of output volumes $Y = \{Y_i\}$, where $Y = f(L, K)$, $(i = \overline{1, n})$ and the corresponding values $K = \{K_i\}$, $L = \{L_i\}$ in value or index calculus, characterizing the functioning of the economic system in question at each moment of time T_i for a certain time interval $[T_1, T_n]$. The values of the uniformity index δ_j: $\delta_j \in [0,1]$ are also set. According to these data, the values of the capital ratio of the economic system under consideration are determined: $k_i = {K_i}/{L_i}$ and the values of the function $g^{\delta_j}(k_i)$ at a fixed value of δ_j:

$$g^{\delta_j}(k_i) = {K_i}/{L_i^{\delta_j}}$$

The values of the function $g^{\delta_j}(k_i)$ can be ordered in ascending order of the values of k_i, then a series g_{lj} $(l = 1, n_l; n_l = 1, n)$ is formed. The values of g_{lj} can be approximated by the functions $\tilde{g}_{lj}$, which satisfy the requirements:

$$F_j = \sum_{l}^{n_l} \left(g_{lj} - \tilde{g}_{lj}\right)^2 \to min.$$

When constructing δ-homogeneous production functions of type (2) VES-functions, data characterizing the functioning of the economy of the Kyrgyz Republic in the periods from 2001 to 2021 were used.

Conclusion and conclusions

The resulting function has the form:

$$Y = 27{,}32 e^{0{,}21} K^{0{,}49} L^{0{,}51} e^{-3{,}324k}$$

The value $\sigma = 1 + \frac{\alpha k}{[b - \alpha k]^2 - b}$,

where α=3,324; b=0,52.

The approximation characteristics: s^2=0,19; dw=1,31; v=3,32; R^2=0,882, show that the obtained parameter estimates are statistically significant.

The analysis of the coefficient of elasticity of production by fixed assets shows the tendency of this indicator to decrease, and analagically to the behavior of the return on funds. The labor elasticity coefficient increased slightly. The total elasticity of factors is less than one, i.e. production increases more slowly than the costs of factors.

Comparison of the values allows us to conclude that the constructed production function of the VES-function type allows us to obtain a more accurate approximation from 70 to 85% of the values of the final product of the economic system Y and the modified production function $g(k)$ to the corresponding initial data.

The values of the standard deviation for the VES-function type production function constructed by the authors are less than for the previously developed CES-functions. This allows us to conclude that the proposed VES function gives a more "stable" approximation of the calculated values of the magnitude Y and the modified production function $g(k)$.

Based on the results of the conducted research , the following main conclusions can be drawn:

- The decision to prioritize investments in different sectors may not coincide with the optimal value required for overall socio-economic benefits. Any deviation from this optimum should be corrected by the government both through regulatory methods and through direct participation in the economic process.
- The Government of the country should take into account the existence of economic imbalances at the macro level and in the future it is necessary to coordinate the distribution of foreign investment resources at the state level. Optimization measures should be aimed at identifying economic imbalances at an early stage and taking measures to address them in a timely manner.
- The principle of economic well-being should offer an absolutely clear strategy for implementing structural reforms, in order to increase productivity and within the framework of managing imbalances.

The recommendations made will be more quantitative than qualitative in nature, i.e. they will talk about how much and in what proportions labor and investment should be invested in the development of a specific sector of the economy. I.e. at the macro level (industry level), and not at the micro level of a specific enterprise to which a particular a set of labor and investment resources. This is a property of the model, because thanks to this aggregation, we study the economy of the state as a whole, not reducing it to a purely sectoral section, but considering it in general.

REFERENCES

1. Statistical Yearbook of the Kyrgyz Republic 2001-2022. www.stat.kg.

2. Volnykh, E.V. Construction of a δ-homogeneous production VES-function [Text]/ Volnykh, E.V. Siberian Journal of Industrial Mathematics. - 2007. –Vol. X, No. 2 (30). – pp. 31-44.

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SIMULATION OF PARKING OPERATION

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The article considers the functioning of motor vehicle parking as a queuing system. Analytical dependencies for assessing the functioning of parking and the results of field studies of the average duration of parking are given. A numerical example shows the use of the above dependencies to analyze the functioning of parking.

Keywords: analysis of functioning, parking, duration of parking, queuing system, shopping center.

Макалада кезек күтүү тутуму катары унаа токтотуучу жайдын иштеши каралды. Унаа токтотуучу жайдын иштешин баалоо үчүн аналитикалык көз карандылыктар жана унаа токтотуунун орточо узактыгы боюнча табигый изилдөөлөрдүн натыйжалары келтирилген. Унаа токтотуучу жайдын иштешин талдоо үчүн берилген көз карандылыкты колдонуу менен сандык мисал көрсөтүлгөн.

Урунттуу сөздөр: иштешин талдоо, унаа токтотуучу жай, унаа токтотуунун узактыгы, кезек системасы, соода борбору.

В статье рассмотрено функционирование автотранспортной парковки как системы массового обслуживания. Приведены аналитические зависимости для оценки функционирования парковки и результаты натурных исследований средней продолжительности паркирования. На числовом примере показано использование приведенных зависимостей для анализа функционирования парковки.

Ключевые слова: анализ функционирования, парковка, продолжительность парковки, система массового обслуживания, торговый центр.

In all countries, especially in cities, the number of cars has increased significantly in recent years, while there is not only a problem with traffic jams, but also a problem with parking vehicles.

The Bishkek City Mayor's Office, together with the Ministry of Economy and Commerce of the Kyrgyz Republic and the PPP Center, recently announced its intention to implement a public-private partnership project "Development of automated parking space in Bishkek". But, in our opinion, the main requirements for potential operators of future parking lots are not justified by a detailed scientific survey of the parking problem (the actual position of the automobile load of Bishkek city).

The number of parking spaces that must be provided for during the construction of new facilities in the Kyrgyz Republic is regulated by SN KR 30-01:2020 "Planning and development of cities and urban-type settlements" [1].

The specified building standard determines the need for parking spaces depending on the characteristics of the object – the number of simultaneous visits, the number of employees, retail space, etc. For example, for retail facilities with an area of more than 200 m^2 per 100 m^2 of retail space, 5-7 parking spaces should be provided in the parking lot. But the regulatory approach to determining the required number of parking spaces has a number of drawbacks. The following factors affect the needs for parking spaces for institutions, organizations and shopping centers:

- the nature of the work of institutions and organizations;
- specialization of the shopping center (the presence of enterprises in the shopping center that attract and detain visitors for a long time, for example, cinemas, restaurants, entertainment venues);
- uniformity of demand for shopping center visits;
- distance from residential areas, public transport routes.

Assessing the need for parking spaces requires a comprehensive study of the parking process, which mainly depends on two factors – the intensity of demand for parking and the duration of parking.

Car parking can be considered as a system that meets the demand for parking vehicles and has limited opportunities to meet this demand. Therefore, it is possible to consider the parking process as a queuing system, where one parking space is a service channel, and the cars arriving at the parking lot will be an incoming stream of requirements. Considering parking as a queuing system, it is possible to give a quantitative assessment of its functioning.

The limiting probabilities of all parking conditions depending on the number of parking spaces, the intensity of demand and the duration of parking are expressed using the Erlang formula [2]. The Erlang formula is used quite widely, provides for the possibility of an unlimited queue and gives the probability that new demand will have to wait in line due to the use of all seats. The probability that *k* places are occupied by the service:

$$P_n = \frac{\left(\frac{l}{m}\right)^k \frac{1}{k!}}{\sum_{k=0}^{n} \frac{1}{k!}\left(\frac{l}{m}\right)^k} \tag{1}$$

where *l* is the density of the flow of applications (the intensity of demand for parking); *n* is the number of channels (the number of parking spaces); *k* is the number of available places; *m=1/t* is the intensity of the flow of service; *t* is the average time of service requirements in the system (the duration of parking).

Most often, instead of the $\frac{l}{m}$ – ratio, the concept of the reduced intensity of the flow of applications ρ is used, which is the number of applications received during the average service time of one application:

$$\rho = \frac{l}{m} \tag{2}$$

A special case of formula (1) is the probability P_0 that all serving channels are free:

$$P_0 = \frac{1}{\sum_{k=0}^{n} \frac{\rho^k}{k!}} \tag{3}$$

Knowing the probabilities $P_0, P_1, P_2, \dots, P_4$ it is possible to find the characteristics of the parking efficiency – the absolute *A*-capacity, the average number of cars in the parking lot and the probability of denial of service $P_{отказ}$.

The probability of denial of service of a newly received request (lack of free parking space):

$$P_{отказ} = P_0 \frac{\rho^n}{n!} \tag{4}$$

Parking capacity:

$$A = \lambda(1 - P_{отказ}) \tag{5}$$

Average number of free parking spaces (number of free channels):

$$\bar{N}_0$$

cars in the parking lot (number of channels engaged in maintenance):

$$= \sum_{k=0}^{n-1} (n-k)\, P_k = \sum_{k=0}^{n-1} \frac{\rho^k (n-k)}{k!} P_0 \tag{6}$$

Average number

$$N = \rho(1 - P_{\text{отказ}}) \tag{7}$$

Parking utilization rate:

$$K_n = \frac{N}{n} \tag{8}$$

To assess the functioning of the parking lot, it is necessary to know the intensity of the arrival of cars to the parking lot and the average duration of parking in the parking space. If the intensity of arrival is the input of the system under study and is determined by various external factors, then the average duration of parking is an internal characteristic of the queuing system.

To study these parameters, we conducted field surveys in the parking lot of the «GUM» shopping center in Bishkek. The study consisted of recording license plates of parked cars every 15 minutes. The probability density of the parking time distribution is shown in Fig. 1. The average parking time was 47.4 minutes (0.793 hours).

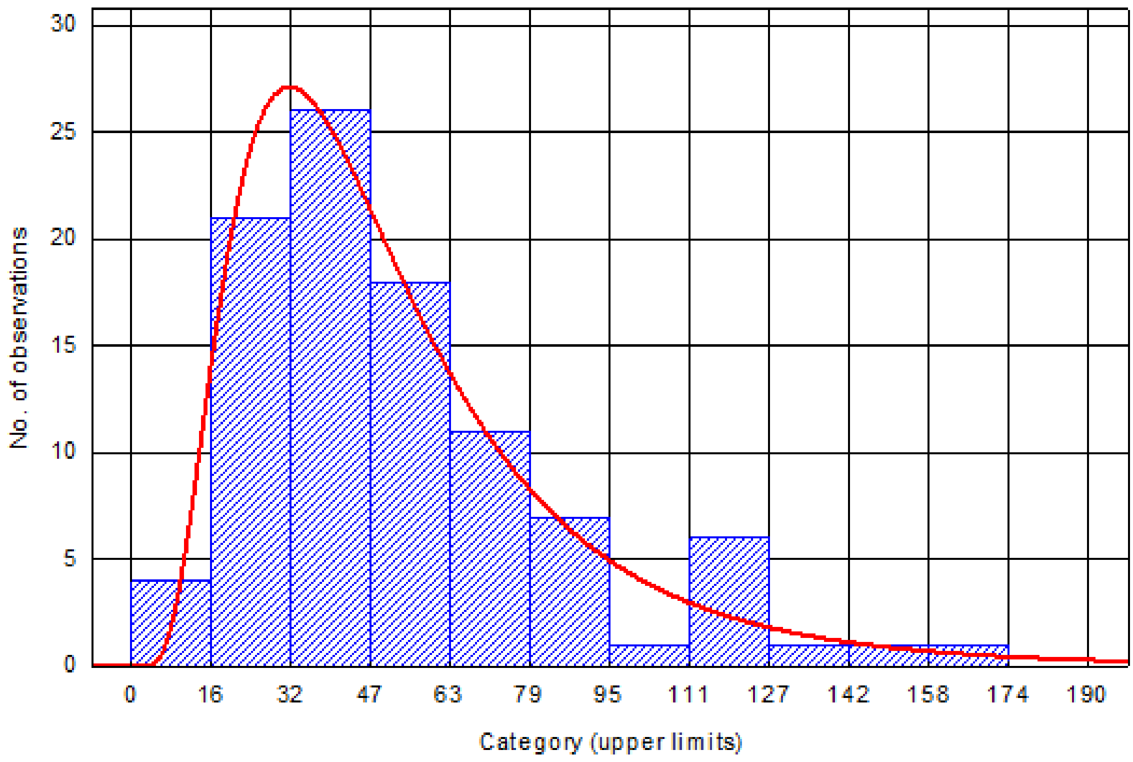

Fig. 1 Probability density of parking time distribution in the area of the «GUM» shopping center in Bishkek

Let's consider the use of the proposed method on the example of analyzing the functioning of parking at a shopping center with a capacity of 500 parking spaces and the intensity of demand for parking 600 vehicles / h. Divide the parking space into m= 5 segments with a capacity of n= 100 parking spaces. The calculation is given in tabular form (Table 1).

Table 1 – The results of calculating the parameters of parking operation

Parameter	Formula for calculations	Value
Demand for parking for a particular segment l_m	$\lambda_m = \frac{d}{m}$	120
The reduced intensity of the flow of applications ρ_m	$\rho_m = \frac{\lambda_m}{\mu}$	107.18
The probability that all service channels are free, P_0^m	$P_0^m = \frac{1}{\sum_{k=0}^{n} \frac{\rho_m^k}{k!}}$	$1.22 \cdot 10^{-46}$

The probability of denial of service in all segments of the parking $P_{отк}$	$P_{отк} = \left(P_0 \frac{\rho^k}{k!}\right)^m$	0.0044
The average number of free parking spaces	$\bar{N}_0 = \sum_{k=0}^{n-1} \frac{\rho^k (n-k)}{k!} P_0$	31.48
The average number of cars in the parking	$\bar{N} = \rho(1 - P_{отк})$	463.68
Parking utilization rate	$K_n = \frac{\bar{N}}{n}$	0.927

As you can see, parking with a capacity of 500 parking spaces satisfies the demand for parking with an intensity of 600 vehicles per hour, while the probability of no free parking space is less than 1%, and the available parking spaces are used by 92.7%.

Using the mathematical apparatus of the theory of queuing allows you to analyze the efficiency of parking, taking into account the intensity of demand for parking and the average time spent in the parking lot. In order to fully use the capabilities of this approach in design practice, it is necessary to perform additional studies to identify the dependencies of the intensity of demand for parking and the time spent in parking for various objects.

REFERENCES

1. SN KR 30-01:2020 "Planning and development of cities and urban-type settlements" http://cbd.minjust.gov.kg

2. Pleskunov, M.A. Theory of mass service: textbook / M.A. Pleskunov; Ministry of Science and Higher Education of the Russian Federation, Ural federal university. Yekaterinburg: Ural university Publishing House, 2022.— 264 p.

3. Wentzel E.S., Ovcharov L.A. Applied problems of probability theory. – M.: Radio and Communications, 1983. – 416 p.

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MATHEMATICAL PROBLEMS AT OLYMPIADS IN INFORMATICS

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The Institute participates in conducting National Olympiads in mathematics and informatic. Each Olympiad task in informatic includes any mathematical items evidently or latently. There are some permissions and bans on such items. Also, almost every task (besides mathematically formulated ones) means any knowledge on real life. Arising problems are discussed in the paper.

Keywords: informatic, mathematics, Olympiad, task, latent knowledge, special knowledge

Институт математика жана информатика боюнча Улуттук олимпиадаларды өткөрүүгө катышат. Информатика боюнча ар бир олимпиадалык маселе ачык же жашыруун түрдө ар кандай математикалык предметтерди камтыйт. Бул суроо боюнча кээ бир уруксаттар жана тыюулар бар. Ошондой эле, дээрлик ар бир маселе (математикалык жактан формулировка-лангандан тышкары) чыныгы жашоо боюнча кандайдыр бир билимди билдирет. Бул мака-лада келип чыккан көйгөйлөр талкууланат.

Урунттуу сөздөр: Информатика, математика, олимпиада, маселе, жашыруун билим, атайын билим

Институт участвует в проведении Национальных олимпиад по математике и инфор-матике. Каждая задача олимпиады по информатике открыто или скрыто касается некоторых математических понятий. По этому вопросу есть некоторые разрешения и запреты. Кроме того, почти каждая задача (помимо тех, которые сформулированы математически) подра-зумевает некоторое знание реальной жизни. В этой статье рассматриваются возникающие проблемы.

Ключевые слова: информатика, математика, олимпиада, задача, скрытые знания, специаль-ные знания

Introduction

Each year the Institute participates in conducting National Olympiads for school-children in mathematics and informatics to choose participants for annual

International Mathematical Olympiad (IMO) and International Olympiad in Informatics (IOI).

Each Olympiad task in informatic includes any mathematical items evidently or latently. So-called *Syllabus* provides all necessary knowledge "*included topics*" in mathematics to be involved in IOI tasks and "forbidden to be necessary" knowledge "*excluded topics*", and it must be taken in account in selection for IOI at National Olympiads. Also, almost every task (besides mathematically formulated ones) means any knowledge on real life. Arising problems are discussed in the paper.

In Section 2 we discuss various mathematical topics including ones which can be discovered by authors of tasks and jury before the Olympiad and by participants during the Olympiad.

Section 3 discusses "special knowledge" in science and technologies to create tasks and to be known by participants to solve such tasks [1].

Section 4 reviews the "latent general knowledge" of life and imagery ability [2] for successfully solving of the tasks.

2. Examples in mathematics

The following tasks use topics which are not mentioned or are explicitly excluded by *Syllabus*. Nevertheless, they can be posted and be solved successfully.

All "numbers" are integer below. [·] means also rounding down to integer.

A popular at initial stages of OIs and of mathematical Olympiads (with a fixed large *N*)

Task 1 "Last digit". Given integer number N in 2..10^{12}, what is the last digit of 7^N? Most of contestants will be successful in such a task.

If the contestant protests: *"Fermat's little theorem is not included in Syllabus that is why I could not solve this task*" then the jury may respond: "*You could count 7^2, 7^3, 7^4, 7^5,… and note a regularity*". That is, the ability to make small mathematical discoveries in addition to knowledge within *Syllabus* is meant at OIs.

Stereometrics is explicitly excluded in *Syllabus*. But

Task 2 "How many soms?". There are three distinct 1000×1000-squares A, B, C. Each square has its own coordinate system 0 ≤ x,y≤ 1000. The cost of passing from a point A(i,j) to points A(i+1,j) or A(i-1,j) or A(i,j+1) or A(i,j-1) (if they belong to the square), as well as for squares B and C, is 3 som. The cost of null-transportations A(i,j)↔B(i,j) or C(i,j)↔B(i,j) is 30 som. Given numbers x, y, u, v in 0..1000, find the minimal cost of passing from the point A(x,y) to the point C(u,v).

Obviously, any way of "common" search in the graph with $3{\cdot}1001^2$ vertices is hopeless. However, most of contestants will be successful in this task.

Remark. If the contestant protests: "*3D-Manhattan metrics is explicitly excluded in Syllabus that is why I could not solve this task*" then the jury may respond: "*Neither a 3D-space nor Manhattan metrics are mentioned in the text of task*".

Euler's formula for planar graphs is not mentioned in *Syllabus*.

Task 3 "Drawing". Call an intersection (point only) of two drawn segments as "node", the drawn segment between two nodes without inner nodes as "fragment" and a drawn triangle without drawn inner points as "domain". Let a triangle UVW be drawn, number N (nodes) is 3, number F (fragments) is 3, number D (domains) is 1. The operation "draw a segment splitting any domain into two domains" was executed some times. Given numbers N and D in 10..1000, N−2≤ D≤ 2N−6, find number F. If it is impossible, output 0. Example. 1st operation: draw segment UP, point P∈VW; 2nd operation: draw segment VQ, point Q∈UP; 3rd operation: draw segment WQ. The result is N=5; F=8; D=4.

After some attempts most of contestants will be successful in this task.

Remark. If the contestant protests: *"Euler's formula for planar graphs is not an included topic in Syllabus, 2022"* then the jury may respond: *"There are other ways to solve the task, constructing a sequence of operations yielding given N and D."*

“Calculus” is explicitly excluded in *Syllabus*. But we [3] proposed tasks of type

Task 4“Minimization”. Given a natural number F in 1..10^{300}*; find such number X that the expression H(X):=*X^3*+F/(X+1) is minimal; if there are some such numbers then output the greatest of them.*

The author of the task is obliged to calculate the derivative *H′(X)* and to prove that it increases for (real) *X>0*. But the contestant who does not know the notion “derivative” can guess that *H(X)* is a unimodal function. Then the algorithm is obvious.

“Non-trivial calculations on floating point numbers, manipulating precision errors; trigonometric functions” are explicitly excluded in *Syllabus*.

Kyrgyzstan and other countries have central-symmetric elements on their coat-of-arms. Corresponding modifications of the following task may be in their honor [4].

Task 5 “Symmetry”. Given integer numbers N in 2..2023, K in 1..64. The center of the regular 64-polygon is in (0;0), the 1st vertex is in (N;0), vertices are numbered counterclockwise. Find such integers X and Y that the Kth vertex is within the square (X, X+2)×(Y,Y+2). If there exist some such pairs output one of them.

The author’s solution of the task is a little rational-numbers-interval-analysis-soft. Trigonometric functions are not used in the solution; coordinates of vertices are calculated such as V[1]:=(N;0); V[17]:=(0;N); V[9]:=(V[1]+V[17])/|(V[1]+V[17])|*N … (Euclidean distances, Pythagorean theorem are included in *Syllabus*).

Such solution takes about half of hour to type even for an experienced contestant but they can write a program for floating point numbers X1, Y1 in a minute:

*Let X1:=N*cos(2.*pi/64.*(K-1)); Y1:=N*sin(2.*pi/64.*(K-1)); X:= floor(X1); If X1-X<0.5 then X:=X-1; Y:= floor(Y1); If Y1-Y<0.5 then Y:=Y-1; Output (X,Y).*

3. Examples involving knowledge in sciences

The following tasks were given by us.

Task 6 "Cattle" (velocity). There are five villages on a straight road. The car started from the beginning of the road at 5.00. From 6.00 to 7.00 the cattle go to the pasture along villages, so the car speed within villages is 6 km/h, otherwise the car speed is 60 km/h. Find the coordinates (in meters) of the car in M minutes after 5.00 am. Input: M < 216; coordinates of villages (km).

Task 7 "Weighing" (Gravitation, the lever law). Points ... −3, −2, −1, 0, 1, 2, ... are marked on a long ruler. The ruler is suspended in the middle, at the point 0. There is plenty of a flour, a light bag for flour, and weights of natural numbers (given) P kg and (given) Q kg. A bag of flour and weights can be hung on a ruler, only at marked points. There cannot be two loads at same point. It is required to weigh (given) Z kg of flour. Find the minimum possible length of the segment occupied by the weights.

Task 8 "Skew scale". Given natural numbers P, Q, Z in 1..2023. The (very light) bowls of the scale are suspended at distances P and Q (horizontally) from the suspension point of the scale. To weigh Z kg how many 1-kg-kettlebells are necessary? Example. P=20, Q=10, Z=1993 $\to$ 998.

Task 9 "Walls" (gravitation, properties of liquid). Given number K in 4..6. K thin rectangular walls were built on the plane in "north-south" or "east-west" directions (two walls can be intersected). It is raining, raining... How much water do the walls hold? ("zero" can also be). Input. The first row contains K. The next K rows contain five numbers X1, Y1, X2, Y2, Z in 1..10, the coordinates of the end-points and height of the i-th wall, i=1..K.

4. General, or common terminology and knowledge

Traditional notions are persons (participants, players, travelers), animals (it is better that their properties in the task be similar to their actual ones; hounds and moths with GPS; warms as mathematical tasks characters from ancient times), ways, roads, crosses, villages (as points or segments on highways) and cities (with rectangular street grid), walls (as segments on a plane), cars (as points), pixels …

We proposed voxels, timexels, 2D-printers, 3D-printers, future (3D+T)-printers for wider using [5].

Probably, the ancient

Task 10 "Heads-feet". Geese and cats together have (given) H heads and (given) F feet. How many geese and how many cats are there? If it is impossible, output 0 0

is popular because it is the first example of latent information.

Various natural [6] tasks can be composed on the notions of (ideal, thin, inextensible) rope and other idealized household items.

Task 11 "Rope". Given a rope of length L and a graph, all its arcs have lengths 1 and are thin tubes. Rope is arranged within arcs, without touching itself. Can Rope be drawn through all arcs without self-touching? If it is possible, find the minimal time to do it.

4. Conclusion

To make Olympiad tasks in informatics and mathematics more interesting and useful we propose to give tasks demanding knowledge in sciences and the ability, declared by us [7].

REFERENCES

1. Pankov P.S. Real Processes as Sources for Tasks in Informatics. *Olympiads in Informatics*, 2010, **4,** pp. 95-103.

2. Pankov P.S. Independent learning for open society. *Collection of papers as results of seminars conducted within the frames of the program "High Education Support"*. Foundation «Soros-Kyrgyzstan», Bishkek, 1996, issue 3, pp. 27-38.

3. Pankov P.S. Tasks in Informatics of Continuous Content. *Olympiads in Informatics*, 2013, **7,** pp. 101-112. *

4. Pankov P.S., Baryshnikov K.A. Representational Means for Tasks in Informatics. *Olympiads in Informatics*, 2009, **3,** pp. 101-111.

5. Pankov P.S., Imanaliev T.M., Kenzhaliev A.A. Automatic Makers as a Source for Olympiad Tasks. *Olympiads in Informatics*, 2021, **15,** pp. 75–82.

6. Pankov P.S. Naturalness in Tasks for Olympiads in Informatics. *Olympiads in Informatics: Country Experiences and Developments*, 2008, **2,** pp. 16-23.
7. Pankov P., Janalieva J., Naimanova A. Inductive and experimental studying of mathematical subjects (mathematical facts and notions which can be discovered independently), *LAP Lambert Academic Publishing*, Saarbrücken, 2015, 70 p.

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