Кыргыз Республикасынын Улуттук илимдер академиясы
Национальная академия наук Кыргызской Республики
National Academy of Sciences of Kyrgyz Republic
ISSN 1694-8173

КР УИА Математика институтунун Кабарлары

Вестник Института математики НАН КР

Herald of Institute of Mathematics of NAS of KR

№2

БИШКЕК – 2023 - BISHKEK

ISSN печатной версии: 1694-8173

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Herald of Institute of Mathematics of NAS of KR, 2023, No 1
Founder of the magazine: Institute of Mathematics of National Academy of Sciences of Kyrgyz Republic. The magazine is published since 2018.

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Website: https://math.kg/index.php/zhurnal
ISSN печатной версии: 1694-8173

MSC: 54E15

GENERALIZATION OF THE MAIN PRINCIPLES OF FUNCTIONAL ANALYSIS

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The main principles of functional analysis, such as the principle of continuation of linear continuous functionals, the principle of openness of linear continuous operators and the principle of uniform boundedness of a sequence of linear bounded operators, employ a key place in functional analysis and find multifold applications in various fields of mathematics.

Keywords: τ–metric, τ–norm space, τ–Banach space.

Үзгүлтүксүз сызыктуу функционалдарды улантоо принципи, үзгүлтүксүз сызыктуу операторлордун ачыктыгынын принципи жана чектелген сызыктуу операторлорунун катарынын бир калыптуу чектендирилгендигинин принципи сыяктуу болгон функционалдык анализдин баш принциптери функционалдык анализде маанилүү ролун ойношот жана ар түрдүү колдонмолонгондуктарына математиканын ар кайсы бөлүмдөрүндө ала жүрүшөт.

Урунттуу сөздөр: τ–метрика, τ–нормалаштырылган мейкиндик, τ–Банах мейкиндиги.

Главные принципы функционального анализа, такие как принцип продолжения непрерывных линейных функционалов, принцип открытости линейных непрерывных операторов и принцип равномерной ограниченности последовательности ограниченных линейных операторов, играют ключевую роль в функциональном анализе и находят множественные приложения в различных областях математики.

Ключевые слова: τ–метрика, τ–нормированное пространство, τ–банахово пространство.

These principles are generalized to the class of multi-normal spaces, which makes it possible to introduce multinorms in any locally convex linear topological space.

Let $R = (-\infty, +\infty), R_+ = [0, +\infty)$, and τ be an arbitrary cardinal number. By R_+^τ and R^τ we denote the Tihonoff product of τ copies of the spaces R_+ and R respectively (with natural topology). In the spaces R_+^τ and R^τ, the scalar addition and multiplication operations, as well as partial ordering, are defined naturally (coordinately). The space R^τ turns into the so-called ”Tychonoff” topological semifield, and R^τ is the positive cone of the semifield R^τ (see [3], [4]). It is proved (see [3], [4]) that any topological semifield can be embedded into a "Tychonoff" topological semifield R^τ.

Let $\{f_a: X \to Y_a, a \in A\}$ be the family of mappings of set X to the family of sets $\{Y_a: a \in A\}$. The mapping $f: X \to \prod\{Y_a, a \in A\}$ defined by the formula $f(x) = \prod\{f_a(x): a \in A\}$ is called the diagonal product of the mapping family $\{f_a(x): a \in A\}$ and is denoted by $\Delta\{f_a(x): a \in A\}$.

Definition 1. ([6]). Let X be a non-empty set. Mapping $\rho_\tau: X \times X \to R_+^\tau$ is called τ–metric or multimetric (if τ is not fixed) on X and the pair (X, ρ_τ) is a τ–metric or multimetric space if the following axioms are satisfied:

1. $\rho_\tau(x, y) = \theta$ if and only if $x = y$, where θ is a point in the space R_+^τ, where all coordinates consist of zeros;
2. $\rho_\tau(x, y) = \rho_\tau(y, x)$ for all $x, y \in X$;
3. $\rho_\tau(x, y) \leq \rho_\tau(x, z) + \rho_\tau(z, y)$ for all $x, y, z \in X$.

Any τ-metric on a set X produces uniformity U_{ρ_τ} and topology T_{ρ_τ}. For each neighborhood $O(\theta)$ of the point θ of the space R_+^τ, we define the set $V_{O_\theta} = \{(x, y): \rho_\tau(x, y) \in O(\theta)\}$.

Consequently, the family $\{V_{O_\theta}: O(\theta)$ runs through the base of neighborhoods of the point θ in the space $R^\tau\}$ forms the base of some uniformity U_{ρ_τ} on the set X. For any point $x \in X$ is defined as $(x) = \{y \in X : \rho_\tau(x, y) \in O(\theta)\}$. This set $\{G(x) : x \in X\}$ creates a topology T_{ρ_τ}, which coincides with the topology generated by the uniformity U_{ρ_τ}. Thus, the topological space (X, T_{ρ_τ}) is a Tychonoff space (see [5]).

If (X, U) is a uniform space of weight τ, then there exists a family of pseudo- metrics $\{\rho : \alpha \in A\}$ on X that are uniformly continuous on (X, U) and generate uniformity U with $|A| = \tau$. Further, the diagonal product $\rho_\tau = \Delta\{\rho_\alpha : \alpha \in A\}$ of this family of pseudometrics $\{\rho_\alpha : \alpha \in A\}$ is a τ–metric on X, while generating uniformity U.

Let (X, ρ_τ) be multimetric space and let $\{x_\alpha : \alpha \in M\}$ be the directionality in (X, ρ_τ). Let's assume that the directionality of $\{x_\alpha : \alpha \in M\}$ converges to a point

$x_0 \in X$ if for any neighborhood V_{O_θ} of point x_0, there exists an index $\alpha_0 \in M$ such that for all $\alpha \geq \alpha_0, \rho_\tau(x_\alpha, x_0) \in V_{O_\theta}$ is satisfied. The Cauchy (fundamental) directionality is defined similarly.

A multimetric space (X, ρ_τ) is called complete if every Cauchy directionality in (X, ρ_τ) converges.

Definition 2. ([2]). A mapping $\|\cdot\|_\tau: L_+ \to R_+^\tau$ of a linear space L (over the real number field) into a space R_+^τ is called a τ-norm or multinorm (if τ is not fixed) on the linear space L and a pair $(L, \|\cdot\|_\tau)$ is τ–normalized or multinormalized space if the following axioms are satisfied:

1. $\|x\|_\tau = \theta$ if and only if x is a zero element of linear space L and θ is a point of space R_+^τ, all coordinates of which are zero;
2. $\|\lambda \cdot x\|_\tau = |\lambda| \cdot \|x\|_\tau$ for any scalar $\lambda \in R$ and any $x \in L$;
3. $\|x + y\|_\tau \leq \|x\|_\tau + \|y\|_\tau$ for all $x, y, z \in L$.

If $(L, \|\cdot\|_\tau)$ is a multinormalized space and $\rho_\tau(x, y) = \|x - y\|_\tau$ then we obtain a τ–metric space, which in turn gives rise to a linear normalized space $(L, \|\cdot\|_\tau)$.

A complete τ-normalized space $(L, \|\cdot\|_\tau)$ is called a τ-Banach or a multiBanach space.

Example 1. Let *(L, T)* be an arbitrary locally convex linear topological space. Let $\mathcal{P}$ be a separating family of continuous seminorms on L generating the topology of T (for the existence of such a family of $\mathcal{P}$ seminorms see [7]). Let $|\mathcal{P}| = \tau$. Then the diagonal product of the family of half-norms $\mathcal{P}$, $\|\cdot\|_\tau = \Delta\{p : p \in \mathcal{P}\}$ is a τ-norm on L generating the topology T.

Inversely, if $(L, \|\cdot\|_\tau)$ is an arbitrary τ-normalized space and the mapping $\|\cdot\|_\tau: L \to R_+^\tau = \prod\{R_a: a \in A\}$ is a τ-norm, where $|A| = \tau$ and $\pi_a: R_\tau \to R_a$ is a projection for any $a \in A$, then assuming $p_a = \pi_a \cdot \|\cdot\|_\tau$, for each $a \in A$ we obtain a separating family of continuous half-norms $\mathcal{P} = \{p_a: a \in A\}$ on L, which gives rise to the topology of τ -normed space $(L, \|\cdot\|_\tau)$, and the topology will be locally convex.

Remark 1. Let $(X, \|\cdot\|_\tau)$ be an arbitrary τ–normalized space, $\|\cdot\|_\tau: X \to R_+^\tau =$

$\prod\{R_a : a \in A\}$, where $|A| = \tau$.

Let $\|\cdot\|_\tau{}' = \Delta\{q_b : b \in B\}$, which also satisfies all the axioms of the τ-norm.

1. Let $\|x\|_\tau{}' = \theta$, then $\Delta\{q_b(x) : b \in B\} = \theta$. This means that $q_b(x) = 0$ for all $b \in B$. Since the family of seminorms $\{q_b : b \in B\}$ separates the points of space X, i.e., for every non-zero $x \in X$ there is an element $b \in B$ such that $\prod q_b(x) \neq 0$. This is impossible, so x is the zero element of the linear space X. Inversely, if x is the zero element of linear space X, then $p_b(x) = 0$ for all $b \in B$. Then $\|x\|_\tau{}' = \Delta\{p_b : b \in B\} = \theta$;

2. $\lambda\|x\|_\tau{}' = \Delta\{q_b(x) : b \in B\} = \Delta\{|\lambda| q_b(x) : b \in B\} = |\lambda|\Delta\{q_b(x) : b \in B\} = \lambda\|x\|_\tau{}'$; $\|\lambda \cdot x\|_\tau{}' = \Delta\{q_b(\lambda \cdot x) : b \in B\} = \Delta\{|\lambda| \cdot q_b(x) : b \in B\} = |\lambda|\Delta\{q_b(x) : b \in B\} = |\lambda| \cdot \|x\|_\tau{}'$.

3. Consider $\|x + y\|_\tau' = \Delta\{q_b(x + y) : b \in B\} \leq \Delta\{q_b(x) + q_b(y) : b \in B\} = \Delta\{q_b(x) : b \in B\} + \Delta\{q_b(y) : b \in B\} = \|x\|_\tau' + \|y\|_\tau{}'$, so $\|x + y\|_\tau' \leq \|x\|_\tau' + \|y\|_\tau{}'$.

Let $\mathcal{P} = \{p_a : a \in A\}$ and $\mathcal{P}' = \{p_b : b \in B\}$. They give rise to the same locally convex topology on the linear space X.

Now we introduce the partial order relation in set B. Consider that $a \leq b \Leftrightarrow pb(x) \geq pa(x)$ for all $x \in X$. For any $a, b \in B$, there is such a $c \in B$, for all $x \in X$. Hence, the set B is a direction.

In the following, without limiting generality, we will replace the τ-norm $\|\cdot\|_\tau$ with the τ-norm $\|\cdot\|_\tau{}'$.

The following theorem will play an important role later on.

Theorem 1. ([2]). Any τ–Banach spaces and only these are limits of projective spectra of length τ composed of Banach spaces.

Proof 1. Let $(L, \|\cdot\|_\tau)$ be an arbitrary τ-Banach space, and $\|\cdot\|_\tau : L \to R_+^\tau$ be a τ–norm, where $R_+^\tau = \prod\{R_+^a : a \in A\}, R_+^a = R_+$, for each $a \in A$ and $|A| = \tau$.

Let $q_a : R_+^\tau \to R_+^a$ be the natural projection onto the a-th factor where $a \in A$. Let $p_a = q_a \cdot \|\cdot\|_\tau$. Then p_a is a semi-norm on L. We introduce a partial order on the set A. Consider that $a \leq b$ if and only if $p_b(x) \geq p_a(x)$ for every $x \in L$. Hence, the identity mapping $i_a^b : (L, p_a) \to (L, p_b)$ is uniformly continuous. We introduce an equivalence relation on L: $x \sim^a y <=> p_a(x-y) = 0$.

Let $L_a = L/\sim^a$ be a quotient set, $\phi_a : L \to L/\sim^a$ be a quotient mapping, and $\|\cdot\|_a$ be the norm on L_a obtained by factorizing the half-norms p_a for each $a \in A$. Then $(L_a, \|\cdot\|_a)$ – normalized (Banach) space. Note that the relation $x \sim^b y$ follows from the relation $x \sim^a y$. Then a natural mapping $\pi_a^b : L_b \to L_a$ is defined, where the mapping $\pi_a^b : (L_b, \|\cdot\|_b) \to (L_a, \|\cdot\|_a)$ –is uniformly continuous. If $a < b < c$, then $\pi_a^c = \pi_a^c \cdot \pi_a^b$. As a result, we have projective spectrum $\{L_a, \pi_a^b, A\}$ of length τ composed of normalized (Banach) spaces. Let $L' = \underleftarrow{\lim}\{L_a, \pi_a^b, A\}$. On L', define the τ–norm $\|\cdot\|_\tau = \{\|\cdot\|_a : a \in A\}$ and establish a bijection f between sets L and L'. Define a mapping $f : L \to L'$ at each point $x \in L$ that puts the singular thread $f(x) = \{\phi_a(x) : a \in A\}$.

If $x_1 \neq x_2$, then there exists a semi-norm p_a such that $p_a(x_1 - x_2) = 0$. Then $\phi_a(x_1) \neq \phi_a(x_2)$. This shows that the mapping is reciprocal. Now we show that the mapping f is a surjection. Let $y' = \{\phi_a(y) : a \in A\}$ be an arbitrary thread. Then $(y) = \{\phi_a(y): a \in A\} = y'$. Hence, the mapping f is a bijection. By the definition of the τ-norm $\|\cdot\|_\tau$ we have $\|x\|_\tau = \|f(x)\|_\tau{}'$.

Inversely, let $\{(L_a, \|\cdot\|_a), \pi_a^b, A\}$ be a projective spectrum of length τ composed of normalized (Banach) spaces. Let $L = \underleftarrow{\lim}\{L_a, \pi_a^b, A\}$ be the limit of the projective spectrum. We define the τ-norm on L as follows: for each $x \in L$, let $\|x\|_\tau = \{\|x\|_a; a \in A\}, x = \{x_a, a \in A\}, x_a \in L_a$ for any $a \in A$. Let us show that all axioms of the τ-norm are satisfied:

1. If x is a zero element of the linear space L, then $\|x_a\|_a = 0$ for any $a \in A$ and $\|x\|_\tau = \theta$. Conversely, if $\|x\|_\tau = \theta$, then $\|x_a\|_a = 0$ for every $a \in A$ and x –zero element of linear space L;
2. Let $x \in L$ and $\lambda \in R$-an arbitrary scalar. Then $\|\lambda x\|_\tau = \|\lambda x_a\|_a = \{|\lambda| \cdot \|x_a\|_a, a \in A\} = |\lambda| \cdot \|x\|_\tau$ for each $x \in L$;
3. Let $x, y \in L$. Then $\|x+y\|_\tau = \{\|x_a + y_a\|_a : a \in A\} \leq \{ \|x_a\|_a +$

$\|y_a\|_a : a \in A\} = \|x\|_\tau + \|y\|_\tau$.

If the projective spectrum $(L_a, \|\cdot\|_a, \pi_a^b, A)$ is composed of Banach spaces, Then its projective limit $(L, \|\cdot\|_\tau) = \varprojlim\{(L_a, \|\cdot\|_a), \pi_a^b, A\}$ is a closed subspace of the product of complete spaces. Therefore, the τ-normalized space (L, $\|\cdot\|_\tau$) is a τ-Banach space. Theorem 1 is proved.

The above three principles of functional analysis have been generalized to different classes of linear topological spaces (see, e.g., [8]).

We extend these principles from the class of Banach spaces to the class of τ–Banach spaces (which are quite wide, see Example 1), in a natural way, while preserving the formulation in Banach spaces.

The continuation principle of linear bounded functionals for normalized spaces is formulated as follows (see, e.g., [9]): Let (E, $\|\cdot\|$) be a valid normalized space, L be its subspace and f_0 be a bounded linear functional on L. This bounded linear functional can be extended to some bounded linear functional f on the whole space E without increasing the norm, i.e., $\|f_0\|_{onL} = \|f\|_{onE}$.

First we introduce the notion of τ–functional.

Definition 3. A linear mapping $f : E \to R^\tau$ of a linear space E into a linear space R^τ is called a linear τ–functional.

If (E, $\|\cdot\|$) is a τ–normalized space, then the continuity of the linear τ–functional f is naturally determined.

Example 2. Let $f_a: (E, \|\cdot\|_a) \to R$ be a linear continuous functional for each $\in A, |A| = \tau$. Consider the diagonal product $f = \Delta\{f_a: a \in A\}$ of the family of linear continuous functionals $\{f_a: a \in A\}$. Then the mapping $f: (E, \|\cdot\|_\tau) \to R^\tau$ is a linear continuous τ-functional.

Inversely, if $f: (E, \|\cdot\|_\tau) \to R^\tau$ is an arbitrary continuous τ–functional, then suppose that $f_a = \pi_a \cdot f$, where $\pi_a : R^\tau \to R_a$ is projection for each $a \in A, |A| = \tau$. Then we obtain the family $\{\{f_a: a \in A\}$ of continuous linear functionals where $f = \Delta\{f_a: a \in A\}$.

Now let us define the norms of the continuous linear τ–functional.

Let $f: (E, \|\cdot\|_\tau) \to R^\tau$ – a linear continuous τ -functional. The smallest element $C \in R_+^\tau$ satisfying the inequality $|f(x)| \leq C \cdot \|x\|_\tau$ for each $x \in X$ is called the τ -norm of the linear continuous τ–functional, where $|f(x)|$ is an element of R_+^τ consisting of the module of the coordinates of the element $f(x) \in R^\tau$.

For any continuous linear τ-functional *f*, its τ-norm always exists. Indeed, let $f: (E, \|\cdot\|_\tau) \to R^\tau$ be an arbitrary linear continuous τ -functional. Let us represent it as the diagonal product of the family of ordinary continuous linear functionals $\{f_a: a \in A\}$ (see Example 2), where $f_a: (E, \|\cdot\|_a) \to R = R_a, a \in A, |A| = \tau, f_a = \pi_a \cdot f$, where $\pi_a : R^\tau \to R_a$ is the natural projection for each $a \in A$. Then, it is easy to check that the τ-norm of the continuous linear τ–functional f is equal to $\|f\|_\tau = \{\|f_a\|_\tau : a \in A\}$.

Theorem 2. (on the continuation of continuous linear τ–functions). Let *f:* $(E, \|\cdot\|_\tau) \to R$, $(E, \|\cdot\|_\tau)$ be an arbitrary τ-normalized space, L be its subspace, and f_0 be a continuous linear τ–functional on L. This continuous linear functional can be extended to some continuous linear τ–functional f on the whole space $(E, \|\cdot\|_\tau)$ without increasing τ–norm, i.e., $\|f_0\|_{onL} = \|f\|_{onE}$.

Proof 2. Let the conditions of the theorem be satisfied. By Theorem 1, given τ-normalized space $(E, \|\cdot\|_\tau)$, let us represent it as the limit of the projective spectrum $\{(E_a, \|\cdot\|_a), \pi_\alpha^\beta, A\}$ of length τ composed of normalized spaces.

Let $\pi_a: E \to E_a$ be a natural projection for each $a \in A$ and let $L_a = \pi_a(L)$. Let us represent the linear continuous mapping f_0 as the diagonal product of the mappings of the family of linear continuous functionals $\{f_0^a ; a \in A\}, f_0^a: L \to R_a, a \in A, |A| = \tau$ (see Example 2).

Then the mapping $f_0^a: L_a \to R_a$ is a continuous linear functional on $(L_a, \|\cdot\|)$. By the principle of continuation of linear functionals, it continues to some linear continuous functional $f_a: (E, \|\cdot\|) \to R_a$. Then the diagonal product $f = \Delta\{f_a: a \in A\}$ is a linear continuous τ-functional on $(E, \|\cdot\|_\tau)$, with f being an extension of τ–functional f_0 and $\|f_0\|_{\tau\, on\, L} = \|f\|_{\tau\, on\, E}$. The openness principle for Banach spaces is formulated as follows (see [9]): the linear continuous mapping A of the Banach space

E onto the Banach space E_1 is open.

Theorem 3. (about the open mapping). Let $f: (E, \|\cdot\|_\tau) \to (L, \|\cdot\|_\tau)$ be a linear continuous mapping the τ–Banach space $(E, \|\cdot\|_\tau)$ to the τ–Banach space $(L, \|\cdot\|_\tau)$. Then the mapping *f* is open.

Proof 3. By Theorem 1, the τ-Banach spaces $(E, \|\cdot\|_\tau)$ and $(L, \|\cdot\|_\tau)$ decompose, respectively, into projective spectra of length τ consisting of Banach spaces, i.e., $(E, \|\cdot\|_\tau) = \lim\limits_{\leftarrow}\{E_a, \pi_a^b, M\}, (L, \|\cdot\|_\tau) = \lim\limits_{\leftarrow}\{L_a, p_a^b, M\}$, where $|M| = \tau$. The linear continuous mapping $f : (E, \|\cdot\|_\tau) \to (L, \|\cdot\|_\tau)$ induces the linear continuous mapping $f_a: (E_a, \|\cdot\|) \to (L_a, \|\cdot\|)$, with $p_a \cdot f = f_a \cdot \pi_a$ for every $a \in M$, where $\pi_a: E \to E_a$ and $p_a: L \to L_a$ are natural projections. Let us show that the mapping *f* is open. Note that (see e.g. [3]) the family $\{\pi_a^{-1}\ (U_a): U_a\ is\ an\ open\ set\ in\ (E_a, \|\cdot\|_a), a \in M\}$ forms the base of the τ–Banach space $(E, \|\cdot\|_\tau)$, and the family $\{p_a^{-1}\ (V_a): V_a\ is\ an\ open\ set\ in\ (L_a, \|\cdot\|_a), a \in M\}$ forms the base of the τ–Banach space $(L, \|\cdot\|_\tau)$. By the openness principle, the linear continuous mapping $f_a: (E, \|\cdot\|) \to (L, \|\cdot\|)$ is open for every $a \in M$. Note that $f(\pi_a^{-1}\ (U_a)) = p_a^{-1}\ (f_a(U_a))$. The right-hand side of the equality is an open set in $(L, \|\cdot\|_\tau)$. It follows that the mapping *f* is open.

Theorem 4. (about the inverse operator). Let $A: (E, \|\cdot\|_\tau) \to (L, \|\cdot\|_\tau)$ be a linear continuous operator, mutually identically mapping τ-Banach space $(E, \|\cdot\|_\tau)$ to τ-Banach space $(L, \|\cdot\|_\tau)$. Then the inverse operator A^{-1} is continuous.

Proof 4. Let $A: (E, \|\cdot\|_\tau) \to (L, \|\cdot\|_\tau)$ be a linear continuous operator, mutually uniquely mapping τ–Banach space $(E, \|\cdot\|_\tau)$ to τ–Banach space $(L, \|\cdot\|_\tau)$. By Theorem 1, let us represent these τ–Banach spaces as limits of projective spectra of length τ composed of Banach spaces, i.e., $(E, \|\cdot\|_\tau) = \lim\limits_{\leftarrow}\{E_a, \pi_a^b, M\}$, $(L, \|\cdot\|_\tau) = \lim\limits_{\leftarrow}\{L_a, p_a^b, M\}$, where $|M| = \tau$. Then the operator A induces a linear continuous operator $A_a: (E, \|\cdot\|_a) \to (L, \|\cdot\|_a)$, mutually identically mapping the Banach space $(E, \|\cdot\|_a)$ to the Banach space $(L, \|\cdot\|_a)$ for every $a \in M$. Then by the inverse operator theorem (see e.g. [5]) the inverse operator ${A_a}^{-1}$ a is continuous. Then it is easy to see that ($A^{-1} = \Delta\{{A_a}^{-1} \cdot P_a : a \in M\}$). The operator A^{-1} as a diagonal product of continuous mappings $\{{A_a}^{-1} \cdot P_a : a \in M\}$ is continuous. Theorem 4 is proved.

Theorem 5. (about a closed graph). Let $(E, \|\cdot\|_\tau)$ and $(L, \|\cdot\|_\tau)$ be τ-Banach spaces and the mapping $A : E \to L$ is linear and its graph $G = \{(x, Ax) : x \in E\}$ is closed in $(E, \|\cdot\|_\tau) \times (L, \|\cdot\|_\tau)$. Then the mapping *A* is continuous.

Proof 5. Let's put $\|(x, y)\|_\tau = \{\|x\|_\tau, \|y\|_\tau\}$. Then $((E \times L), \|\cdot,\cdot\|_\tau)$ is a τ-Banach space (see [8]). Since the mapping *A* is linear, its graph G is a subspace in $(E, \|\cdot\|_\tau) \times (L, \|\cdot\|_\tau)$. Closed subspaces of τ-Banach spaces are τ-Banach spaces. Next, let us define the mappings $g_1 : G \to E$ and $g_2 : E \times L \to L$, assuming $g_1(x, Ax) = x, g_2(x, y) = y$. Then g_1 is a continuous linear mutually unambiguous mapping of τ-Banach space *G* onto τ-Banach space *E*. It follows from Theorem 4 on inverse mapping that the inverse mapping $g_1^{-1} : E \to G$ is continuous. But $A = g_2 \circ g_1^{-1}$, and the mapping g_2 is continuous. Hence, the mapping *A* is continuous. Theorem 5 is proved.

The principle of uniform boundedness of operators for normalized spaces can be formulated as follows [1]: If a sequence of linear bounded operators $\{A_n\}$ mapping a Banach space E to a normalized space L converges pointwise to a linear operator *A*, then the sequence of norms $\{\|A_n\|\}$ of these operators is bounded, hence $\lim\limits_{x\to 0} A_n x = 0$ is uniformly *n = 1, 2, ...* and the operator $Ax = \lim\limits_{n\to\infty} A_n x$ is bounded (here 0 is the zero element of the space *E*).

Let $A : (E, \|\cdot\|_\tau) \to (L, \|\cdot\|_\tau)$ be a linear continuous operator, mapping τ-normalized space $(E, \|\cdot\|_\tau)$ into τ–normalized space $(L, \|\cdot\|_\tau)$. Let us define τ-norm and boundedness of the mapping *A*.

Definition 4. The linear operator $A : (E, \|\cdot\|_\tau) \to (L, \|\cdot\|_\tau)$ will be called bounded if there exists such an element $C \in R_+^\tau$, that for any $x \in E$ the inequality $\|Ax\|_\tau \leq C \cdot \|x\|_\tau$ holds.

Note that the boundedness of a linear operator, in general, does not imply its continuity. But continuous linear operators are bounded. For normalized spaces, continuity and boundedness of linear operators are equivalent (see, for example, [9]).

The smallest element of $C \in R_+^\tau$ that satisfies the inequality $\|Ax\|_\tau \le C \cdot \|x\|_\tau$, for any $x \in X$, is called τ-norm of operator A and is denoted by $\|A\|_\tau$.

We show that τ-norm of a linear continuous operator $A : (E, \|\cdot\|_\tau) \to (L, \|\cdot\|_\tau)$ always exists. Indeed, by Theorem 1, the τ-normalized spaces $(E, \|\cdot\|_\tau)$ and $(L, \|\cdot\|_\tau)$ are represented as limits of projective spectra of length τ composed of normalized spaces, i.e., $(E, \|\cdot\|_\tau) = \varprojlim\{E_a, \pi_a^b, M\}$ and $(L, \|\cdot\|_\tau) = \varprojlim\{L_a, \pi_a^b, M\}$. The linear continuous operator $A : (E, \|\cdot\|_\tau) \to (L, \|\cdot\|_\tau)$ for every $a \in M, |M| = \tau$ induces a linear continuous operator. $A_a : (E_a, \|\cdot\|_a) \to (L_a, \|\cdot\|_a)$. The norm of such an operator exists and we denote it by $\|A_a\|$. Then it is easy to check that $\|A_a\|_\tau = \{\|A_a\| : a \in M\} \in R_+^\tau$.

Theorem 6. (on the uniform boundedness of a sequence of continuous operators). If a sequence of linear continuous operators $\{A_n\}$ mapping a τ-Banach space $(E_n, \|\cdot\|_\tau)$ to a τ-Banach space $(L_n, \|\cdot\|_\tau)$ converges pointwise to linear operator $A : (E, \|\cdot\|_\tau) \to (L, \|\cdot\|_\tau)$, then the sequence of τ-norms $\{\|A_n\|_\tau\}$ of these operators is bounded in R_+^τ, $\lim_{x\to 0} A_n x = \theta$ is uniform with respect to *n = 1, 2, ...* and the operator $Ax = \lim_{n\to\infty} A_n x$ is bounded.

Proof 6. By Theorem 1 we have $(E, \|\cdot\|_\tau) = \varprojlim\{E_a, \|\cdot\|, \pi_a^b, M\}$ and $(L, \|\cdot\|_\tau) = \varprojlim\{L_a, \|\cdot\|, \pi_a^b, M\}$, where $(E_a, \|\cdot\|_a)$ is a Banach space, and $(L_a, \|\cdot\|_a)$ are normalized spaces for each $a \in M, |M| = \tau$. Continuous linear operators A and A_n induce also linear continuous (bounded) operators $A_a: (E_a, \|\cdot\|) \to (L_a, \|\cdot\|)$ and $A_n^a: (E_a, \|\cdot\|) \to (L_a, \|\cdot\|)$. Since $A_n x \to Ax$ for every $x \in E$ at $n \to \infty$, then $A_n^a y \to A_a y$ for every $y \in E_a$ and $a \in M$ at $n \to \infty$. By virtue of the principle of uniform boundedness of operators, the sequence of operators $\|A_n^a\|$ is bounded for every $a \in A$. Then, it is easily verified that the sequence of τ -norm operators $\{\|A_n\|_\tau$ is bounded in R_+^τ. $\lim_{x\to 0} A_n x = \theta$ is uniformly $n = 1, 2, \ldots$. and the operator $Ax = \lim_{n\to\infty} A_n x$ is bounded (i.e., its τ -norm is bounded in R_+^τ). Theorem 6 is proved.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 13, 2023
Revised: December 8, 2023
Accepted: January 15, 2024

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SOME PROPERTIES OF THE MAPPINGS, PRESERVING THE INDEX OF COMPLETENESS TOWARDS THE IMAGE

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This article discusses the issue of preserving the index of completeness towards the image by uniformly continuous mappings of complete uniform spaces. The study of complete uniform spaces occupies a very important part in the theory of uniform spaces. A method for a more refined classification of complete uniform spaces [1] was presented by academician A.A. Borubaev, and with the help of this method it becomes possible to assign a certain cardinal number to each complete uniform space. This cardinal number characterizes the least powerful subsystem of the family of uniform coverings, relative to which the space continues to remain complete in a certain sense. However, the question naturally arises about what properties uniformly continuous mappings must have in order to be able to preserve the index of completeness of a uniform space. This article attempts to determine the necessary and sufficient features of mappings capable of preserving the index of completeness towards the image.

Keywords: index of completeness of complete uniform spaces, uniformly continuous mappings, H-Cauchy filter, H-completeness, composition of uniformly continuous mappings, uniform coverings.

Бул макалада толук бир калыптуу мейкиндиктерди бир калыптуу үзгүлтүксүз чагылгандыктар аркылуу сүрөттөлүштүн толуктугунун индексин сактоо маселеси талкууланат. Толук бир калыптуу мейкиндиктерди изилдөө бир калыптуу мейкиндиктер теориясында абдан маанилүү орунду ээлейт. Толук бир калыптуу мейкиндиктердин кыйла такталган классификациясынын усулу академик А.А.Бөрүбаев тарабынан сунушталган [1], жана бул усулдун жардамы менен ар бир толук бир калыптуу мейкиндикке белгилүү кардиналдык санды ыйгаруу мүмкүн болот. Бул кардиналдык сан мейкиндик белгилүү бир мааниде толук бойдон кала берген салыштырмалуу бир калыптуу жабуулардын үй-бүлөсүнүн эң аз кубаттуу подсистемасын мүнөздөйт. Бирок, табигый түрдө, бир калыптуу мейкиндиктин толуктук индексин сактап калуу үчүн бир калыптуу үзгүлтүксүз чагылгандыктар кандай касиеттерге ээ болушу керек деген суроо туулат. Бул макалада сүрөттөлүштүн толуктугунун индексин сактоого жөндөмдүү чагылгандыктардын керектүү жана жетиштүү өзгөчөлүктөрүн аныктоого аракет кылынат.

Урунттуу сөздөр: толук бирдиктүү мейкиндиктердин толуктук индекси, бир калыптуу үзгүлтүксүз чагылгандыктар, Н-Коши фильтри, Н-толуктук, бир калыптуу үзгүлтүксүз чагылгандыктардын композициясы, бир калыптуу жабдуулар.

В данной статье рассматривается вопрос сохранения равномерно непрерывными отображениями индекса полноты полных равномерных пространств в сторону образа. Исследование полных равномерных пространств занимает очень важную часть в теории равномерных пространств. Метод более тонкой классификации полных равномерных пространств [1] представлен А.А.Борубаевым. Соответственный метод даёт возможность каждому полному равномерному пространству поставить в соответствие определённое кардинальное число. Это кардинальное число характеризует подсистему с наименьшей мощностью семейства равномерных покрытий, относительно которой пространство продолжает оставаться полным в некотором смысле. Однако естественным образом возникает вопрос о том, какими свойствами должны обладать равномерно непрерывные отображения для того, чтобы сохранять индекс полноты равномерного пространства. В данной статье делается попытка определить необходимые и достаточные признаки отображений, способных сохранять индекс полноты в сторону образа.

Ключевые слова: индекс полноты полных равномерных пространств, равномерно непрерывные отображения, Н-фильтр Коши, Н-полнота, композиция равномерно непрерывных отображений, равномерные покрытия.

Let us consider the uniformed space (X, U) and choose some subset of U, which will be named as H.

Every filter F, which has non empty intersection with any covering α from H, is called as (see [1]) Cauchy H - filter:$(\forall\alpha \in H)$: $\alpha \cap F \neq \emptyset$.

A space (X, U) with $H \subset U$ is called as H -complete, and the system $H$is called compete, if every Cauchy H -filter has at least one condensation point (see [1]).

If the space (X, U) is complete, then the least of all such cardinal numbers τ, that there is such a system $H \subset U$, that (X, U) is H -complete, and $|H| = \tau$, is called the index of completeness of (X, U) (see [1]).

Additionally to these general definitions we will need more ones, that are following

Definition 1. Let X be a non-empty set. Then if W is some system of coverings of X, for which work following conditions:

1) if for α from W exists such cover β of *X*, that$\beta \prec \alpha$, then $\beta \in W$;

2) if $\beta, \alpha \in W$, then there exists such $\gamma \in W$, that $\beta \prec \gamma, \alpha \prec \gamma$,

then *(X,W)* is called a generalized uniform space, and W is called a generalized uniformity on *X*.

Let us consider a map *f* mapping one generalized uniform space *(X,U)* to another one *(Y,G)*. If for any $\alpha \in G$ exists such $\beta \in U$, that $\alpha \prec f(\beta)$, then the map *f* is called uniformly continuous map.

Let *F* is a filter of generalized uniformly space *(X,U)*, for which that for any $\alpha \in U$filter *F* has non-empty intersection with α. Then *F* is called generalized Cauchy-filter of a generalized uniform space *(X,U)*.

For following consideration, we need to introduce the topology, that in some meaning could be inducted by generalized topology.

Let us consider the system τ of subsets of *X*, that we are going to introduce by the following way: $\tau = \{O: \forall x \in O \exists \alpha \in U: \alpha(x) \subset O\}$. Then, if $O_1, O_2 \in \tau$, then for $x \in O_1 \cap O_2 \neq \emptyset$ there is truth that $x \in O_1$ and $x \in O_2$. But from this we obtain that due to the definition of τ there exist such covers α and β, for which that $\alpha(x) \subset O_1, \beta(x) \subset O_2$ and $\alpha, \beta \in U$ is true.

On the other hand, for considering by us covers α and β exists such generalized uniformly cover γ, which is inscribed in any of them, because $\alpha$and β are elements of *U*. Let us choose some such coverγ and consider the set $\gamma(x)$. And because of that $\gamma \succ \alpha, \gamma \succ \beta$, for every $\Gamma \in \gamma: x \in \Gamma$ we get that exist $A \in \alpha$,B $\in \beta$, for which $\Gamma \subset$ A,Γ $\subset B$. From this by the definition of star we get that $\gamma(x) \subset \alpha(x), \gamma(x) \subset \beta(x)$. It means that $\gamma(x) \subset \alpha(x) \cap \beta(x)$ by the definition of intersecting of sets. It means that $\gamma(x) \subset \alpha(x) \cap \beta(x) \subset O_1 \cap O_2$. I. e. for $x \in O_1 \cap O_2$ exists such γ, that the star of x in the relationship to γ lies on the set $O_1 \cap O_2$, that means that: $\gamma(x) \subset O_1 \cap O_2$. That is why the system τ is closed in the relationship to final intersection of contained by this sets.

Now let us consider a subsystem Σ of the system τ, and for this we significate $\tilde{O} = \cup \{O: O \in \Sigma\}$. Let us select some $x \in \tilde{O}$. Then exists $O \in \Sigma$, for which $x \in O$. But by the property, that defines the τ for selected $x \in O$ exists covering α, for which, that

$\alpha(x) \subset O$, and $\alpha \in U$, is true. And for the corresponding covering an interrelation $\alpha(x) \subset O$ means, that $\alpha(x) \subset \tilde{O}$. And because of undefined selecting by us of $x \in \tilde{O}$, we get, that for any $x \in \tilde{O}$ exists such covering $\alpha \in U$, that $\alpha(x) \subset O$. And because of undefined selecting of Σ as a subsystem of the system τ, the system τ is closed in the relationship to the operation of union of any subsystem of sets had by the τ.

For any $\alpha \in U$: $\alpha(x) \subset X$, so $X \in \tau$. For any $x \in X$and $\alpha \in U$: $x \in \emptyset =>$ $\alpha(x) \subset \emptyset$. From that it is implicated, that such *X* as $\emptyset$ are elements of system τ.

So, considered by us τ corresponds for all three axioms of topology, that is why it induce some topology on the set *X*.

Let topology in the relationship to *U*, noted by us above, being considered in the role of some generalized uniform structure, be called by us as inducted by generalized uniformity *U* topology.

If for a generalized uniform space *(X,U)* every its generalized Cauchy filter has at least one point of condensation, the such generalized uniform space will be called by us as complete space.

In the generalized uniform space *(X,U)* we select some subsystem of *U*, that we are going to call as H.

Every filter *F*, that has non-empty intersection with any cover α from H, is called Cauchy$H$-filter: $(\forall \alpha \in H)$: $\alpha \cap F \neq \emptyset$. The space (X, U) is called H-complete, if every its Cauchy H -filter at least one condensation point.

If (X, U) is complete, the least among such cardinal numbers τ, for which exists such $H \subset U$, that (X, U) is H-complete, and $|H| = \tau$, is called as index of completeness of (X, U).

Let us suppose that the generalized uniform space *(X,V)* is *H*-complete.

We can notice, that from the system *H* it is possible to make a structure of a base for some other generalized uniform space *(X,V)* on the same set *X*: to do this it is necessary to make the system *H* bigger by the following coverings:

$\bigcup_{n \in N}\{\{\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n\}: \alpha_1, \alpha_2, \ldots, \alpha_n \in H\}$.

Now we can see, that every Cauchy H -filter *F* is a generalized Cauchy filter of got by us in this case generalized uniform space *(X,V)* because of filters' properties. Actually, for any cover α from the system *V* exists a cartage $\{\alpha_1, \alpha_2, \ldots \alpha_n\} \subset H: \alpha = \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n$. And for any cartage $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ $\exists \{A_1, A_2, \ldots A_n\}: A_i \in \alpha_i, A_i \in F$, by the definition of Cauchy H -filter. It means, that there exists such cartage $\{A_1, A_2, \ldots A_n\}$, for that $\bigcap_{i=1}^{n} A_i = A$ is element of considered by us filter by the definition of filter, because of that $A_i \in F$. And because of considering by us not concreted cover$\alpha$of a system *V*, considered by us Cauchy H -filter *F* is intersected with any cover of the system *V*, and because of this it is generalized Cauchy filter for generalized uniformity *V*. From this it is implemented, that any Cauchy H -filter of a space *(X,U)* is generalized Cauchy filter of generalized uniformity *V*.

Existing of condensation points of filter *F* in the topology, inducted by the generalized uniformity of *(X,U)*, means existing of condensation points of this filter in the topology, inducted by *(X,V)*, because this topology is weaker then this one on *(X,U)*. It means, that canonical injection from *(X,U)* to *(X,V)*, that is identity mapping *f*, **conserves for any** Cauchy$H$ -filter of the space *(X,U)* its property of having non-empty set of its condensation points. But it is important, that every Cauchy H -filter of the space *(X,U)* is generalized Cauchy filter of generalized uniformity *V* in the corresponding to that, what is proved above. Let us notice, that every Cauchy H -filter of the space *(X,U)* has non-empty set of its condensation points. So, it means, that every generalized Cauchy filter of the space *(X,V)* in the space *(X,V)* has non-empty set of its condensation points, and its inverse image with the considering in the relationship with the mapping *f* has non-empty set of its *(X,U)*. So, we fix this important for us property of the mapping *f*, which we will need soon.

Now let us notice: every element of *V* is cover of *X*, which is got as intersec-tion of covers from the system *H*. But *H* is subsystem of *U*, that's why all finite intersections of covers, which are contained in *H*, are contained in *U* because of that *U* is closed in the relationship to operation of finite intersection of covers, contained in this. So, we notice, that *V* is subset of *U*. Corresponding to the definition of uniformly continuous mapping, it a priori means, that signified higher mapping of identity *f* is uniformly continuous mapping.

So, the natural injection $f:(X,U)\rightarrow(X,V)$ is a uniformly continuous mapping. Above we have so designed the property of this mapping, which is, that every generalized Cauchy filter of (X,V) conserves the set of its condensation points being non-empty in the returning to its inverse image with considering of this in the relationship to the mapping *f*. It means conserving by *f* in the direction to the inverse image of the property of being non-empty for the set of the condensation points for any generalized Cauchy filter of the space (X,V). We can see more: f conserves the property of completeness in the direction of its image, because (X,V) is complete, so as (X,U) is, by our considering.

We have come to such intermediate result: mapping *f* considered by us has the following properties:

- uniformly continuity;

- conserving of completeness in the direction of its image;

- with considering in the relationship to mapping f inverse image of any generalized Cauchy filter, having non-empty set of its condensation points, inducts such filter, which has non-empty set of its condensation points.

Let us imagine the inversed sentence: some generalized uniform space (X,U) has bijective mapping f into some generalized uniform space (Y,V), and f is a uniformly continuous mapping, conserving:

- conserving of completeness in the direction of its image;

- with considering in the relationship to mapping f inverse image of any generalized Cauchy filter, having non-empty set of its condensation points, inducts such filter, which has non-empty set of its condensation points, i. e. if F is a generalized Cauchy filter in the space (Y,V), the inducted by $f^{-1}(F)$ filter has non-empty set of its condensation points.

We suppose that *V* has a base *H'* with cardinal number of itτ. It means, that we can eject a system of covers of the space (X,U): $H=\{f^{-1}(\alpha):\alpha\in H'\}$ and consider some Cauchy *H*-filter *F*.

Then *f(F)* for every element α from *H'* has intersection with it as follows:

$\alpha \cap f(F) = f(f^{-1}(\alpha) \cap F)$, but because of, that F is Cauchy H-filter, and H is defined as the system of inverse images of sets, contained in H', we get, that: $f^{-1}(\alpha) \cap F \neq \emptyset$, from what is implicated, that: $\alpha \cap f(F) = f(f^{-1}(\alpha) \cap F) \neq f(\emptyset) = \emptyset$, i. e.: $\alpha \cap f(F) \neq \emptyset$.

And because of that we consider undefined Cauchy H-filter F, we get, that *f(F)* induces generalized Cauchy filter in the space (Y, V) for any Cauchy H-filter F in (X, U). And that means, that because of bijectivity of *f* any Cauchy H-filter F is inverse image of some generalized Cauchy filter *f(F)*: $F = f^{-1}(f(F))$. Let us return to the property of *f* which is to conserve inverse image of any generalized Cauchy filter as that filter, which had non-empty set of its condensation points: let us notice, that from $F = f^{-1}(f(F))$ is implicated, that F is a filter, which has non-empty set of its condensation, because *f(F)* is generalized Cauchy filter of the complete space. It means, that because of that we consider F as some Cauchy H-filter, there is such class $H \subset U$, that, as an implication from the bijectivity of *f*, H has cardinal number of itself τ, and (X, U) is H-complete.

Thus, we have just proved the following lemma.

Lemma. That (X, U) has index of completeness τ, can be interpreted as that τ is infimum among cardinal numbers of all bases of generalized uniform spaces (Y, V), for each of which exists bijective uniformly continuous mapping $f_V: (X, U) \to (Y, V)$, conserving:

1) a property of completeness in the direction of its image;

2) for any generalized Cauchy filter of the space (Y, V), having non-empty set of its condensation points, a property of having by it non-empty set of condensation points in the direction to its inverse image.

So, if we suppose that $g: (X, U) \to (Y, V)$ is mapping one complete generalized uniform space to another one, then due to Lemma exist bijective maps $x: (X, U) \to (X, U')$ and $y: (Y, V) \to (Y, V')$, where *(Y, V'), (X, U')* are generalized uniform spaces, *x* and *y* are uniformly continuous maps, having such properties as 1) and 2). It means, that there exists uniformly continuous bijection *y* with the property 2), that

$yg: (X, U) \to (Y, V')$ is map, conserving the property of completeness.

Now we suppose that the space *(X,U)* has its index of completeness not less then the index of completeness of the space *(Y,V)*. Because of the definition of index of completeness: for any $H \subset U$, for which *(X,U)* is H-complete there exists such $G \subset V, |G| = |H|$, that *(Y, V)* is G-complete. It means, that for every such system $H \subset U$, for which *(X,U)* is *H*-complete, there exists uniformly continuous mapping $y: (Y, V) \to (Y, V')$ with the properties 1) and 2), for which *V′* has a base, which is equivalent as a set to the system *H*.

From this we obtain that there exists such *V′*, having its weight equivalent to *H*, for which exists such uniformly continuous mapping $y: (Y, V) \to (Y, V')$ with the property 2), that *yg* conserves the property of completeness. It is so, because correspondingly to considering *g* conserves a property of completeness, because we imagine this mapping as a mapping of one complete space to another one. Let us fix got by us properties of the mapping *y*.

And now let us suppose that $g: (X, U) \to (Y, V)$ is a map, mapping one complete generalized uniform space to another. Here let for any $H \subset U$ such a mapping *y* exist, for which is true its being a uniformly continuous bijection, having the property 2), and for which: *yg* conserves the property of generalized completeness and maps *(X,U)* to a generalized uniform space, having a base, equivalent as a set to *H*. Then we get, that *yg(X,U)=(Z,V′)* is generalized complete. It means, that *y* conserves completeness during the mapping of the complete generalized uniform space *(Y,V)=g(X,U)* to the complete generalized uniform space *(Z,V′)*. As an implication of this we can watch, that due to higher written Lemma the generalized uniform space *(Y,V)* has an index of completeness, not revaluing (strongly) the index of completeness of *(X,U)*.
So, we have just proved the following sentence.

Preposition 1. Let $g: (X, U) \to (Y, V)$, where only complete spaces are considered, and let us suppose, that $|V| \geq ic(U)$. Then the condition of that $ic(V) \leq ic(U)$ is equivalent to condition of that:
- there exists such τ, that for any *H*: $|H| \leq \tau$, for which *(X,U)* is *H*-complete, such an uniformly continuous bijection $y: (Y, V) \to (Y, V')$, having the property 2) and having

as image of mapping by this a space *(Z,V')* with its weight $w(V') = |H|$, exists, that *yg* conserves the property of generalized completeness.

By analogy to considered above it is possible to note that if for any such uniformly continuous bijection $y: (Y, V) \to (Y, V')$, having the property 2) and the space *(Z,V')* as image of mapping by it, for which *yg* conserves the property of generalized completeness, there exists such $H \subset U$, that *(X,U)* is H-complete and $w(V') = |H|$, then $ic(U) \leq ic(V)$. So as an addition to written higher it is possible to extraduce by the way, which is analogical to the way of considering higher after first sentence the sentence, which is inverse to it, the sentence, which is inverse in the relationship to corresponding of being considered now sentence. This sentence is concluded in that $ic(U) \leq ic(V)$ means being completed of the condition, that for any such having the property 2) and the space *(Z,V')* as the image of the mapping by it uniformly continuous bijection $y: (Y, V) \to (Y, V')$, for which *yg* conserves the property of generalized completeness, there exists such $H \subset U$, that *(X,U)* is H-complete and $w(V') = |H|$. Then we are able to note that there are some implications for this result and for the result of Proposition 1, which are following and both of which had been proved by us just now.

Proposition 2. Let $g: (X, U) \to (Y, V)$, where only complete spaces are considered. Then the condition, that $ic(U) \leq ic(V)$, is equivalent to the condition of that: - for any such having the property 2) and the space *(Z,V')* as the image of the mapping by it uniformly continuous bijection $y: (Y, V) \to (Y, V')$, for which *yg* conserves the property of generalized completeness, there exists such $H \subset U$, that *(X,U)* is H-complete and $w(V') = |H|$.

Proposition 3. Let $g: (X, U) \to (Y, V)$, where only complete spaces are considered, and let $|V| \geq ic(U)$. Then the condition $ic(V) = ic(U)$ is equivalent to the system of the following conditions:

- there exists such τ, that for any H: $|H| \leq \tau$, for which *(X,U)* is H-complete, such an uniformly continuous bijection $y: (Y, V) \to (Y, V')$, having the property 2) and having as image of mapping by this a space *(Z,V')* with its weight $w(V') = |H|$, exists, that *yg* conserves the property of generalized completeness;

- for any such having the property 2) and the space *(Z,V')* as the image of the mapping by it uniformly continuous bijection $y:(Y,V) \to (Y,V')$, for which *yg* conserves the property of generalized completeness, there exists such $H \subset U$, that *(X,U)* is H-complete and $w(V') = |H|$.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 8, 2023

Revised: December 12, 2023

Accepted: January 15, 2024

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ON QUASI-IDENTITIES OF CERTAIN FINITE MODULAR LATTICE

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A lattice L has no finite basis of quasi-identities if a quasivariety generated by L is not finitely axiomatizable. We prove that so called lattice $L^{+}_{a-b}(M_3, c)$ has no finite basis of quasi-identities.

Keywords: lattice, quasivariety, finite basis of quasi-identities.

L торчосунда теңдешсымалдардын чектүү негизи болбойт, эгерде L тарабынан түзүлгөн түспөлдүксымалы чектүү аксиоматизацияланбаса. $L^{+}_{a-b}(M_3, c)$ деп аталган торчонун теңдешсымалдардын чектүү негизи жок экенин далилдейбиз.

Урунттуу сөздөр: торчо, түспөлдүксымалы, теңдешсымалдардын чектүү негизи.

Решётка L не имеет конечного базиса квазитождеств, если квазимногообразие, порождённое L, не конечно-аксиоматизируемо. Мы докажем, что так называемая решётка $L^{+}_{a-b}(M_3, c)$ не имеет конечного базиса квазитождеств.

Ключевые слова: решётка, квазимногообразие, конечный базис квазитождеств.

1. Introduction

The finite basis problem for varieties (quasivarieties) asks: Which finite algebras generate finitely axiomatizable varieties (quasivarieties)? Nowadays a lot of results are obtained for different classes of algebras. Among of them are groups, rings, lattices, topoboolean algebras, etc.

In a paper [1] R.McKenzie proved that any finite lattice has a finite basis of identities. V.P.Belkin [2] proved that the similar result is not true for quasi-identities. He proved that there is a finite lattice that has no finite basis of quasi-identities. In particular, the ten-element modular lattice M_{3-3} is the smallest lattice that has no finite basis of quasi-identities. In this regard, the following question naturally arises: Which finite lattices have finite bases of quasi-identities? This problem was suggested by V.A.Gorbunov and D.M.Smirnov [3] in 1979. Sufficient condition consisting of two parts under which the locally finite quasivariety of lattices has no finite (independent) basis of quasi-identities was found by V.I.Tumanov [4] in 1984. He conjectured that a finite modular lattice has a finite basis of quasi-identities if and only if a quasivariety generated by this lattice is a variety. In general, the conjecture is not true. W.Dziobiak [5] found a finite lattice that generates finitely axiomatizable proper quasivariety. Also we would like to point out that Tumanov's problem is still unsolved for modular lattices.

In a paper [6] it was found that the lattice $L_{a-b}(M_3, c)$ does not satisfy one of Tumanov's conditions and has no finite basis of quasi-identities. To prove this the authors used the lattice $L^+_{a-b}(M_3, c)$ (see Figure 1 below). It is natural to ask if $L^+_{a-b}(M_3, c)$ has a finite basis of quasi-identities? The main goal of this paper is to prove that $A = L^+_{a-b}(M_3, c)$ has no finite basis of quasi-identities.

2. Basic definitions and preliminaries

A quasivariety is a class K of algebras closed under subalgebras, direct products and ultraproducts. Equivalently, a quasivariety is a class K of algebras which can be axiomatized by quasi-identities. A quasi-identity is a universal Horn sentence with non-empty positive part, that is of the form

$$(\forall \bar{x})[p_1(\bar{x}) \approx q_1(\bar{x}) \wedge \dots \wedge p_n(\bar{x}) \approx q_n(\bar{x}) \to p(\bar{x}) \approx q(\bar{x})]$$

where $\bar{x} = (x_1, x_2, \dots x_m)$ and $p, q, p_1, q_1, \dots p_n, q_n$ are terms of the signature σ. A class K of algebras is called a variety if it is closed under subalgebras, homomorphic images, and direct products. By Birkhoff theorem, a variety is a class of similar algebras axiomatized by a set of identities, where by identity we mean a sentence of the form $(\forall \bar{x})[s(\bar{x}) \approx t(\bar{x})]$ for some $s(\bar{x})$ and $t(\bar{x})$.

A quasivariety K has a finite basis of quasi-identities if there is a finite set Σ of quasi-identities such that $K = Mod(\Sigma) = \{A \mid A \vDash \Sigma\}$ A lattice L has a finite basis of quasi-identities if the quasivariety $Q(L)$ generated by L is finitely based, and if $Q(L)$ is not finitely based then lattice L has no finite basis of quasi-identities.

Let $V(K)(Q(K))$ denote the smallest variety (quasivariety) containing a class K. We say that $V(K)(Q(K))$ is the variety (quasivariety) generated by K. If K is a finite family of finite algebras then $Q(K)$ is called finitely generated. If K has a single member A

we write simply $Q(A)$.

An algebra A is finite if $|A|$ is finite, and trivial if $|A| = 1$. A subalgebra B of algebra A is proper if it is not trivial and $B \neq A$. A congruence θ on an algebra A is trivial if it is the least or the greatest element in the congruence lattice $Con\ A$. A homomorphism $h\colon A \to B$ from algebra A to B is proper if the kernel congruence $\ker h$ is not trivial.

Let $L = M_{3,3}$, (a,b) be a semi-splitting pair and c a coatom of M_3 (see Figures 1,2). $L_{a-b}(M_3,c) = \{(x,p),(y,c),(z,t),(b,q) \quad | \, x,y,z \in L,\ p,q \in M_3,\ p < c,\ x \leq b,$ $y \leq a, b \leq z, q < 1,\ q \nleq c\}$. This construction was suggested by Tumanov [9]. It is easy to check that $L_{a-b}(M_3,c) = L^+_{a-b}(M_3,c) - \{d\}$ (see Figure 1). The following folklore lemma provides no finite axiomatizability.

Lemma 2.1. A locally finite quasivariety K is not finitely axiomatizable if for any positive integer n there is a finite algebra A_n such that $A_n \notin K$ and every proper subalgebra of A_n belongs to K.

3. Main result

Let $A = L^+_{a-b}(M_3,c)$ be a lattice drawn in the Figure 1 and S be a non-empty subset of a lattice L. Denote $\langle S\rangle$ the sublattice of L generated by S.

Theorem 3.1. Quasivariety $Q(A)$ generated by lattice A is not finitely based.

To prove this theorem, by Lemma 2.1, we need to construct an infinite set of finite lattices $\{A_i \notin K \mid i > 0\}$ such that any proper sublattice of A_i belongs to K.

At first we define a lattice T_n by induction on n:

$n = 1$. $T_1 \cong \mathrm{M}_3$ and $T_1 = \langle a_1, b_1, c_1\rangle$.

$n > 1$. $T_n = \langle T_{n-1} \cup a_n, b_n, c_n\rangle$ where $b_n = c_{n-1}$ and $c_n, a_n \notin T_{n-1}$, $a_n \wedge c_i = c_n \wedge c_i \notin T_{n-1}$ and $a_n \vee c_i = c_n \vee c_i \notin T_{n-1}$ for all $i < n$ (see Figure 2).

We define a modular lattice A_n as $A_n = \langle T_n \cup a_0, b_0, c_0\rangle$ where $a_n \wedge a_2 = c_0 \vee a_0$, $a_n \wedge a_1 = b_0$ and $1_{T_n} = 1_{A_n}$ (see Figure 3).

Lemma 3.2. For any $n > 1$ and a non-trivial congruence $\theta \in ConA_n$ there is $1 < m < n$ such that $A_n/\theta \cong A_m$ or $A_n/\theta \cong M_{3,3}$ provided $(a_1,b_1) \notin \theta$, otherwise $A_n/\theta \cong T_m$.

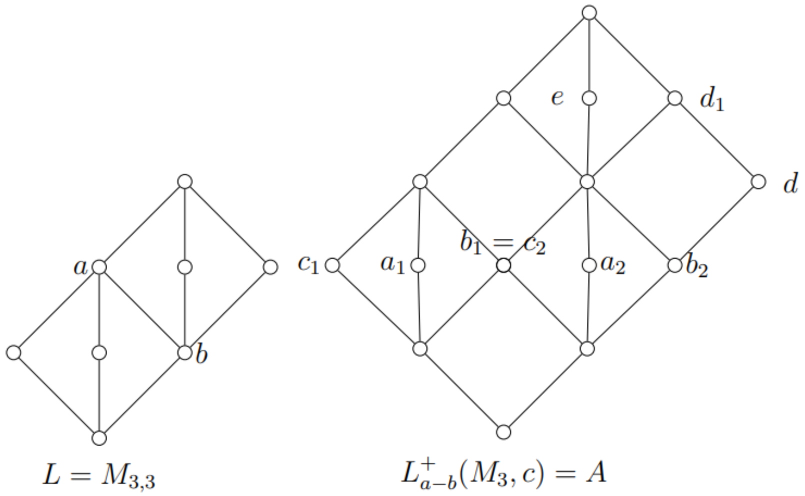


FIGURE 1. Lattices L, A

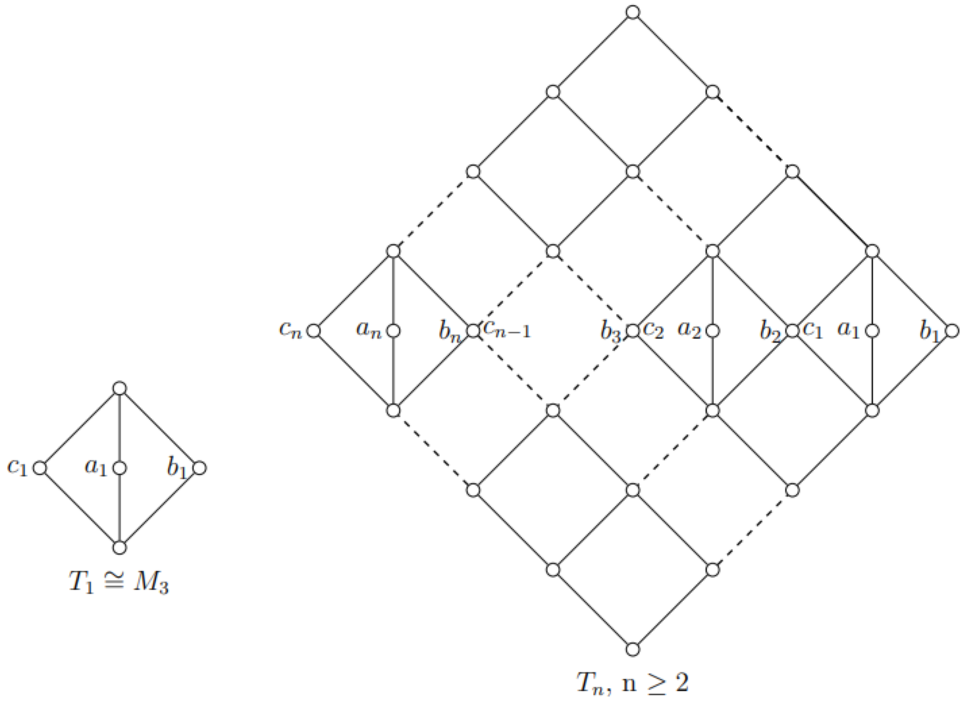
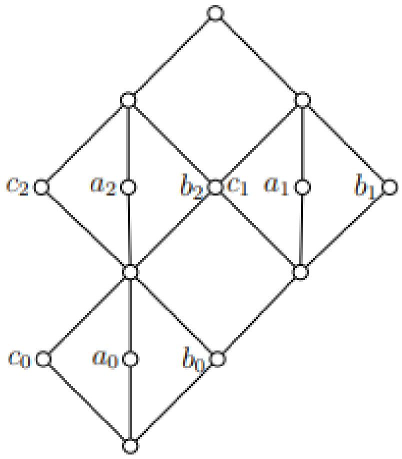


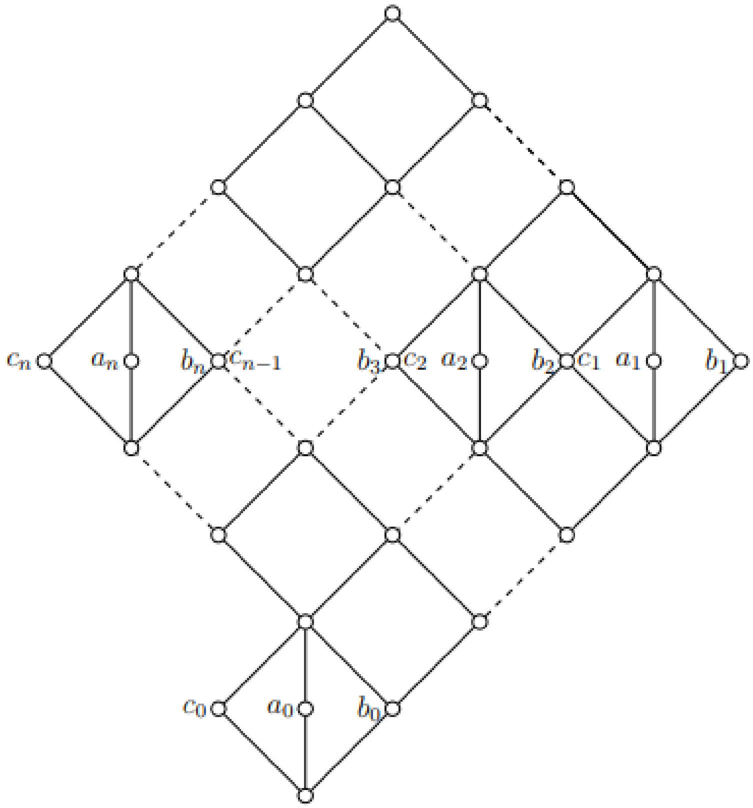
FIGURE 2. Lattices T_1, T_n

Proof of Lemma 3.2. We prove by induction on $n > 2$. One can check that it is true for $n = 3$ because of $A_3/\theta \cong A_2$ or $A_3/\theta \cong M_{3,3}$ whether $(a_1, b_1) \notin \theta$, otherwise $A_3/\theta \cong A_2$ or $A_3/\theta \cong M_3$ for any non-trivial congruence $\theta \in ConA_3$.

Let $n > 3$ and let u cover v in A_n and $\theta(u, v) \subseteq \theta$. By construction of A_u we have $A_n/\theta(u, v) \cong A_{n-1}$ or $T_n/\theta(u, v) \cong T_{n-1}$.



A_1



A_n, n ≥ 2

FIGURE 3. Lattices A_1, A_n

Continuation of Proof of Lemma 3.2.

Assume $(a_1, b_1) \notin \theta$. Since for every non-trivial congruence $\theta \in ConA_n$ there are $(u, v) \in A_n$ such that u covers v and $\theta(u, v) \subseteq \theta$, we get

$$A_n/\theta \cong (A_n/\theta(u, v))/(\theta/\theta(u, v))$$

Since $A_n/\theta(u, v) \cong A$ we obtain

$$A_n/\theta \cong \big(A_n/\theta(u, v)\big)/\big(\theta/\theta(u, v)\big) \cong A_{n-1}/\theta',$$

for some $\theta' \in ConA_{n-1}$. And, by induction, $A_{n-1}/\theta' \cong A_m$ or $A_{n-1}/\theta' \cong M_{3,3}$ for some $m > 0$. Thus $A_n/\theta \cong A_m$ or $A_n/\theta' \cong M_{3,3}$.

Now assume $(a_1, b_1) \in \theta$. Then $\theta(a_1, b_1) = \theta(u, v)$ and $A_n/\theta(u, v) \cong T_n$. Hence $A_n/\theta \cong \big(A_n/\theta(u, v)\big)/\big(\theta/\theta(u, v)\big) \cong T_{n-1}/\theta'$,

for some $\theta' \in Con(T_n)$. It is not difficult to check that $T_n/\theta \cong T_m$ for some $m > 0$. Thus $A_n/\theta \cong A_m$ or $A_n/\theta \cong T_m$.

Corollary 3.3. There is no proper homomorphism from A_n to $M_{3-3}, A, L_{a-b}(M_3, c)$.

Proof. We provide the proof for a homomorphism from A_n into M_{3-3}. It is not difficult to see that there is no homomorphism from A_n into A and $L_{a-b}(M_3, c)$.

Assume $h: A_n \to M_{3-3}, n > 1$, is a proper homomorphism. Hence ker$h$ is not a trivial congruence on A_n. By Lemma 3.2, $A_n/\text{ker}h \cong A_m$ or $A_n/\theta \cong M_{3,3}$or $A_n/\text{ker}h \cong T_m$ for some $m > 1$. Thus $A_m \cong h(A_n) \leq M_{3-3}$. It is impossible because by definition of A_m, $|A_m| > |M_{3-3}|$ for all $m > 1$, hence A_n is not a sublattice of M_{3-3}. Obviously, $M_{3,3}$ and T_m are not sublattices of M_{3-3}. thus there is no such homomorphism h.

Lemma 3.4. For every $n > 2$, a lattice A_n has the following properties:

i) $A_n \notin Q(A)$;

ii) Every proper subalgebra of A_n belongs to $Q(A)$.

Proof. i). Suppose $A_n \in Q(A)$ for some $n > 1$. Then A_n is a subdirect product of subdirectly $Q(A)$-irreducible algebras. Since every $Q(A)$-irreducible algebra is a subalgebra of A, we get that A_n is a subdirect product of subalgebras of A. By Corollary 3.3, there is no proper homomorphism from A_n onto $M_{3-3}, A, L_{a-b}(M_3, c)$. Hence $A_n \in Q(M_3)$ for all $n > 1$. It is impossible because $M_{3-3} \leq A_n$ and $M_{3-3} \notin Q(M_3)$.

ii). We prove by induction on n. It is true for $n \leq 2$ by manual checking. Let $n > 2$ and let S be a maximal sublattice of A_n. Since the lattice A_n is generated by

the set of double irreducible elements $a_1, \dots, a_n, a_0, b_1$, there is $0 < i \leq n$ such that $a_i \notin S$ or $b_1 \notin S$ or $a_0 \notin S$.

Suppose $b_1 \notin S$. One can see that $\langle S \rangle \leq_s 2 \times M_3 \times T_{n-1}$. Since $T_{n-1} \leq_s M_3^{n-1}$ we get $\langle S \rangle \in Q(M_3) \subset Q(A)$.

Suppose $a_0 \notin S$. Then $\langle S \rangle \leq_s 2 \times T_n \leq_s 2 \times M_3^n \in Q(M_3) \subset Q(A)$.

Suppose $a_i \notin S$. Since $n > 2$ and S is a maximal sublattice, then there are $i \neq k \neq l$ such that $\theta(b_k, c_k), \theta(b_l, c_l) \in ConA, \theta(b_k, c_k) \cap \theta(b_l, c_l) = \Delta$

and $A_n/\theta(b_k, c_k) \cong A_n/\theta(b_l, c_l) \cong A_{n-1}$ or $\{A_n/\theta(b_k, c_k), A_n/\theta(b_l, c_l)\} = \{A_{n-1}, T_{n-1}\}$ or $n \geq 3$.

We proved for the case when $A_n/\theta(b_k, c_k) \cong A_n/\theta(b_l, c_l) \cong A_{n-1}$. This isomorophisms mean that $A_n \leq_s A_{n-1} \times A_{n-1}$ and $S \leq A_{n-1} \times A_{n-1}$. Let $h_k: A_n \to A_{n-1}$ and $h_l: A_n \to A_{n-1}$ are homomorphisms such that $kerh_k = \theta(b_k, c_k)$ and $kerh_l = \theta(b_l, c_l)$. Since $(a_i, b_i) \notin \theta(b_k, c_k) \cup \theta(b_l, c_l))$ then $h_k(S), h_l(S)$ are proper sublattices of A_{n-1}. By induction $h_k(S), h_l(S) \in Q(A)$. As $b_k, c_k, b_l, c_l \in S$, the restrictions of congruences $\theta(b_k, c_k)|_S$ and $\theta(b_l, c_l)|_S$ are not trivial congruences on S. More $\theta(b_k, c_k)|_S \cap \theta(b_l, c_l)|_S = \Delta_S$. Hence $S \leq_S h_k(S) \times h_l(S)$. It means $S \in Q(A)$. Since every maximal proper subalgebra of A_n belongs to $Q(A)$ then every proper subalgebra of A_n belongs to $Q(A)$.

It is easy to check that for $\{A_n/\theta(b_k, c_k), A_n/\theta(b_l, c_l) = \{A_{n-1}, T_{n-1}\}$ the same arguments apply.

Proof of the Theorem. By Lemma 3.4(i), $A_n \notin Q(A)$ for all $n > 1$. By Lemma 3.4(ii), every proper subalgebra of A_n belongs to $Q(A)$. Hence, by Lemma 2.1, $Q(A)$ is not finitely axiomatizable.

Acknowledgments

The author thanks professor A.M. Nurakunov for setting the problem, his attention and useful remarks.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 27, 2023

Revised: December 22, 2023

Accepted: January 15, 2024

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u-UNIFORMLY ČECH COMPLETE SPACES

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A.A. Borubaev in one of his seminars posed a problem: What uniform spaces can be perfectly mapped onto complete separable metrizable uniform spaces? Therefore, it is very important to find a class of uniform spaces that could be perfectly mapped onto complete separable metrizable uniform spaces. This class of spaces is proposed in this work. In this article, *u*-uniformly Čech complete space are introduced and studied. The problem of A.A. Borubaev is solved. Various characteristics of *u*-uniformly Čech complete space are established. In particular, the stability of *u*-uniformly Čech complete properties respect to the hyperspace of uniform spaces is showed.

Key words: countable covering, *u*-uniformly Čech complete space, separable metrizable space, perfect mapping.

А.А. Бөрүбаев өзүнүн семинарларынын биринде төмөнкүдөй маселени койгон: Кайсыл бир калыптуу мейкиндиктер толук сепарабелдүү метризацияланган мейкиндиктерге жеткилең чагылат? Ошондуктан толук сепарабелдүү метризацияланган мейкиндиктерге жеткилең чагылган бир калыптуу мейкиндиктердин классын табуу аракети маанилүү. Мейкиндиктердин мындай классы бул иште сунушталат. Бул илимий макалада Чех маанисиндеги *u*-бир калыптуу толуктук түшүнүгү киргизилет жана изилденет. А.А. Бөрүбаев койгон маселе чечилет. Чех боюнча *u*-бир калыптуу толук мейкиндиктердин түрдүү мүнөздөмөлөрү тургузулат. Чех маанисиндеги *u*-бир калыптуу толуктук ксиетинин бир калыптуу мейкиндиктердин гипермейкиндигине карата туруктуулугу көрсөтүлөт.

Негизги сөздөр: санактуу жабдуу, Чех маанисиндеги *u*-бир калыптуу толук мейкиндик, сепарабелдүү метризацияланган мейкиндик, жеткилең чагылдыруу.

А.А. Борубаев на одном его семинаре поставил задачу: какие равномерные пространства могут быть совершенно отображены на полные сепарабельно метризуемые пространства? Поэтому весьма важно попытаться найти класс равномерных пространств, который мог бы совершенно отображен на полные сепарабельно метризуемые пространства. Такой класс пространств предлагается в настоящей работе. В данной статье вводятся и исследуются *u*-равномерно полные в смысле Чеха пространства. Решается проблема А.А. Борубаева. Устанавливаются различные характеристики u-равномерно полных в смысле Чеха пространств. В частности, показывается устойчивость u-равномерно полных в смысле Чеха свойств относительно гиперпространства равномерных пространств.

Ключевые слова: счетное покрытие, u-равномерно полное в смысле Чеха пространство, сепарабельно метризуемое пространство, совершенное отображение.

Let D be a property of uniform coverings of the uniform space (X, U). The property D of uniform coverings can be: coverings of order $\leq n$, locally finite coverings, star finite coverings, point finite coverings, countable coverings.

Definition 1. Let (X, U) be a uniform space and $H \subset U$ be a subsystem consisting of coverings with the property D. Uniform space (X, U) is called $D - H$ -complete, if any H -Cauchy filter F has at least one cluster point, i.e. $\cap \{[M]: M \in F\} \neq \emptyset$.

Definition 2. Let(X, U) be a complete uniform space. Smallest cardinal numberτ called D -index complete of uniform space (X, U), if a system $H \subset U$ with the property D exists, such that $|H| = \tau$ and (X, U) is $D - H$ -complete.

D -index completeness of the space (X, U) is denoted by $D - ic(U)$. Let D be a countable covering. In this case D -index complete of the uniform space (X, U) is denoted by $c - ic(U)$.

The uniform space (X, U) is called D -uniformly locally compact, if exist a uniform covering with the property D, consisting of compact subsets.

Let D be a countable covering. We will simply call such D -uniformly locally compact spaces - strongly uniformly locally compact. It is clear that any strongly uniformly locally compact space is uniformly locally compact, and the converse is, generally speaking, not true, for example, a discrete uniform space D_τ cardinality $\tau > \aleph_0$ is uniformly locally compact, but it is not strongly uniformly locally compact.

Proposition 1. For a complete uniform space(X, U), the following conditions are equivalent:

1. $D - ic(U) = 1$;
2. (X, U) is D -uniformly locally compact.

Proof. Let the system H consists of a single uniform covering $\alpha \in U$ with property D and the uniform space (X, U) is $D - H$ -complete. Then there is a uniform covering $\gamma \in U$ with the property D consisting of closed subsets and $\gamma \succ \alpha$. Let us prove that $\gamma \in U$ consists of compacts. Let $G \in \gamma$ be some set and F_G be some filter on G. Then the system $F = \{M \subset X: \exists M_G \in F_G: M_G \subset M\}$ is H–Cauchy filter. By virtue of $D - H$ -completeness of space (X, U) the filter F has a cluster point $x \in G$ and that's why $x \in G$ is cluster point of F_G . So G is compact.

If a uniform covering $\alpha \in U$ with the property D consists of compacts, then every H -Cauchy filter is such that $H = \{\alpha\}$ has a cluster point. Therefore, $D - ic(U) = 1$.

Corollary 1. For complete uniform space (X, U) the following conditions are equivalent:

1. $c - ic(U) = 1$;
2. (X, U) is c -uniformly locally compact.

Proposition 2. Let $f: (X, U) \to (Y, V)$ be a perfect uniformly continuous mappings of uniform space (X, U) to the space (Y, V). Then (X, U) also complete and $c - ic(U) \leq c - ic(V)$.

Proof. Let $P \subset V$ be a system of countable coverings such that the space (Y, V) is $c - P$ -complete and $c - ic(V) = |P|$. Let $H = \{f^{-1}\gamma: \gamma \in P\}$. Let us show that the space (X, U) is $c - H$ -complete. Let us choose an arbitrary H -Cauchy filter Fin (X, U). Then fF is a P -Cauchy filter in (Y, V) and by virtue of P -completeness of space (Y, V) fF has a cluster point $y \in Y$. Since the mapping f is perfect, then the filter F has a cluster point $x \in f^{-1}y$. Therefore, (X, U) is H -complete. Since $|H| \leq |P|$, then $c - ic(U) \leq c - ic(V)$.

Lemma 1. Let (X, U) be a complete uniform space and $c - ic(U) \leq \tau$ ($\tau \leq \aleph_0$). Then there exists a pseudouniformity $B \subset U$consisting of countable coverings, such that $w(B) \leq \tau$and the system B is $c - B$ -complete.

Proof. Let $H \subset U$ be a system of countable coverings, such that $c - ic(U) = |H|$. For each covering $\gamma \in U$, let us choose the following normal sequence of countable uniform coverings $\phi_\gamma = \{\gamma_n\}$, t$\gamma_1 = \gamma$. Put $H' = \cup \{\phi_\gamma: \gamma \in H\}$.

We denote by H''all the finite internal intersections from H', B be a family consisting of all coverings, in each of which can be inscribed a covering from H''. Clearly, the family B is complete pseudouniformly and $w(B) \leq \aleph_0$.

Uniform space (X, U) is called a u -uniformly complete Čech space, if it is complete and $c - ic(U) \leq \aleph_0$.

The following theorem is a solution to the above problem posed by A.A. Borubaev.

Theorem 1. For a uniform space(X, U) the following conditions are equivalent:

1. (X, U) is a u -uniformly Čech complete space;
2. The uniform space (X, U) is mapped onto some complete separable metrizable uniform space (Y, V) by a perfect uniformly continuous mapping;
3. A uniform space (X, U) is the limit of the inverse system $\{(X_a, U_a), h_a^b, M\}$, consisting of complete separable metrizable uniform spaces (X_a, U_a) and perfect uniformly continuous mappings "onto" h_a^b.

Proof. 1. $\Rightarrow$ 2. Let (X, U) be a complete uniform space and $c - ic(U) \leq \aleph_0$. Then by Lemma 1 there exists a pseudouniformity $B \subset U$ consisting of countable coverings such, that $w(B) \leq \aleph_0$ and the system B is strongly complete. Then according to Lemma 1.2.13. (see [1], p. 42) for a pseudouniform space (X, B) there exists a uniformly continuous onto mapping $f: (X, B) \to (Y, V)$ such that the family $\{f^{-1}\gamma: \gamma \in V\}$ forms the base of the pseudouniformity B and $w(B) \leq \aleph_0$. We show that the uniform space (Y, V) is complete. Let F_Y be an arbitrary Cauchy filter in (Y, V) and consisting of countable coverings, which $F = \{M \subset X: M \supset f^{-1}L \text{for some } L \in F_Y\}$. Since the family $\{f^{-1}\gamma: \gamma \in V\}$ is a base for the pseudouniformity B. F is a B -Cauchy filter in (X, U). Due to the B -completeness of the space (X, U) we conclude that the filter F has a cluster point in (X, U). From uniform continuity of the mapping f and $fF = F_Y$ follows that F_Y has a cluster point in (Y, V). Therefore, (Y, V) is complete. Next, we prove that the mapping $f: (X, B) \to (Y, V)$ is perfect. Let $F_{f^{-1}y}$be some filter in$f^{-1}y$,$y \in Y$. Put$F_X = \{M \subset X: M \supset M_{f^{-1}y} \forall M_{f^{-1}y} \in F_{f^{-1}y}\}$. Since $f^{-1}y = \cap \{\alpha(f^{-1}y): \alpha \in B\}$, then F_X is the Cauchy B -filter in (X, B). Due to the B -

completeness of the space (X, U), the filter F_X has a cluster point $x \in f^{-1}y$. The point $x \in f^{-1}y$ is a cluster point of the filter $F_{f^{-1}y}$, i.e. $f^{-1}y$ is compact. Let $y \in Y$ be an arbitrary point and O be any open set of the (X, U) such, that $O \supset f^{-1}y$. We show that there exists a covering $\alpha \in B$, such that $\alpha(f^{-1}y) \subset O$. Suppose $L_\alpha = \alpha(f^{-1}y) \cap (X \setminus O) \neq \emptyset$ for any $\alpha \in B$ and $y \in Y$. It's easy to see that $\eta = \{L_\alpha: \alpha \in B\}$ is centered system. Put$F = \{C \subset X: C \supset L_\alpha \forall L_\alpha \in \eta\}$. It is clear that F is a filter in X. Let $\alpha \in B$ be an arbitrary uniform covering. Then there is a uniform covering $\beta \in B$ such that$\beta * \succ \alpha$, i.e. exists $A \in \alpha$ and $x \in f^{-1}y$ such that $\beta(x) \subset A$. But $\beta(x) \subset \beta(f^{-1}y)$, so, $A \in F$ i.e. F is Cauchy B -filter in (X, U). Since (X, U) is B -complete, then$\cap \{[M]: M \in F\} \neq \emptyset$. But $\cap [[M]: M \in F\} \subset \cap \{[\alpha(f^{-1}y)] \cap (X \setminus O): \alpha \in B\} = \ = \cap \{\alpha(f^{-1}y) \cap (X \setminus O): \alpha \in B\} = \emptyset$. A contradiction, consequently, the mapping f is closed. The implication $2. \Rightarrow 1.$ follows from Proposition 2.

$1. \Rightarrow 3.$ Let$\{(X_a, B_a): a \in M\}$ be the family of all complete separable metrizable pseudouniform spaces (X_a, B_a) such that $B_a \subset U$ for all $a \in M$ and each B_a contains countable covering. According by Lemma 1. $\{B_a: a \in M\} \neq \emptyset$. For the set M introduced a partial order relation as follows:$a < b$ if and only if $B_a \subset B_b$ and$B_a \neq B_b$. Note that if $a, b \in M$, then the $B_c = sup\{B_a, B_b\} \subset U$ is complete separable metrizable pseudouniformity, therefore, $M$is directed. According to Lemma 1.1.13 (see [1], p. 42), a complete separable metrizable uniform space (X_a, U_a) can be constructed for each pseudouniformity $B_a$and perfect uniform continuous mapping$f_a: (X, U) \to (X_a, U_a)$. Put$X_a = \{[x]_a: x \in X\}$, $[x]_a = \cap \{\alpha(x): \alpha \in B_a\}$, $f_a x = [x]_a$, $x \in X$and $U_a = \{f_a^{\#}\alpha: \alpha \in B_a\}$, $f_a^{\#}\alpha = \{f_a^{\#}A: A \in \alpha\}$, $f_a^{\#}A = X_a \setminus f_a(X \setminus A)$. If $a < b$ then the mapping $f_a^b: X_b \to X_a$ is determined by the formula: $f_a^b([x]_b) = [x]_a$, $x \in X$. It is easy to see that the mapping $f_a^b: (X_b, U_b) \to (X_a, U_a)$ is uniformly continuous. We proof that f_a^b is perfect. Since $f_a = f_a^b \circ f_b$ and f_a is perfect mapping, then the map f_a^b is also perfect mapping. It's easy to see that if $a < b < c$, then $f_a^c = f_a^b \circ f_b^c$. We have got the inverse system $S = \{(X_a, U_a), f_a^b, M\}$ consisting of complete separable metrizable uniform spaces and perfect uniformly continuous mappings "onto" f_a^b. Let $(\hat{X}, \hat{U}) = \varprojlim$. We prove that there is a uniform homeomorphism $f: (X, U) \to (\hat{X}, \hat{U})$.

Let f be the diagonal product of the family of mappings f_a, i.e. $fx = \{f_a x\}$, $x \in X$. Therefore, $f: (X, U) \to \prod\{(X_a, U_a): a \in M\}$is uniformly continuous "onto" mapping. It $f_a x = f_a^b(f_b x)$ follows that $\{f_a x\}_{a \in M}$ is a thread of the spectrum inverse system S and $fx \in \hat{X}$ i.e. $f: (X, U) \to (\hat{X}, \widehat{U})$ is a uniformly continuous mapping. From the separation of space (X, U) it follows that mapping$f$ is bijective. It is clear that $\cup \{B_a: a \in M\}$ forms the basis of the uniformity U. It is easy to prove that the mapping f is a uniform homeomorphism. Hence, $f: (X, U) \to (\hat{X}, \widehat{U})$ is a uniform homeomorphism.

3. $\Rightarrow$ 1. Let$(X, U) = \underleftarrow{lim_{a}{}_{a_a}^{b}}$ be inverse system where (X_a, U_a) is a complete separable metrizable uniform space for each $a \in M$ and f_a^b is a perfect uniformly continuous mapping "onto". Then the projection $f_a: (X, U) \to (X_a, U_a)$ is uniformly continuous mapping. Since it is f_a^ba perfect uniformly continuous mapping "onto", it is also f_a a perfect uniformly continuous mapping "onto". From Proposition 1. follows that (X, U) is complete and $D - ic(U) \leq \aleph_0$.

Proposition 3. Any u-uniformly Čech complete space is uniformly paracompact.

Proof. Let (X, U) be a u-uniformly Čech complete space. Then there exist a complete separable metrizable space (Y, V) and a perfect uniformly continuous "onto" mapping $f: (X, U) \to (Y, V)$. Choose a countable base $\{\beta_n\}$ of the separable metrizable space (Y, V) consisting of countable coverings. Put $\alpha_n = f^{-1}\beta_n$. We show that the sequence $\{\alpha_n\}$ of countable uniform coverings for any finite additive open coverings λ of the space (X, U) satisfies conditions (BP) [2]. Let's choose an arbitrary point $x \in X$. Because $f^{-1}y$ is compact, then there exist a finite number of elements L_i of coverings λ such that $f^{-1}y \subset \overset{i=1}{\underset{n}{\cup}} L_i$. It is clear that $\overset{i=1}{\underset{n}{\cup}} L_i \in \lambda$. Let $L = \overset{i=1}{\underset{n}{\cup}} L_i$. Due to the closedness of the mapping f the set $O = Y \backslash f(X \backslash L)$ is a neighborhood of the point$y \in Y$. Then there is such $n \in N$ that $\beta_n(y) \subset O$. Then$\alpha_n(x) \subset L$, $x \in f^{-1}y$. Hence, (X, U) is uniformly paracompact.

Theorem 2. For a uniform space (X, U) the following conditions are equivalent:

1. (X, U) is a u -uniformly Čech complete space;
2. $(exp_c X, exp_c U)$ is a u -uniformly Čech complete space.

Proof. Let (X, U) be complete and $c - ic(U) \leq \aleph_0$. Then, according to Theorem 1, the space (X, U) is mapped onto some complete separable metrizable space (Y, V), i.e. there is a perfect uniformly continuous mapping $f: (X, U) \to (Y, V)$ of a uniform space (X, U) to a complete separable metrizable space (Y, V). It is known that the mapping $exp_c f: (exp_c X, exp_c U) \to (exp_c Y, exp_c V)$, $exp_c(f)(F) = f(F)$, $F \in exp_c X$ is a perfect uniformly continuous mapping "onto". Then according to Theorem 0.6.5. (see [1], p. 30) the uniform space $(exp_c Y, exp_c V)$ is complete, and according to Corollary 0.6.2. (see [1], p. 30) and conclusion 0.6.4. (see [1], p. 30), space is separable metrizable $(exp_c Y, exp_c V)$. It follows by Proposition 1. that $(exp_c X, exp_c U)$ is complete and $c - ic(exp_c U) \leq \aleph_0$.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 20, 2023

Revised: December 18, 2023

Accepted: January 15, 2024

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INDEX OF SEQUENTIAL COMPLETENESS OF UNIFORM SPACES AND UNIFORMLY CONTINUOUS MAPPINGS

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In this article, we study the sequential completeness of uniform spaces and uniformly continuous mappings. Index of sequential completeness $\mathrm{ic_s(U)}$ of a uniform space $\mathrm{(X, U)}$ and index of sequential completeness $\mathrm{ic_s(f)}$ of a uniformly continuous mapping $\mathrm{f: (X, U) \to (Y, V)}$ between uniform spaces $\mathrm{(X, U)}$ and $\mathrm{(Y, V)}$, and also sequential Dieudonne completeness of a spaceX are defined. In particular, it is established $\mathrm{ic_s(U) = 1}$ iff $\mathrm{(X, U)}$ is uniformly locally countable compact; $\mathrm{ic_s(f) = 1}$ iff a mapping f is uniformly locally quasi-perfect. Tychonoff space X is sequential Dieudonne complete iff a uniform space $\mathrm{(X, U_X)}$ with the universal uniformity $\mathrm{U_X}$ is sequentially complete.

Key words: sequential completeness, countable compactness, countable paracompactness, uniformly local paracompactness, uniform space, mappings.

Бул макалада бир калыптуу мейкиндиктердин жана бир калыптуу үзгүлтүксүз чагылдыруулардын секвенциалдуу толуктугу изилденет. $\mathrm{(X, U)}$ бир калыптуу мейкиндиктердин $\mathrm{ic_s(U)}$ секвенциалдуу толуктуулук индекси жана$\mathrm{(X, U)}$, $\mathrm{(Y, V)}$ бир калыптуу мейкиндиктердин арасындагы $\mathrm{f: (X, U) \to (Y, V)}$ бир калыптуу үзгүлтүксүз чагылдыруулардын $\mathrm{ic_s(f)}$ секвенциалдуу толуктуулук индекси жана $\mathrm{ic_s(U) = 1}$ болот качан гана $\mathrm{(X, U)}$ бир калыптуу мейкиндик бир калыптуу локалдуу санактуу компактуу болгондо гана делген жана $\mathrm{ic_s(f) = 1}$ болот качан гана f чагылдыруу бир калыптуу локалдуу квазижеткилең болгондо гана делген натыйжалар далилденет. Ошондой эле X тихоновдук мейкиндик Дьедонне боюнча секвенциалдуу толук болот качан гана $\mathrm{(X, U_X)}$ бир калыптуу мейкиндик $\mathrm{U_X}$ универсалдуу бир калыптуулугу менен секвенциалдуу толук болгондо гана делген натыйжалар дагын далилденет.

Урунттуу сөздөр: секвенциалдуу толуктуулук, санактуу компактуулук, санактуу паракомпактуулук, бир калыптуу локалдуу паракомпактуулук, бир калыптуу мейкиндик, чагылдыруулар

В данной статье изучается секвенциальная полнота равномерных пространств и равномерно непрерывных отображений. Вводятся индекс секвенциальной полноты $\mathrm{ic_s(U)}$ равномерных пространств $\mathrm{(X, U)}$ и индекс секвенциальной полноты $\mathrm{ic_s(f)}$ равномерно непрерывных отображений $\mathrm{f: (X, U) \to (Y, V)}$ равномерных пространств $\mathrm{(X, U)}$ и $\mathrm{(Y, V)}$, а также,

секвенциально полные в смысле Дьедонне пространств X. В частности, доказываются, что $ic_s(U) = 1$ в том и только в том случае, если (X, U) является равномерно локально счетно компактным, и, $ic_s(f) = 1$ в том и только в том случае, если отображение f является равномерно локально квазисовершенным. Также, доказывается, что тихоновское пространство X является секвенциально полным по Дьедонне тогда и только тогда, когда равномерное пространство (X, U_X) с универсальной равномерностью U_X является секвенциально полным.

Ключевые слова: секвенциальная полнота, счетная компактность, счетная паракомпактность, равномерно локальная паракомпактность, равномерные пространства, отображенияяse

1. Uniformly Locally Countably Compactness

A Tychonoff space X is called:

(1) countable compact if every of its countable open covering has a finite open refinement [1];

(2) countable paracompact if every of its countable open covering has a locally finite open refinement [1];

(3) Locally countable compact if for every point of the space X there exists a neighborhood whose closure is countable compact [1].

A uniform space (X, U) is called:

(i) uniformly locally compact if there exists a uniform covering consisting of compact subsets [6];

(ii) uniformly countable paracompact if every of its countable open covering has a uniformly locally finite open refinement [7];

(iii) sequential complete, if every Cauchy filter with countable base converges in it [3].

The covering α is called finitely additive, if $\alpha^{\angle} = \alpha$, $\alpha^{\angle} = \{\cup\, \alpha_0 : \alpha_0 \subset \alpha \text{ is finite}\}$. For coverings $\alpha$and β of the set X, the symbol $\alpha \succ \beta$ means that the covering α is a refinement of the covering β, i.e. for any $A \in \alpha$ there exist $B \in \beta$ such that $A \subset B$. Let (X, U) be a uniform space.

Definition 1.1. A uniform space (X, U) is called uniformly locally countable compact if there exists a uniform covering such that the closures of all its elements are countable compact.

Proposition 1.1. Any uniformly locally compact uniform space (X, U) is uniformly locally countable compact.

Proof. Let (X, U) be a uniformly locally compact space. Then there exists a uniform covering such that the closures of all its elements are compact i.e. countable compact. Consequently, (X, U) is uniformly locally countable compact.

Proposition 1.2. Any uniformly locally countable compact space is uniformly countable paracompact.

Proof. Let (X, U) be a uniformly locally countable compact space and α be countable finitely additive open cover. Then there exists such a uniform covering $\lambda \in U$ the closure of every element of which is countable compact. Then it is easy to see that $\lambda \succ \alpha$, i.e. the covering λ is a refinement of the covering α. Consequently, (X, U) is uniformly countable paracompact.

Proposition 1.3. If a uniform space (X, U) is uniformly locally countable compact then the space (X, τ_U) is locally countable compact and countable paracompact.

Proof. Assume there exists a uniformity U such that the space (X, U) is uniformly locally countable compact. By Proposition 1.2. the space (X, U) is uniformly countable paracompact, therefore (X, τ_U) is countable paracompact. From the definition of uniform local countable compactness of the space (X, U) it follows local countable compactness of the space (X, τ_U).

Proposition 1.4. If for a uniform space (X, U) the topological space (X, τ_U) is countable compact, then the uniform space (X, U) is uniformly locally countable compact.

Proof. Let $\alpha \in U$ be an arbitrary uniformly covering. The space (X, τ_U) countable compact, therefore the closures of all elements covering α are countable compact. Consequently, (X, U) is uniformly locally countable compact.

Proposition 1.5. For a precompact uniform space (X, U) the following conditions are equivalent:

1. (X, U) is uniformly locally countable compact;
2. (X, τ_U) is countable compact.

Proof. $1 \Rightarrow 2$. Let (X, U) be a precompact uniformly locally countable compact and α be an arbitrary countable cover of the space (X, τ_U). Then, due the precompactness and uniform local countable compactness of the space (X, U), there exists a finite uniform covering $\beta \in U$ such that the closure of each its element is countable compact. Every $B \in \beta$, by virtue of countable compactness of the subspace $[B]$, is covered by a finite number of elements of the open covering α. Since the cover of $\beta \in U$ is finite, all systems of such finite subfamilies form a finite subcover of α. Thus, the countable compactness of (X, τ_U) is proved.

$2 \Rightarrow 1$ follows directly from Propositions 1.3.

Theorem 1.1. Any locally countable compact topological group $(G, \cdot, \tau)$ is countable paracompact.

Proof. Let $(G, \cdot, \tau)$ be a locally countable compact topological group. Then for every point of the space G there exists a neighborhood whose closure is countable compact. It is easy to see that the uniform space (G, U) is uniformly locally countable compact. Then, according to Proposition 1.2. the group $(G, \cdot, \tau)$ is countable paracompact.

2. Index of μ -completeness of uniform spaces

The following theorem characterizes countable compact spaces by means of uniform structures.

Theorem 2.1. A Tychonoff space X is countable compact if and only if for each uniformity U the uniform space (X, U) is countable complete.

Proof. Let Tychonoff space (X, τ) be countable compact and a Cauchy filter F of the space (X, U) having a countable base. Then F has an adherent point of the space (X, U). Since in a uniform space of concept the point of adherent and limit points are equivalent, this implies the F convergence in (X, U). Hence, (X, U) is sequentially complete.

Conversely, let F be an arbitrary ultrafilter, which has a countable base and (X, U)

- any such uniform space that $\tau_U = \tau$. Then there exists a precompact uniformity U_P, that $\tau_{U_P} = \tau_U = \tau$. Clearly, F is a Cauchy filter in (X, U_P). This, it converges to some points in (X, U_P). Thus, the ultrafilter F has a countable pre-base converges to (X, τ).

Any complete space is sequentially complete, however, the following theorem shows that in the class of metrizable spaces the concepts of completeness and sequential completeness coincide.

Theorem 2.2. If uniform space (X, U) is sequentially complete and metrizable, then the space (X, U) is complete.

Theorem 2.3. Let $f: (X, U) \rightarrow (Y, V)$ be a perfect uniformly continuous mapping of a uniform space (X, U) onto sequentially complete space (Y, V). Then the uniform space(X, U) is also sequentially complete and $ic_s(U) \leq ic_s(V)$.

Proof. Let $P \subset U$ be a system of covers such that the space (Y, V) is P -sequentially complete and $ic_s(V) = |P|$. We put $H = \{f^{-1}(\beta): \beta \in P\}$. We show that the space (X, U) is H -sequentially complete. Let F be any Cauchy H -filter in (X, U) having countably base. Then the filter fF is Cauchy P -filter in (Y, V) and due to the P -sequential completeness of the space (Y, V) has a condensation point $y \in Y$. Since the mapping f is perfect, then the filter Fhas a condensation point $x \in f^{-1}y$. Therefore, the space (X, U) is H -sequentially complete. It is clear that sequential $|H| \leq |P|$ so $ic_s(U) \leq ic_s(V)$.

Definition 2.1. Tychonoff space$X$is called sequential Dieudonne complete space, if there is a uniformity Uagreed with the topology of a space X, such that a uniform space (X, U) is sequentially complete.

Theorem 2.4. Tychonoff space X is sequentially Dieudonne complete iff a uniform space (X, U_X) with the universal uniformity U_X is sequentially complete.

Proof. Necessity. Let X be is sequential Dieudonne complete space and F be an arbitrary Cauchy filter in (X, U_X) with a countable base. By virtue of the sequentially Diedonne completeness of the space X, there exists a sequentially complete uniformity of U . Since $U_X \supset U$, then F is a Cauchy filter in (X, U). Then F converges to some point x in (X, U). Therefore, the uniform space (X, U_X) is sequentially complete.

Sufficiency. Let a uniform space (X, U_X) be sequentially complete. Then, by the definition of sequential Diedonne completeness the Tychonoff space X is a sequentially Diedonne complete.

Let (X, U) be uniform space and $H \subset U$ the system of uniform covers. The filter F is called H -Cauchy filter if $\alpha \cap F \neq \emptyset$ for any $\alpha \in H$ [2].

Definition 2.2. A uniform space (X, U) is called H -$\aleph_0$-complete if every H -Cauchy filter F having a countable base has at least one touch point.

Definition 2.3. The smallest cardinal number η is called index of sequential completeness of a uniform space (X, U) if there exists a system $H \subset U$ such that $|H| = \eta$ and (X, U) is H -$\aleph_0$-complete. The index of sequential completeness of uniform space (X, U) is denoted by $ic_s(U)$.

Theorem 2.5. For a sequentially complete uniform space (X, U) the following conditions are equivalent:

1. $ic_s(U) = 1$;
2. The uniform space (X, U) is uniformly locally countable compact.

Proof. $1 \Rightarrow 2$. Let $H = \{\alpha\}$ and (X, U) be H -sequentially complete. Then there is such a uniform covering $\beta \in U$ consisting of closed subsets. We show that every element B of β consists of countable compact subsets. Let B be an element of β and F_B be an arbitrary Cauchy filter in subspace (B, U_B) with countable base. Suppose $F = \{N \subset X$: there is $N_B \in F_B$ such that $N_B \subset N\}$. We prove that F is a H -sequently-Cauchy filter in (X, U). There exists $A_B \in \alpha_B$ such that $A_B \in F_B$, since F_B is an H -sequential-Cauchy filter in (B, U_B). From $A_B \subset A$ it follows that $A \in F$. Hence, F is an H -sequential-Cauchy filter. Then F has a cluster point $x \in B$. Therefore, a point $x \in B$ is the cluster point of the Cauchy filter F_B, i.e. B is countably compact. Consequently, (X, U) is uniformly locally countable compact.

$2 \Rightarrow 1$. Let (X, U) be uniformly locally countable compact. Then there is a uniform covering $\alpha \in U$ consisting of countable compact subsets. Let $H = \{\alpha\}$. Then any H -Cauchy filter F having a countable base has a cluster point. Hence, $ic_s(U) = 1$.

Let $f: (X, U) \to (Y, V)$ be uniformly continuous mapping of uniform space (X, U)

to uniform space (Y, V). The pseudouniformity $U_f \subset U$ is called the base of the uniformly continuous mapping f, if for any $\alpha \in U$ there are $\beta \in V$ and $\gamma \in U_f$, such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [2].

Every uniformly continuous map possesses, generally speaking, many bases. The least cardinal number τ that is the weight of some base U_f of the mapping f, is called the weight of the mapping $f$and is denoted by $w(f)$.

Recall, a uniform continuous mapping $f: (X, U) \to (Y, V)$ between the uniform spaces (X, U) and (Y, V) is called complete, if for any Cauchy filter F of a uniform space (X, U) for it the image fF converges in (Y, V), converges in(X, U) [2]; a uniform continuous mapping $f: (X, U) \to (Y, V)$ between the uniform spaces (X, U) and (Y, V) is called sequentially complete, if for any Cauchy filter F with countable base of a uniform space (X, U) for it the image fF converges in (Y, V), converges in (X, U) [5]; a uniform continuous mapping $f: (X, U) \to (Y, V)$ between the uniform spaces (X, U) and (Y, V) is called metrizable, if f have countable base [3].

3. Index of μ -completeness of uniformly continuous mappings

Any complete uniformly continuous mapping is sequentially complete.

Theorem 3.1. If uniform continuous mapping $f: (X, U) \to (Y, V)$ is sequentially complete and metrizable, then the map f is complete.

Definition 3.1. A uniform continuous mapping$f: (X, U) \to (Y, V)$ between the uniform spaces (X, U) and (Y, V) is called H -sequentially complete, if for any H -Cauchy filter F with countable base of a uniform space (X, U) for it the image fF converges in (Y, V), converges in(X, U).

Definition 3.2. The smallest cardinal number η is called index of sequential completeness of a uniformly continuous mapping f if there exists a system $H \subset U_f$ such that $|H| = \eta$ and f is H -sequentially complete. The index of sequential completeness of uniform continuous mapping f denoted by $ic_s(f)$.

Proposition 3.1. If $f: (X, U) \to (Y, V)$ - uniformly continuous mapping of the sequentially complete space (X, U) to uniform space (Y, V), then mapping f is sequentially complete, conversely, if mapping f of the uniform space (X, U) to the one point space $Y = \{y\}$ is sequentially complete, then a uniform space (X, U) is sequentially complete.

Theorem 3.2. If mapping f and space (Y, V) is sequentially complete, then a uniform space (X, U) is sequentially complete.

Definition 3.3. A uniformly continuous mapping $f: (X, U) \to (Y, V)$ of the uniform space (X, U) to a uniform space (Y, V) called uniformly locally quasi-perfect, if there exists a uniform covering $\alpha \in U$, such that the mapping $f|_A: (A, \tau_{U_A}) \to (Y, \tau_V)$ is quasiperfect for all $A \in \alpha$.

Theorem 3.3. For sequentially complete uniformly continuous mapping $f: (X, U) \to (Y, V)$ of the uniform space (X, U) to a uniform space (Y, V) the following conditions are equivalent:

1. $ic_s(f) = 1$;
2. f is uniformly locally quasiperfect.

Proof. $1 \Rightarrow 2$. Let $H = \{\alpha\} \subset U$ and f be H - sequentially complete. Then exists a uniform covering $\beta \in U$, such that $\beta \succ \alpha$. We show that $f|_B: (B, \tau_{U_B}) \to (Y, \tau_V)$ is quaziperfect for all $B \in \beta$. Let F_B be an arbitrary Cauchy filter F with countable base of a uniform space (B, U_B), such that $f|_B F_B$ is convergent in (Y, V). It is easy to see that the family $F = \{M \subset X: M_B \in F_B, M_B \subset M\}$ is H -filter Cauchy having a base cardinality $\leq \mu$ in (X, U). Due to H -sequentially completeness of the mapping f, it converges to some point belonging to B. It follows that F_B converges in (B, U_B). So the mapping f is quaziperfect.

$2 \Rightarrow 1$. Let the mapping $f: (X, U) \to (Y, V)$ be uniformly locally quaziperfect. Then there exists $\alpha \in U$ such that $f|_A$ is quaziperfect for all $A \in \alpha$. Put $H = \{\alpha\}$. Let F be an arbitrary H -Cauchy filter having a countable base in (X, U). Then there exists $A \in \alpha$ such that $A \in F$. $B_A = \{A \cap N: N \in F\}$. Then it is easy to see that B_A is the base of some Cauchy filter F_A in (A, U_A). By condition F_A converges in (A, U_A). Consequently, the Cauchy H –filter Fconverges. So f is H - sequentially complete and $ic_\mu(f) = 1$.

Theorem 3.4. Let $f: (X, U) \to (Y, V)$ be a uniformly open mapping and $ic_s(f) \leq \tau$. If $ic_s(V) \leq \tau$, then $ic_s(U) \leq \tau$.

Proof. Let the condition of the theorem be satisfied. Then exists $H_f \subset U$ such that $|H_f| \leq \aleph_0$ and mapping f is H_f-sequentially complete. Show that $ic_s(U) \leq \tau$. Let F be an arbitrary H_f-filter Cauchy in (X, U), having a countable base. According to the conditions of the theorem there exists $H_Y \subset V$ such that $|H_Y| \leq \aleph_0$ and uniform (Y, V) is H_Y- sequentially complete. Let $H_{f^{-1}Y} = f^{-1}H_Y$. It is clear that$H_{f^{-1}Y} \subset U$ and it consists of open coverings. Now let the system H_X consist of all elements of the systems $H_{f^{-1}Y}$ and H_f. Obviously $|H_X| \leq \aleph_0$. Let F_X be an arbitrary H_X-filter Cauchy in (X, U) having a countable base. Since f is a uniformly open mapping, then $fH_X \subset V$. fF_X converges in (Y, V), since F_X is H_f-filter Cauchy in (X, U) having countable base and fF_X is H_Y-filter Cauchy in (Y, V). Then due to H_f- sequentially completeness of the mapping f F_X converges in (X, U). So, is $ic_s(U) \leq \tau$.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 22, 2023

Revised: December 18, 2023

Accepted: January 15, 2024

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MSC 53C44, 37C05, 37C10

ON DYNAMICS OF THE NORMALIZED RICCI FLOW ON THE WALLACH SPACES

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We study a system of nonlinear differential equations that can be obtained as a result of reduction of the normalized Ricci flow equation on generalized Wallach spaces. The main idea based on an observation that Einstein metrics are exactly singular points of the obtained differential system and vice versa. Therefore to describe how the Ricci flow evolves Riemannian metrics it suffices to determine types of singular points, their stable and unstable manifolds responsible for the behavior of trajectories. We prove that on the Wallach spaces singular points are all of hyperbolic types and conclude that the Ricci flow possesses the simple dynamics on these spaces.

Keywords: generalized Wallach space, Wallach space, Riemannian metrics, Einstein metrics, normalized Ricci flow, dynamical system, hyperbolic singular point, semi-hyperbolic singular point, nilpotent singular point, linearly zero singular point.

Биз нормалдаштырылган Риччи агымы теңдемесин жалпыланган Уоллах мейкиндиктерине редукциялоо учурунда алынчу сызыктуу эмес дифференциалдык теңдемелер системасын изилдейбиз. Башкы идея Эйнштейн метрикалары алынган дифференциалдык системанын өзгөчө чекиттери болоорун жана тескерисинче экенин байкоого негизделген. Демек, Риччи агымы Риман метрикаларын кандайча өзгөртөөрүн сыпатташ үчүн ошол өзгөчө чекиттердин тибин жана траекториялардын жүрүш-турушуна жооптуу алардын туруктуу жана туруксуз көптүспөлдүүлүктөрүн аныктоо жетиштүү. Биз Уоллах мейкиндиктеринде өзгөчө чекиттердин бардыгы гиперблалык типте экенин далилдейбиз жана ушул мейкиндиктерде Риччи агымы жөнөкөй динамикага ээ экенин корутундулайбыз.

Урунттуу сөздөр: жалпыланган Уоллах мейкиндиги, Уоллах мейкиндиги, Риман метрикасы, Эйнштейн метрикасы, нормалдаштырылган Риччи агымы, динамикалык система, гиперболалык өзгөчө чекит, жартылай гиперболалык өзгөчө чекит, нильпотенттик өзгөчө чекит, сызыктуу нөлдүк өзгөчө чекит.

Мы изучаем систему нелинейных дифференциальных уравнений, которая получается в результате редукции уравнения нормализованного потока Риччи на обобщённых пространствах Уоллаха. Главная идея основана на наблюдении того, что все эйнштейновы метрики являются особыми точками полученной дифференциальной системы и, наоборот.

Следовательно, чтобы описать, как поток Риччи переводит римановы метрики, достаточно определить типы особых точек, их устойчивые и неустойчивые многообразия, ответственные за поведение траекторий. Мы доказываем, что в пространствах Уоллаха все особые точки имеют гиперболический тип и заключаем, что поток Риччи обладает простой динамикой на этих пространствах.

Ключевые слова: обобщённые пространства Уоллаха, пространства Уоллаха, риманова метрика, эйнштейнова метрика, нормализованный поток Риччи, динамическая система, гиперболическая особая точка, полу-гиперболическая особая точка, нильпотентная особая точка, линейно нулевая особая точка.

The study of the normalized Ricci flow equation $\partial_t g(t) = -2Ric_g + 2n^{-1}g(t)\,S_g$, was initiated by R.Hamilton in [1] for an one-parametric family of Riemannian metrics $g(t)$ on a Riemannian manifold M^n, where Ric_g and S_g are the Ricci tensor and the scalar curvature of the metric g, and has a long history since 1982. An interesting class of homogeneous spaces was introduced in [2,3] called *three-locally symmetric spaces* or *generalized Wallach spaces*. Such spaces are characterized as compact homogeneous spaces G/H with an isotropy representation that can be decomposed into a direct sum of three irreducible modules and can be characterized by a triple of real parameters $(a_1, a_2, a_3) \in (0, 1/2)^3$ (see [2-4] and references therein for details). Note that classifications of generalized Wallach spaces were proposed recently by Yu.Nikonorov in [4] and independently by Chen Z., Kang Y. and Liang K. in [5]. On generalized Wallach spaces the normalized Ricci flow equation can be reduced to the following system of ODEs studied in [6-11]:

$$\dot{x}(t) = f_1(x,y,z), \quad \dot{y}(t) = f_2(x,y,z), \quad \dot{z}(t) = f_3(x,y,z), \tag{1}$$

where $x = x(t) > 0,\ y = y(t) > 0,\ z = z(t) > 0, \quad (a_1, a_2, a_3) \in (0,\ 1/2)^3$,

$$f_1(x,y,z) = -1 - a_1 x\left(\frac{x}{yz} - \frac{y}{xz} - \frac{z}{xy}\right) + xB,$$

$$f_2(x,y,z) = -1 - a_2 y\left(\frac{y}{xz} - \frac{x}{yz} - \frac{z}{xy}\right) + yB,$$

$$f_3(x,y,z) = -1 - a_3 z\left(\frac{z}{xy} - \frac{y}{xz} - \frac{x}{yz}\right) + zB,$$

$$B=\left(\frac{1}{a_1x}+\frac{1}{a_2y}+\frac{1}{a_3z}-\frac{x}{yz}-\frac{y}{xz}-\frac{z}{xy}\right)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)^{-1}.$$

It should be noted that singular points of the system (1) are exactly *Einstein metrics* of generalized Wallach spaces and vice versa. In [6-9] we studied singular (equilibria) points of (1). In particular, a special algebraic surface was found in [6] that provides degenerate singular points to (1) and its topological structure was described in [8]; in [9] two-parametric bifurcations were studied. In [10] we proved theorems covering some results of [12,13] related to the evolution of positively curved invariant Riemannian metrics on the Wallach spaces $SU(3)/T^2$, $Sp(3)/Sp(1)^3$ and $F_4/Spin(8)$ that was introduced in [14] and can be considered as partial cases $a_1=a_2=a_3\coloneqq a$ of generalized Wallach spaces corresponding to the values $a=1/6$, $a=1/8$ and $a=1/9$ respectively. Thus suppose $a_1=a_2=a_3\coloneqq a\in(0,1/2)$. It is not difficult to prove that the function $V=xyz$ is a first integral of (1): $f_1V_x+f_2V_y+f_3V_z=0$. Therefore any surface $V=c$ is invariant under (1). Using this fact and the unit volume condition $V=1$ we can eliminate $z=(xy)^{-1}$, for example, and get the planar system

$$\dot{x}(t)=xy^{-1}+x^2y-2-2ax(2x^2-y^2-x^{-2}y^{-2}),$$
$$\dot{y}(t)=yx^{-1}+y^2x-2-2ay(2y^2-x^2-x^{-2}y^{-2}),\qquad(2)$$

more convenient to the analysis. A singular point is said to be *hyperbolic* if both of eigenvalues of the corresponding Jacobian matrix J have nonzero real parts. In [6,7] we also proved that the *nilpotent case* (eigenvalues both equal zero, but $J\neq 0$) never can occur for (2), and, hence all its degenerate singular points are *semi-hyperbolic* ($J\neq 0$ and only one of eigenvalues equals zero) for all $a\in(0,1/2)\setminus\{1/4\}$. At $a=1/4$ we have the *linear zero* case with $J=0$ (see Remark 1 below).

The present paper has the aim to show that the normalized Ricci flow has the simple dynamic on the Wallach spaces $SU(3)/T^2$, $Sp(3)/Sp(1)^3$ and $F_4/Spin(8)$ In this way we will give a justification of some properties of the stable, and unstable manifolds of singular points of (2) observed and used in [6,10] but omitted there. As it follows from Theorem 6 in [6] for every fixed $a\in(0,1/2)\setminus\{1/4\}$ the system (2) admits exactly 4 distinct singular points of the hyperbolic type such that one of them is a node (stable or unstable depending on a) and the other points are all saddles. But possessing a qualitative character based on topological ideas this theorem could not find singular points in analytic way (to show their coordinates). The following theorem answers that question:

Theorem 1. The following assertions hold for the system (2):

a) For $a \in (0,1/2)\backslash\{1/4\}$ the system (2) admits exactly four distinct hyperbolic equilibria (singular) points $o_0 = (1,1)$, $o_1 = (qk, q)$, $o_2 = (q, qk)$ and $o_3 = (q, q)$, where $q = \sqrt[3]{k^{-1}}$, $k = \frac{1-2a}{2a}$. Moreover, o_0 is a single node and the other points o_1, o_2, o_3 are all saddles. In addition, o_0 is unstable if $a < 1/4$ and is stable if $a > 1/4$.

b) The curves $x = y^{-2}$, $y = x^{-2}$ and $y = x$ denoted respectively by c_1, c_2, and c_3 are invariant under (2) for all $x > 0$, $y > 0$.

c) There is no center manifolds and in the case $a < 1/4$ the possible manifolds and eigenspaces are $W_0^u = Span\{(1,0),(0,1)\}$, $E_1^u = Span\{(0,1)\}$, $E_2^u = Span\{(1,0)\}$, $E_3^u = Span\{(-1,1)\}$, $E_1^s = Span\{(-2, q^3)\}$,$E_2^s = Span\{(q^3, -2)\}$, $E_3^s = Span\{(1,1)\}$.

Moreover $\{(x,y) \in c_1 \mid x > 1\} \subset W_1^s$, $\{(x,y) \in c_2 \mid 0 < x < 1\} \subset W_2^s$, $\{(x,y) \in c_3 \mid 0 < x < 1\} \subset W_3^s$, $\{(x,y) \in c_1 \mid 0 < x < 1\} \subset W_0^u$,

$\{(x,y) \in c_2 \mid x > 1\} \subset W_0^u$, $\{(x,y) \in c_3 \mid x > 1\} \subset W_0^u$, where W_i^s, W_i^u mean the stable and the unstable manifolds of o_i with corresponding tangent spaces E_i^s, E_i^u.

Remark 1. The case $a = 1/4$ is very original. It is also a bifurcation value (see [8]) when all singular points of (2) merge to the single point $o_0 = (1,1)$. Indeed $k = q = 1$ at $a = 1/4$. But o_0 is not hyperbolic now, since $J = 0$ implying that all possible eigenvalues are equal to zero. According to classification given in [15] this case contains a rich collection (65 types) of topologically different phase portraits. In [6] we proved that only the possibility takes place among them that is the saddle $(1,1)$

with 6 hyperbolic sectors around it (see the left panel of Figure 1).

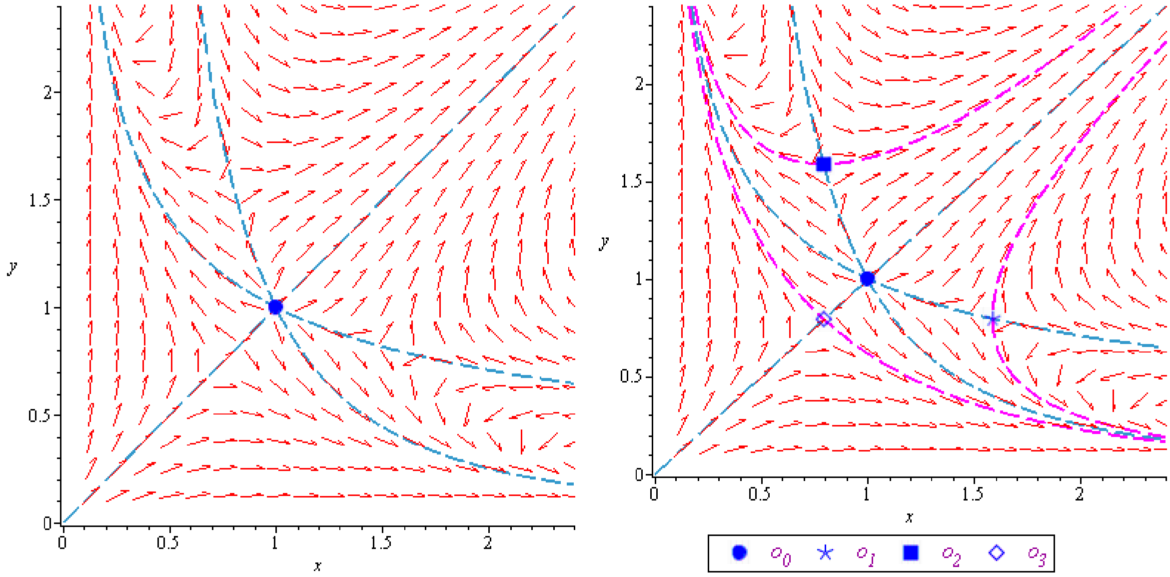

Figure 1. The phase portraits of the system (2): at $a = 1/4$ (the left panel) and at $a = 1/6$ (the right panel) which corresponds to the case $a \in (0, 1/2)\backslash\{1/4\}$.

Proof of Theorem 1. a) Let be J_i the Jacobian matrix corresponding to the singular point o_i of (2). Introduce the characteristic parameters δ=det(J), ρ=trace(J), σ=$\rho^2 - 4\delta$. Using results of [6] we obtain δ=$(1-4a)^2$, ρ=2(1-4a), σ= 0 for $o_0 = (1,1)$ and $\delta = \frac{(2a+1)(4a-1)^2}{(2a-1)q^2} < 0$ for the other singular points $o_1 = (qk, q)$, $o_2 = (q, qk), o_3 = (q, q)$. Therefore o_i is a hyperbolic saddle for each $i = 1,2,3$ and o_0 is a hyperbolic node. Depending on the sign of ρ it is stable (attracting) or unstable (repelling). In the right panel of Figure 1 singular points are depicted corresponding to $a = 1/6$.

b) It is obvious.

c) Indeed there is no zero eigenvalues of the Jacobian matrix at o_i, $i = 0,1,2,3$. Two coincided eigenvalues equal to $1-4a$ correspond to o_0 with eigenvectors (1,0) and (0,1). Since $1-4a > 0$ the manifold of o_0 is repelling. Among the saddles consider only $o_2 = (q, qk)$. Eigenspaces for other saddles can be found by similar way. At o_2 the Jacobian matrix admits two distinct eigenvalues $-\frac{(4a-1)^2(2a+1)q^2}{2a} > 0$ and $-\frac{(4a-1)^2(2a-1)q^2}{2a} < 0$ with a linearly independent eigenvectors (1,0) and

$(a, 2a-1)$ that form respectively the unstable (repelling) and the stable (attracting) subspases E_2^u and E_2^s. In the left panel of Figure 2 E_2^u and E_2^s are depicted as the saddle separatrices (the pale green and green straight lines) of the corresponding linearized system of ODEs

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = -\frac{(4a-1)q^2}{2a}\begin{pmatrix} 2a+1 & q^3 \\ 0 & 2a-1 \end{pmatrix}\begin{pmatrix} x(t)-q \\ y(t)-qk \end{pmatrix}. \tag{3}$$

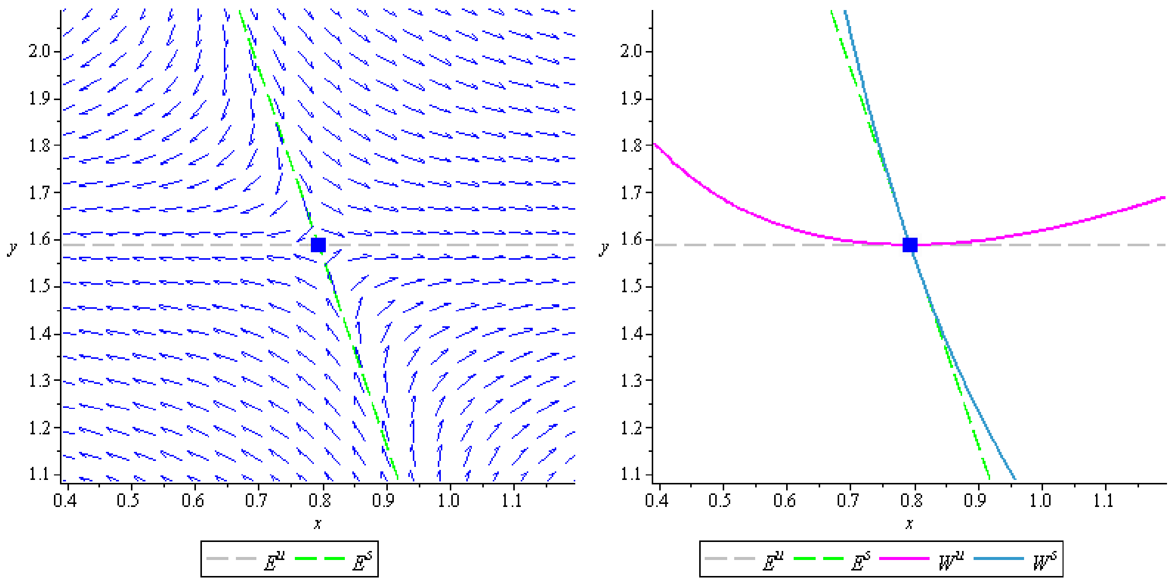

Figure 2. The phase portrait of the system (3) (the left panel) and the sets E_2^u, E_2^s , W_2^u and W_2^s (the right panel) at $a = 1/6$

Note that E_2^u and E_2^s are tangent to the curves W_2^u and W_2^s at the singular point o_2 (the blue solid box) as it is shown in the right panel of Figure 2.
W_2^s is exactly the part of the invariant curve c_2 defined by the equation $y = x^{-2}$ for $0 < x < 1$ (the curve in steel blue color). *Theorem 1 is proved*.

Remark 2. Not always we can find the unstable manifolds W_i^u for $i = 1,2,3$ excepting the case $a = 1/6$ corresponding to the Wallach space $SU(3)/T^2$ where W_i^u coincide with the Käeler metrics, for example, W_2^u can be defined as $y - x = \frac{1}{xy}$ (see the curve in magenta color in the right panel of Figure 2). The other two Käeler metrics are $x - y = \frac{1}{xy}$ and $x + y = \frac{1}{xy}$ shown in the right panel of Figure 1.

Conclusion. Since each Wallach space $SU(3)/T^2$, $Sp(3)/Sp(1)^3$ and $F_4/Spin(8)$ satisfies $a < 1/4$ the normalized Ricci flow's dynamics is completely determined by Einstein metrics of these spaces homothetic to $(1,1,1)$, (qk, q, q), (q, qk, q) and (q, q, qk). As a dynamical system the normalized Ricci evolves invariant Riemannian metrics of the Wallach spaces along trajectories which can not be closed or admit self intersections. This confirms the general information on the normalized Ricci flow's behavior on compact homogeneous spaces known from the property of the scalar curvature to increase unless metrics are Einstein.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 14, 2023

Revised: December 18, 2023

Accepted: January 15, 2024

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BOUNDEDNESS OF SOLUTIONS OF A CLASS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER ON THE SEMIAXIS

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In this paper, the issues of the limitation of solutions for linear differential equations of the second order on the semi-axis are investigated. Here we use the solution formulas for one class of second-order linear differential equations with variable coefficients obtained by the authors. Based on this formula and the method of integral transformations, sufficient conditions for the limitation of solutions for second-order linear differential equations on the semiaxis are established.

Keywords: Formula, linear differential equation, second order, boundedness, semi-axis.

Бул макалада жарым октогу экинчи тартиптеги сызыктуу дифференциалдык теңдемелердин чыгарылыштарынын чектелиши изилденди. Мында авторлор тарабынан алынган өзгөрүлмө коэффициенттери бар экинчи тартиптеги сызыктуу дифференциалдык теңдемелердин бир классы үчүн чыгарылыштарынын формуласы колдонулду. Бул формуланын негизинде жана интегралдык өзгөртүүлөр методу менен жарым октогу экинчи тартиптеги сызыктуу дифференциалдык теңдемелердин чыгарылыштарынын чектелишинин жетиштүү шарттары көрсөтүлгөн.

Урунттуу сөздөр: Формула, сызыктуу дифференциалдык теңдеме, экинчи тартиптеги, чектелгендик, жарым ок.

В данной работе исследуются вопросы ограниченности решений для линейных дифференциальных уравнений второго порядка на полуоси. Здесь используется формула решения для одного класса линейных дифференциальных уравнений второго порядка с переменными коэффициентами, полученными авторами. На основе этой формулы и методом интегральных преобразований установлены достаточные условия ограниченности решений для линейных дифференциальных уравнений второго порядка на полуоси.

Ключевые слова: Формула, линейное дифференциальное уравнение, второго порядка, ограниченность, полуось.

Consider the linear differential equation

$$y'' + p(t)y' + q(t)y = f(t), t \geq t_0, \tag{1}$$

with initial conditions

$$y(t_0) = m,\ y'(t_0) = n,\ m,n \in R,, \tag{2}$$

where $p(t), q(t), f(t)$ are the known functions.

The questions of boundedness and asymptotic stability of solutions on the semiaxis for differential and integro-differential equations were studied in [1-8]. In this paper, on the basis of the results of [9] and the method of transformations, we establish sufficient conditions for the boundedness of solutions of the differential equation (1) on the semiaxis.

Assume the following conditions are met:

$$a)\ q(t) = q_0(t) + q_1(t), t \geq t_0, \tag{3}$$

$$q_0(t) = a(t)[p(t) - a(t)] - l^2(t) + a'(t),\ t \geq t_0, \tag{4}$$

$$l(t) = r\, exp\left\{\int_{t_0}^{t}[2a(s) - p(s)]ds\right\}, \tag{5}$$

where $a(t),\ a'(t),\ q_1(t), p(t), f(t)$ are known continuous functions on $[t_0, \infty), r \in R, r \neq 0$;

b) $a(t) + l(t) \geq 0$ and $a(t) - l(t) \geq 0$ for all $t \in [t_0, \infty)$,

$$|f(t)|\left\{exp\left[-\int_{t_0}^{t}(a(\tau) + l(\tau) - p(\tau))d\tau\right] + \right.$$

$$\left. + exp\left[-\int_{t_0}^{t}(a(\tau) - l(\tau) - p(\tau))d\tau\right]\right\} \leq l_1(t)$$

and

$$|q_1(t)|\left\{exp\left[-\int_{t_0}^{t}\big(a(\tau) + l(\tau) - p(\tau)\big)d\tau\right] + exp\left[-\int_{t_0}^{t}(a(\tau) - l(\tau) - \right.$$

$$\left. p(\tau))d\tau\right]\right\} \leq l_2(t)$$

for all $t \in [t_0, \infty)$, where $l_1(t), l_2(t) \in L_1[t_0, \infty), l_1(t) \geq 0$ and $l_2(t) \geq 0$ for all $t \in [t_0, \infty)$.

Substituting (3) into (1) we have

$$y'' + p(t)y' + q_0(t)y = f(t) - q_1(t)y,\ t \geq t_0. \tag{6}$$

Taking into account the corollary of Theorem 1 from [9], from (6) we obtain,

$$y(t) = \frac{y_1(t)}{2l(t_0)}[m(l(t_0) - a(t_0)) - n] + \frac{y_2(t)}{2l(t_0)}[n + m(l(t_0) + a(t_0))] + y_3(t), \tag{7}$$

where

$$y_1(t) = exp\left[-\int_{t_0}^{t}(a(s)+l(s))ds\right], \tag{8}$$

$$y_2(t) = exp\left[-\int_{t_0}^{t}(a(s)-l(s))ds\right], \tag{9}$$

$$y_3(t) = \int_{t_0}^{t}[f(s)-q_1(s)y(s)][2l(s)y_1(s)y_2(s)]^{-1}[y_1(s)y_2(t)-y_1(t)y_2(s)]ds,$$

$$t \ge t_0. \tag{10}$$

Substituting (8), (9), (10) into (7) and taking into account (5) we get

$$y(t) = \frac{1}{2l(t_0)}[m(l(t_0)-a(t_0))-n]\, exp\left\{-\int_{t_0}^{t}[a(s)+l(s)]ds\right\}+$$

$$+\frac{1}{2l(t_0)}[n+m(l(t_0)+a(t_0))]\, exp\left\{-\int_{t_0}^{t}[a(s)-l(s)]ds\right\}+$$

$$+\int_{t_0}^{t}\frac{1}{2r}exp\left[\int_{t_0}^{s}p(\tau)d\tau\right][f(s)-q_1(s)y(s)]\left\{exp\left[-\int_{t_0}^{s}(a(\tau)+l(\tau))d\tau\right]\times\right.$$

$$exp\left[-\int_{t_0}^{t}(a(\tau)-l(\tau))d\tau\right]-exp\left[-\int_{t_0}^{t}(a(\tau)+l(\tau))d\tau\right]\times$$

$$\left.exp\left[-\int_{t_0}^{t}(a(\tau)-l(\tau))d\tau\right]\right\}ds, \quad t \ge t_0. \tag{11}$$

We introduce the notation:

$$F(t) = \frac{1}{2r}[m(r-a(t_0))-n]\, exp\left\{-\int_{t_0}^{t}[a(s)+l(s)]ds\right\}+$$

$$+\frac{1}{2r}[n+m(r+a(t_0))]\, exp\left\{-\int_{t_0}^{t}[a(s)-l(s)]ds\right\}+$$

$$+\frac{1}{2r}\int_{t_0}^{t}f(s)\left\{exp\left[-\int_{t_0}^{s}(a(\tau)+l(\tau)-p(\tau))d\tau\right]exp\left[-\int_{t_0}^{t}(a(\tau)-l(\tau))d\tau\right]-\right.$$

$$\left.-\,exp\left[-\int_{t_0}^{s}(a(\tau)-l(\tau)-p(\tau))d\tau\right]exp\left[-\int_{t_0}^{t}(a(\tau)+l(\tau))d\tau\right]\right\}ds, \tag{12}$$

$$K(t,s) = -\frac{q_1(s)}{2r}\left\{exp\left[-\int_{t_0}^{s}(a(\tau)+l(\tau)-p(\tau))d\tau\right]exp\left[-\int_{t_0}^{t}(a(\tau)-l(\tau))d\tau\right]\right.$$

$$\left.-\,exp\left[-\int_{t_0}^{s}(a(\tau)-l(\tau)-p(\tau))d\tau\right]exp\left[-\int_{t_0}^{t}(a(\tau)+l(\tau))d\tau\right]\right\}, \tag{13}$$

$(t,s) \in G = \{(t,s): t_0 \le s \le t < \infty\}$.

Taking into account notations (12) and (13), we write equation (11) in the form

$$y(t) = F(t) + \int_{t_0}^{t} K(t,s)y(s)ds, \quad t \geq t_0, \tag{14}$$

Theorem. Let conditions a) and b) be satisfied. Then the solution of the Cauchy problem (1)-(2) satisfies the following estimate:

$$|y(t)| \leq c_1, \quad t \geq t_0, \tag{15}$$

where

$$c_1 = c_2 \, exp\left[\frac{1}{2|r|}\int_{t_0}^{\infty} l_2(s)ds\right],$$

$$c_2 = \frac{1}{2|r|}\left[|m(r-a(t_0)) - n| + |n + m(r+a(t_0))| + \int_{t_0}^{\infty} l_1(s)ds\right].$$

Proof. Considering condition b), from (12) and (13) we have:

$$|F(t)| \leq c_2, t \geq t_0, \tag{16}$$

$$|K(t,s)| \leq \frac{l_2(s)}{2|r|}, (t,s) \in G. \tag{17}$$

Then, by virtue of (16) and (17) from (14) we obtain:

$$|y(t)| \leq c_2 + \frac{1}{2|r|}\int_{t_0}^{t} l_2(s)|y(s)|ds, \quad t \geq t_0. \tag{18}$$

Further, due to the Gronwall-Bellman inequality, from (18) we have estimate (15). Theorem is proven.

Example. Consider problems (1) - (2) for $t_0 = 0,\ p(t) = 2(1+t)$, $q(t) = (1+t)^2 + 2\,exp[-t^2/2 - 4t]$, $f(t) = 3\,exp[-t^2/2 - 3t]$, $t \in [0,\infty)$.

That is, consider the following linear differential equations

$$y'' + 2(1+t)y' + [(1+t)^2 + 2\,exp(-t^2/2 - 4t)]y = 3\,exp[-t^2/2 - 3t], t \geq t_0.$$

In this case, conditions *a)* are satisfied for

$a(t) = 3 + t, r = 1, q_1(t) = 2\,exp[-t^2/2 - 4t],\ l(t) = 1,\ t \in [0,\infty)$.

In addition, conditions *b)* are satisfied for

$l_1(t) = 6e^{-t},\ l_2(t) = 4e^{-2t},\ t \in [0,\infty)$.

Indeed $a(t) + l(t) = 2 + t \geq 0,\ a(t) - l(t) = t \geq 0,\ t \in [0,\infty)$,

$$|f(t)|\left\{exp\left[-\int_0^t (a(\tau) + l(\tau) - p(\tau))d\tau\right] + exp\left[-\int_0^t (a(\tau) - l(\tau) - p(\tau))d\tau\right]\right\}$$

$$= 3e^{-(t^2+3t)}\left(e^{t^2/2} + e^{t^2/2+2t}\right) \leq 6e^{-t} = l_1(t),\ t \in [0,\infty),$$

$$|q_1(t)|\left\{exp\left[-\int_0^t (a(\tau) + l(\tau) - p(\tau))d\tau\right] + exp\left[-\int_0^t (a(\tau) - l(\tau) - p(\tau))d\tau\right]\right\} =$$

$= 2e^{-(t^2/2+4t)}\left[e^{t^2/2} + e^{t^2/2+2t}\right] \leq 4e^{-2t} = l_2(t) \in [0,\infty).$

Thus, all conditions are met. In this case

$$c_2 = \frac{1}{2}\left[|m(r-3) - n| + |n + m(r+3)| + \int_0^\infty l_1(s)ds\right], \tag{19}$$

$c_1 = c_2 \, exp\left[\frac{1}{2|r|}\int_{t_0}^\infty l_2(s)ds\right].$

In this case, estimates (15) are valid for the Cauchy problem (1)-(2), where the number c_2 is determined by formula (19).

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 16, 2023

Revised: December 12, 2023

Accepted: January 15, 2024

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MSC 45A05, 45B 05

ONE CLASS OF FREDHOLM LINEAR INTEGRAL EQUATIONS OF THE THIRD KIND ON A SEMI-AXIS

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In the present article the theorem is proven about uniqueness of the linear integral equations of the third kind, with method of nonnegative quadratic forms and functional analysis methods.

Key words: linear, integral equations, Fredholm, third kind, uniqueness.

Бул макалада терс эмес квадраттык формалар усулунун, функционалдык анализдин усулдарынын жардамы менен, үчүнчү түрдөгү сызыктуу интегралдык теңдемелердин чыгарылыштарынын жалгыздыгы далилденди.

Урунттуу сөздөр: сызыктуу, интегралдык теңдемелер, үчүнчү түрдөгү, жалгыздык.

В данной работе, с помощью метода неотрицательных квадратичных форм, методами функционального анализа доказывается единственность решений линейных интегральных уравнений третьего рода.

Ключевые слова: линейный, интегральное уравнение, третьего рода, единственность.

1. Introduction

The notion of correctness in the works of A.N. Tikhonov [1], M.M. Lavrent'ev [2] and V.K. Ivanov [3], different from the classical one, provided a means for studying ill-posed problems and stimulated interest in integral equations first kind with large applied value.

The fundamental results for Fredholm integral equations of the first kind were obtained by M.M. Lavrentiev in [4], [5], where the regularizing operators in the sense of M.M. Lavrentiev. In [6], [7] results were obtained in the field of nonclassical Volterra equations of the first kind and numerical methods for solving nonclassical linear Volterra equations of the first kind, which describe the processes of aging and replacement of elements in developing systems.

In [8]- [10] for linear Volterra integral equations of the first and third kind, the uniqueness and stability of solutions are obtained.

In [11] - [14], the uniqueness theorems were proved and regularizing operators were built in the sense of Lavrentiev for linear Fredholm integral equations of the first kind.

In this work, we apply the method of integral transformation to prove uniqueness theorems for the new class of Fredholm linear integral equations of the third kind in the semi-axis

$$Ku \equiv m(t)u(t) + \int_{-\infty}^{a} K(t,s)\ u\ (s)ds = f(t), t \in (-\infty, a], \tag{1}$$

where the $u(t)$is the desired function on $(-\infty, a)$, the given functions $m(t)$ and $f(t)$ is the continuous on$(-\infty, a)$, $m(t)$ -is equal to zero at least at one point on$(-\infty, a)$,

$$\int_{-\infty}^{a}\int_{-\infty}^{a} |K(t,s)|^2\, dsdt < \infty,$$

$$K(t,s) = \begin{cases} A(t,s), & -\infty < s \le t \le a, \\ B(t,s), & -\infty < t \le s \le a. \end{cases} \tag{2}$$

and $B(t,s)$ and $A(t,s)$ are known functions, respectively on the domains

$$\{(t,s): -\infty < s \le t \le a\}, \{(t,s): -\infty < t \le s \le a\}.$$

$u(t) \in L_2(-\infty, a), (-\infty, a)$, where $L_2(-\infty, a)$- the space of square-summable functions in$(-\infty, a)$.

Equation (1) by virtue of relation (2) can be expressed as

$$m(t)u(t) + \int_{-\infty}^{t} A(t,s)\ u\ (s)ds + \int_{t}^{ta} B(t,s)\ u\ (s)ds = f(t). \tag{3}$$

Taking the multiplication of both sides of the equation (1) with $u(t)$, integrating the results $-\infty < t \le a$, we obtain

$$\int_{-\infty}^{a} m(t)|u(t)|^2 dt + \int_{-\infty}^{a}\int_{-\infty}^{t} A(t,s)\ u\ (s)u(t)dsdt +$$

$$+ \int_{-\infty}^{a}\int_{t}^{a} B(t,s)\ u\ (s)u(t)dsdt = \int_{-\infty}^{a} f(t)dt.$$

$$\int_{-\infty}^{a} m(t)|u(t)|^2 dt + \int_{-\infty}^{a}\int_{-\infty}^{t} A(t,s)\ u\ (s)u(t)dsdt +$$

$$+ \int_{-\infty}^{a}\int_{s}^{a} B(s,t)\ u\ (t)u(s)dtds = \int_{-\infty}^{a} f(t)u(t)dt. \tag{4}$$

Using the Dirichlet formula, we have

$$\int_{-\infty}^{a}\int_{s}^{a} B(s,t)\,u(t)\;u\;(s)dtds = \int_{-\infty}^{a}\left[\int_{-\infty}^{t} B(s,t)u(s)dt\right]u(t)dt\,.$$

Then from (4) we obtain

$$\int_{-\infty}^{a} m(t)|u(t)|^2dt + 2\int_{-\infty}^{a}\int_{-\infty}^{t} H(t,s)\,u(s)dsu(t)dt = \int_{-\infty}^{a} f(t)u(t)dt, \qquad (5)$$

where

$$H(t,s) = \frac{1}{2}[A(t,s) + B(s,t)], (t,s) \in \{(t,s){:} -\infty < s \le t \le a\}, \qquad (6)$$

$$\int_{-\infty}^{a}\int_{-\infty}^{t} H^2\,(t,s)dtds < +\infty.$$

We introduce a new function $M(t,s)$ as follows

$$M(t,s) = \begin{cases} H(t,s), & -\infty < s \le t \le a, \\ H(s,t), & -\infty < t \le s \le a. \end{cases} \qquad (7)$$

It's evident that $M(t,s) = M(s,t), \quad (t,s) \in -\infty, a \times -\infty, a.$

It is easy to verify the validity of the equality

$$\int_{-\infty}^{a}\int_{-\infty}^{a} |M(t,s)|^2\,dsdt < +\infty.$$

Then, it is known that

$$M(t,s) = \sum_{i=1}^{\infty} \frac{\varphi_i(t)\varphi_i(s)}{\lambda_i}\,, \qquad (8)$$

where, λ_i are the characteristic numbers of the matrix kernel $M(t,s)$, which are arranged in ascending order of their modules, and $0 < |\lambda_1| \le |\lambda_2| \le . \dots$ and the corresponding orthonormal eigenfunctions.

Theorem. Suppose $m(t) \in L_2(-\infty,a), m(t) \ge 0$ for all $t \in (-\infty,a)$, $M(t,s)$ is a complete kernel and $0 < \lambda_1 \le \lambda_2 \le . \dots$ Then the solution of the equation (1) is unique in $L_2(-\infty,a)$.

Proof. Let equation (1) have a nonzero solution

$u(t) \in L_2(-\infty,a)$ for $f(t) \equiv 0.$

Then

$$m(t)u(t) + \int_{-\infty}^{a} K(t,s)\,u(s)ds = f(t) \equiv 0, t \in (-\infty,a].$$

Hence, by virtue of (5) we have

$$\int_{-\infty}^{a} m(t)|u(t)|^2 dt + 2\int_{-\infty}^{a}\int_{-\infty}^{t} H(t,s)\,u(s)dsu(t)dt \equiv 0.$$

Taking into account (7) and (8) from (9) we get

$$\int_{-\infty}^{a} m(t)|u(t)|^2 dt + \sum_{i=1}^{\infty}\frac{2}{\lambda}\int_{-\infty}^{a} \varphi_i(t)u(t)\int_{-\infty}^{t}\varphi\,(s)u(s)dsdt = 0.$$

$$\int_{-\infty}^{a} m(t)|u(t)|^2 dt + \sum_{i=1}^{\infty}\frac{1}{\lambda}\int_{-\infty}^{a}|\varphi_i(t)u(t)|^2 = 0.$$

Then $\int_{-\infty}^{a}\varphi_i(t)u(t)dt = 0,\quad (i = 1,2,\dots).$

Therefore, $u(t) \equiv 0,\ t \in (-\infty, a]$. The theorem is proved.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 13, 2023

Revised: December 8, 2023

Accepted: January 15, 2024

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MSC 34K20, 45J05

ON A CLASS OF NONLINEAR FREDHOLM INTEGRAL EQUATIONS OF THE THIRD KIND ON THE SEGMENT

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On the basis of the new approach it is shown that solving a class of nonlinear Fredholm integral equations of the third kind on the segment are equivalent to solving systems of nonlinear algebraic equations. The questions of existence and uniqueness of the solution for these nonlinear integral equations are studied.

Keywords: Solution, nonlinear integral equation, Fredholm, third kind, equivalence, system of nonlinear algebraic equations.

Жаңы ыкманын негизинде кесиндидеги үчүнчү түрдөгү Фредгольмдун сызыктуу эмес интегралдык теңдемелердин бир классы үчүн чыгарылыштарын чыгаруу сызыктуу эмес алгебралык теңдемелер системасын чыгарууга эквиваленттүүлүк экендиги көрсөтүлгөн. Бул сызыктуу эмес интегралдык теңдемелер үчүн чыгарылыштары бар экендиги жана жалгыздыгы жөнүндөгү маселелер изилденет.

Урунтуу сөздөр: Чыгарылыш, сызыктуу эмес интегралдык теңдеме, Фредгольм, үчүнчү түрдөгү, эквиваленттүүлүк, сызыктуу эмес алгебралык теңдемелер системасы.

На основе нового подхода показано, что решения для одного класса нелинейных интегральных уравнений Фредгольма третьего рода на отрезке эквивалентны к решению систем нелинейных алгебраических уравнений. Изучаются вопросы существования и единственности решения для этих нелинейных интегральных уравнений.

Ключевые слова: Решение, нелинейное интегральное уравнение, Фредгольм, третий род, эквивалентность, система нелинейных алгебраических уравнений.

Consider the nonlinear integral equation of the third kind

$$p(t)u(t) = \int_a^b \sum_{i=1}^n a_i(t)b_i(s)\,K_i(s,u(s))ds + f(t), t \in [a,b], \tag{1}$$

where $p(a)$=0, $p(t), a_i(t), b_i(t)$ and $f(t)$ are known continuous functions on $[a,b]$, *K(t,u)* is known continuous functions on $[a,b] \times R, i = 1,2,\dots,n,$ *u(t)* is the desired continuous function on $[a,b]$.

Many questions for integral equations were investigated in $[1-12]$. In particular, regularizing operators according to M.M.Lavrentiev were constructed in [3] to solve linear integral Fredholm equations of the first kind. In [5-6], uniqueness theorems were proved for systems of nonlinear Volterra integral equations of the third kind and for systems of linear integral Fredholm equations of the third kind and regularizing operators according to M.M.Lavrentiev were constructed.

In this paper, based on the approach proposed in [7], the questions of the existence and uniqueness of a solution for one class of nonlinear Fredholm integral equations of the third kind on the segment are investigated.

Everywhere we will assume that

$$p(t) = \prod_{i=1}^{m} p_i(t), p_i(a) = 0, \; p_i(t) \in C[a,b], \; i = 1,2,\dots,m. \tag{2}$$

Let

$$c_i = \int_a^b b_i(s) K_i(s, u(s)) ds, i = 1,2,\dots,n. \tag{3}$$

Then from (1) we get

$$p(t)u(t) = \sum_{j=1}^{n} c_j a_j(t) + f(t), t \in [a,b] \;, \tag{4}$$

where c_j are unknown constants, $j = 1,2,\dots,n$. Assuming $t=a$, we have from (4)

$$\sum_{j=1}^{n} c_j a_j(a) + f(a) = 0. \tag{5}$$

Subtracting (5) from (4) we have

$$p(t)u(t) = \sum_{j=1}^{n} c_j \left[a_j(t) - a_j(a)\right] + [f(t) - f(a)], t \in [a,b). \tag{6}$$

Assume the following conditions are met:

a) For all $k=1,2,\dots,m$ and for all $i,j = 1,2,\dots,n$, $\alpha_{i,k}(t) \in C[a,b]$, $\alpha_{i,m}(t) b_j(t) \in L_1(a,b)$, where

$$\alpha_{i,k}(t) = \left(\alpha_{i,k-1}(t) - \alpha_{i,k-1}(a)\right)/p_k(t), \alpha_{i,0}(t) = a_i(t), t \in [a,b] \;, \tag{7}$$

$$\alpha_{i,k}(a) = \lim_{t\to a} \alpha_{i,k}(t);$$

b) For all $k=1,2,\dots,m$ and for all $j = 1,2,\dots,n$

$F_k(t) \in C[a,b]$, $F_m(t) b_j(t) \in L_1(a,b)$, where

$$F_k(t) = \frac{F_{k-1}(t) - F_{k-1}(a)}{p_k(t)}, F_0(t) = f(t), t \in [a,b] \;, F_k(a) = \lim_{t\to a} F_k(t). \tag{8}$$

Further, by virtue of (2), (7) and (8), to determine the unknown constants $c_1, c_2, \dots, c_n$ from (6), we obtain the following system

$$\begin{cases} \sum_{j=1}^{n} c_j a_j(a) + f(a) = 0, \\ \sum_{j=1}^{n} c_j \alpha_{j,k}(a) + F_k(a) = 0, k = 1,2, \dots, m-1 \end{cases} \tag{9}$$

Taking into account (9) from (6) we have

$$u(t) = \sum_{j=1}^{n} c_j \alpha_{j,m}(t) + F_m(t), t \in [a,b] \ . \tag{10}$$

Substituting (10) into (3) we get

$$c_i = \int_a^b b_i(s) K_i\left(s, \sum_{j=1}^{n} c_j \alpha_{j,m}(s) + F_m(s)\right) ds, i = 1,2, \dots, n. \tag{11}$$

Thus, we have a system of nonlinear algebraic equations (9) and (11), which defines c_i, $i = 1,2, \dots, n$.

Theorem. Let the conditions (2), a) and b) be satisfied. Then:

1. If the system of nonlinear algebraic equations (9) and (11) has no solutions, then the integral equation (1) has no solutions in the space $C[a,b]$;

2. If the system of nonlinear algebraic equations (9) and (11) has a unique solution $c_1, c_2, \dots, c_n$, then the integral equation (1) has a unique solution in the space $C[a,b]$. This solution is determined by the formula (10);

3. If the system of nonlinear algebraic equations (9) and (11) have a finite number *r* of solutions $c_{1,k}$, $c_{2,k}, \dots, c_{n,k}$, where *k=1,2,...,r*, then the integral equation (1) in the space $C[a,b]$ have a finite number r of solutions $u_1(t)$, $u_2(t), \dots, u_r(t)$ defined by the formula:

$$u_k(t) = \sum_{j=1}^{n} c_{j,k} \alpha_{j,m}(t) + F_m(t), t \in [a,b], \text{ k=1,2,…,r.} \tag{12}$$

Proof.

1. It is evident that if systems (9) and (11) have no solutions, then the integral equation (1) has no solutions in the space $C[a,b]$.

2. Let systems (9) and (11) have a unique solution. Then the integral equation (1) has a unique solution in the space $C[a,b]$. This solution is determined by the formula (10), where $c_1, c_2, \dots, c_n$ is the only solution of the system (9) and (11).

3. Let the system of nonlinear algebraic equations (9) and (11) have a finite number *r* of solutions $c_{1,k}$, $c_{2,k}, \dots, c_{n,k}$, where *k=1,2,...,r*. Then by virtue of (10) for the integral

equation (1) in the space $C[a,b]$ there are *r* solutions defined by the formula (12). The theorem has been proved.

Example. Consider the integral equations (1), where $a=0, b=1$, $n=2, p(t)=t^2, a_1(t)=t+1, b_1(t)=t, a_2(t)=t^2+3, b_2(t)=\alpha t^2$, $K_1(\mathrm{t},u)=u,\ K_2(\mathrm{t},u)=u^2,\ f(t)=t^3+\beta,\ \alpha,\beta,\gamma-$ parameters, $\alpha\neq 0$, that is, consider the equation

$$t^2u(t)=\int_0^1[(t+1)su(s)+\alpha(t^2+3)s^2u^2(s)]ds+\beta t^3+\gamma,\ t\epsilon[0;1]. \tag{13}$$

Let's introduce the following notation

$$\begin{cases} c_1=\int_0^1 su(s)ds\,, \\ c_2=\int_0^1 s^2u^2(s)ds. \end{cases} \tag{14}$$

Taking into account (14), we have from (13)

$$t^2u(t)=c_1(t+1)+c_2\alpha(t^2+3)+\beta t^3+\gamma, t\epsilon[0;1]. \tag{15}$$

Substituting *t=0* from (15) we get

$$c_1+3\alpha c_2+\gamma=0. \tag{16}$$

Subtracting (16) from (15) we get

$$t^2u(t)=c_1t+c_2\alpha t^2+\beta t^3,\ t\epsilon[0;1], \text{ that is,}$$

$$tu(t)=c_1+c_2\alpha t+\beta t^2, t\epsilon[0;1]. \tag{17}$$

Assuming *t=0*, from (17) we have

$$c_1=0. \tag{18}$$

Taking into account (18), we get from (17)

$$u(t)=c_2\alpha+\beta t,\ t\epsilon[0;1]. \tag{19}$$

Substituting (19) into (14) we have

$$\begin{cases} c_1=\int_0^1 s(c_2\alpha+\beta s)ds\,, \\ c_2=\int_0^1 s^2(c_2\alpha+\beta s\,)^2ds. \end{cases}$$

From here we have

$$\begin{cases} c_1=\frac{1}{2}\alpha c_2+\frac{1}{3}\beta, \\ c_2=\frac{1}{3}\alpha^2{c_2}^2+\frac{1}{2}c_2\alpha\beta+\frac{1}{5}\beta^2. \end{cases} \tag{20}$$

Then, solving the system of equations (16), (18) and (20), we obtain

$$\begin{cases} c_1 = 0, \\ c_2 = -\frac{2\beta}{3\alpha}, \\ c_2 = -\frac{\gamma}{3\alpha}, \\ c_2 = \frac{1}{3}\alpha^2 {c_2}^2 + \frac{1}{2} c_2 \alpha\beta + \frac{1}{5}\beta^2. \end{cases} \tag{21}$$

Then the following cases take place:

1. If $\alpha \neq 0, \gamma \neq 2\beta$, then the system of equations (21) has no solution. Therefore, integral equations (13) have no solutions in the space $C[0,1]$.
2. If $\alpha \neq 0, \gamma = 2\beta, \beta = 0$, then the system of equations (21) has a unique solution $c_1 = 0, c_2 = 0.$ Therefore, the integral equation (13) in the space $C[0,1]$ has a unique solution:

   $$u(t) = 0, t \in [0;1].$$
3. If $\alpha \neq 0,\ \gamma = 2\beta,\ \beta = -\frac{45}{\alpha}$, then the system of equations (21) has a solution $c_1 = 0, c_2 = \frac{30}{\alpha^2}$. Therefore, integral equations (13) in the space $C[0,1]$ have a unique solution:

   $$u(t) = \frac{15}{\alpha}(2 - 3t), t \in [0;1].$$
4. If $\alpha \neq 0,\ \gamma = 2\beta,\ \beta \neq -\frac{45}{\alpha}$, then the system of equations (21) has no solution. Therefore, integral equations (13) have no solutions in the space $C[0,1]$.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 13, 2023

Revised: December 8, 2023

Accepted: January 15, 2024

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MSC 34K20, 45J05 DOI: 10.52448/16948173_2023_2_xx

ON REDUCING THE ANALYTICAL SYSTEM OF VOLTERRA INTEGRAL EQUATIONS WITH A REGULAR SINGULAR POINT TO A LINEAR FORM

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Volterra integral equations with a singular point appear in the study of the theory of wave propagation in media with memory and are considered from various points of view in works. In this paper, we consider a nonlinear system of Volterra integral equations in an unbounded domain.
An analytical transformation has been constructed, with the help of which the nonlinear Volterra integral equations with a regular singular point are reduced to a linear system.
Keywords: Volterra integral equation, singular point, vanishing solution, formal solutions, majorant method, implicit function theorem.

Өзгөчө чекити бар Вольтерра интегралдык теңдемелери эс тутумдуу чөйрөдө толкундун таралуу теориясын изилдөөдө пайда болот жана көптөгөн авторлор тарабынан ар кандай көз караштардан каралат. Бул эмгекте биз чексиз аймакта Вольтерра интегралдык теңдемелеринин сызыктуу эмес системасын карап чыгабыз. Аналитикалык өзгөртүп түзүү курулган жана анын жардамы менен регулярдуу өзгөчө чекити бар сызыктуу эмес Вольтерра интегралдык теңдемелери сызыктуу системага келтирилген.
Урунттуу сөздөр: Вольтерра интегралдык теңдемеси, өзгөчө чекит, жок болуп кетүүчү чыгарылыш, формалдуу чыгарылыш, мажорант усулу, айкын эмес функциялар теоремасы.

Интегральные уравнения Вольтерра с особой точкой появляются при изучении теории распространения волн в средах с памятью и рассмотрены с различных точек зрения многими авторами. В настоящей работе рассматривается нелинейная система интегральных уравнений Вольтерра в неограниченной области. Построено аналитическое преобразование, с помощью которого нелинейные интегральные уравнения Вольтерра с регулярной особой точкой сводятся к линейной системе.

Ключевые слова: Интегральное уравнение Вольтерра, особая точка, исчезающее решение, формальные решения, метод мажорант, теорема о неявных функциях.

Introduction

Volterra integral equations with a singular point appear in the study of the theory of wave propagation in media with memory and are considered from various points of view in works [1-14, 18-24]. In this paper, we consider a nonlinear system of Volterra integral equations in an unbounded domain. The asymptotic structure of its solutions for large values of the argument is studied.

Consider the system

$$tu = \Lambda \int_{+\infty}^{t} u(s)ds + \int_{+\infty}^{t} K(t,s,u)ds,\ T < t < +\infty, \tag{1}$$

where $K = (h_{jk}),\ (j,k = 1,n)$ are power series in integer powers $t, s, u_1, \dots, u_n$, and in u starting from terms not lower than the second dimension, $K(t,s,0) = 0$.

Our goal is to construct an analytical transformation with the help of which a nonlinear Volterra integral equations with a regular singular point is reduced to a linear system [33-34]. Let's demand that

1) all elementary divisors of the matrix Λ are simple and $\Lambda\ = diag(\lambda_1, \dots, \lambda_n)$, $Re\,\lambda_k < 0\ (k = 1,2, \dots n)$;

2) the matrix Λ does not have eigenvalues, that differ from each other by positive integers.

Note that, similarly to [9,18], it can be shown that the corresponding linear system of equations

$$tz(t) = \Lambda \int_{+\infty}^{t} z(s)ds, \tag{2}$$

has a general solution of the form

$$z(t) = t^{\Lambda - E} C, \tag{3}$$

where C is an arbitrary column vector.

Theorem. Let conditions 1) and 2) be satisfied. Then there is an analytical transformation

$$u = z + \sum_{|p| \geq 2} P_p(t) z^p(t), \tag{4}$$

where $P_p(t)$ are analytical functions, such that with its help the original system (1) can be reduced to linear form (2).

To prove this, we will use the approach proposed in [15,25]. First, we prove that under our assumptions, there are some formal, generally speaking, divergent power series that satisfy our system (1). Next, we will prove the convergence of the found series.

The existence of formal series.

Note that the matrix $E-<p,\lambda-E_1>^{-1}\Lambda$, due to condition 2), is not degenerate for all $|p|\geq 2$.

Let us make the substitution (4) into (1), and we will keep in mind that the $P_p(t)$ functions are formal power series. Substituting series (4) into (1), we obtain linear systems of Volterra integral equations for the functions $P_p(t)$

$$tP_p(t)z^p(t)=\Lambda\int_{+\infty}^{t}P_p(s)z^p(s)ds+\int_{+\infty}^{t}h_p(t,s,P_r(s))\,z^p(s)ds \tag{5}$$

where h_p is a polynomial with respect to $P_r,\ |r|<|p|$ coefficients depending on s, the function $h_p(t,s,P_r)$ at $|p|\leq 2$ does not depend on P_p.

Lemma. If $f(t)$ is a function holomorphic $f(t)=\sum_{j=0}^{\infty}f_jt^{-j}$ with $T<t<+\infty$, then the following relation holds:

$$\int_{+\infty}^{t}f(s)z^p(s)ds=F(t)z^p(t), \tag{6}$$

where $F(t)$ the function is holomorphic at $T<t<+\infty$, i.e. having a decomposition of the form

$$F(t)\sim\sum_{j=1}^{\infty}F_jt^{-j},\ \text{at}\ \ t\to+\infty\ .$$

Indeed, differentiating (6) with respect to t, taking into account (3)

$$f(t)z^p(t)=F^{'}(t)z^p(t)+\frac{1}{t}<p,\lambda-E_1>F(t)z^p(t),$$

or, shortening by $z^p(t)$

$$tF^{'}(t)+\langle p,\lambda-E_1\rangle F(t)=tf(t), \tag{7}$$

where $a=(p,\lambda-1)$. Equation (7) is a Fuchs type equation [28]. Then it follows from Fuchs' theorem that the solution to equation (7) is some analytical function $F(t)=\sum_{j=1}^{\infty}F_jt^{-j}$, $T<t<+\infty$.

Also note that from condition 2) it follows that

$\sum_{j=1}^{n}p_j\big(\lambda_j-E_1\big)\neq\lambda_i+k$ for everyone $|p|\geq\sum_{j=1}^{n}p_j\geq 2$ and $k\geq 0, i=1,2,\ldots,n$.

Let us assume that from equations (5) we have defined $P_r,\ |r|\leq|p|$, as functions that are holomorphic for $T<t<+\infty$. According to the previous equation (5) for$P_p(t)$ we can write in the form

$$tP_p z^p = \Lambda \int_{+\infty}^{t} P_p(s) z^p\,(s) ds + Q_p(s) z^p, \tag{8}$$

where Q_p is a holomorphic function for$T < t < +\infty$,$Q_p(t) = \sum_{k=1}^{\infty} Q_{p1} t^{-k}$.

Let us show that equation (8) is formally satisfied by the power series

$$P_p(t) = \sum_{k=0}^{\infty} P_{pk} t^{-k}. \tag{9}$$

In fact, we have

$$\Lambda \int_{+\infty}^{t} \left(\sum_{k=0}^{\infty} P_{pk} s^{-k}\right) z^p(s) ds = B_p(t) z^p(t), \tag{10}$$

where $B_p(t) = t^{-1} \sum_{k=0}^{\infty} b_{pk} t^{-k}$, $b_{pk} = (< p, \lambda - E_1 > +k + 1)^{-1} \Lambda P_{pk}, k = 0,1,2.$

Then from (8) taking into account (10) and condition 2) we can determine all the coefficients of series (9)

$$P_{p0} = \left(E - \frac{\Lambda}{< p, \lambda - E_1 >}\right)^{-1} Q_{p1},$$

$$P_{pk} = \left(E - \frac{\Lambda}{< p, \lambda - E_1 > +k + 1}\right)^{-1} (Q_{pk+1}), k = 1,2, \ldots$$

since the matrix is $\left(E - \frac{\Lambda}{<p,\lambda-E_1>+k+1}\right)$ is non-degenerate for all $|p| \geq 2$.

Proof of convergence.

So, series (9) formally satisfies equation (8). Let us prove that power series (9) converges at $T < t < +\infty$.

Note that, due to conditions 1), 2), there exists B>0 such that $\|\bar{\Lambda}\| > B$for all where $k = 1,2, \ldots$, the matrix $\bar{\Lambda} \equiv \left(E - \frac{\Lambda}{<p,\lambda-E_1>+k+1}\right)^{-1}, k = 0,1,2, \ldots$.

Consider now the equation

$$BV = \bar{Q}(t), \tag{11}$$

where $\bar{Q}(t)$ is the majorant of the function$Q(t)$. It is clear that equation (11) has a formal solution

$$V = \frac{1}{B} \bar{Q}(t) = \sum_{k=0}^{\infty} \bar{P}_{pk} t^{-k}, \tag{12}$$

and $\bar{P}_{pk}$ are positive constants and

$$|P_{pk}| < \bar{P}_{pk}. \tag{13}$$

Equation (11) by the theorem on implicit functions [14] has a unique holomorphic solution $V = V(t)$ at $T < t < +\infty$.
and it coincides with the formal solution (12). Hence, by virtue of (13), the convergence of series (9) for $T < t < +\infty$. Then, by virtue of condition 2), series (4) converges in $T_1 < t < +\infty$.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 24, 2023
Revised: December 22, 2023
Accepted: January 15, 2024

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MSC 34K20, 45J05

ON THE METHOD OF STUDYING THE ASYMPTOTIC STABILITY OF SOLUTIONS OF A LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATION OF THE SEVENTH ORDER ON THE SEMI-AXIS

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The methodology to determine sufficient conditions of asymptotic stability on the semi-axis of solutions to a linear seventh-order Volterra-type integro-differential equation is proposed. For this, a non-standard method of reducing equations to a system, a method of squaring equations, Volterra's equation transformation me-thod, the method of cutting functions, the method of integral inequalities are developed and Sylvester's criterion and the Lyusternik-Sobolev lemma are applied.

Keywords: linear seventh-order Volterra integro-differential equation, asymptotic stability of solutions, a non-standard reducing to a system, squaring equations, cutting functions, Sylvester's criterion, Lyusternik-Sobolev lemma.

Жетинчи тартиптеги сызыктуу Вольтерра тибиндеги интегро-дифференциалдык теңдеменин жарым окто асимптотикалык турумдуулугунун жетиштүү шарттарын аныктоо усулдугу сунушталат. Бул үчүн теңдемени системага стандарттык эмес келтирүү усулу, теңдемелерди квадратка көтөрүү усулу, В. Вольтерранын теңдемелерди өзгөртүп түзүү усулу, кесүүчү функциялар усулу, интегралдык барабарсыздыктар усулу өнүктүрүлөт, Сильвестрдин критерийи жана Люстерник-Соболевдин леммасы колдонулат.

Урунттуу сөздөр: Вольтерра тибиндеги сызыктуу жетинчи тартиптеги интегро-дифференциалдык теңдеме, чыгарылыштардын асимптотикалык турумдуулугу, теңдемени системага стандарттык эмес келтирүү, теңдемени квадратка көтөрүү, кесүүчү функциялар, Сильвестрдин критерийи, Люстерник-Соболевдин леммасы.

Предлагается методика определения достаточных условий асимптотической устойчивости на полуоси решений линейного интегро-дифференциального уравнения седьмого порядка типа Вольтерра. Для этого развиваются нестандартный метод сведения к системе, метод возведения уравнений в квадрат, метод преобразования уравнений В. Вольтерра, метод срезывающих функций, метод интегральных неравенств, применяются критерий Сильвестра и лемма Люстерника-Соболева.

Ключевые слова: линейное вольтеррова интегро-дифференциальное уравнение седьмого порядка, асимптотическая устойчивость решений, нестандартное сведение к

системе, возведение уравнений в квадрат, срезывающие функции, критерий Сильвестра, лемма Люстерника-Соболева.

All functions involved are continuous, and the relationships hold under $t \geq t_0, t \geq \tau \geq t_0; I = [t_0, \infty)$; "IDE" stands for integro-differential equation, and "DE" stands for differential equation. As for the asymptotic stability of solutions to a linear seventh-order Volterra integro-differential equation (IDE), it refers to the tendency of all solutions and their derivatives up to the sixth order, inclusive, to approach zero. "AS" stands for asymptotic stability.

PROBLEM. To establish sufficient conditions for the AS of solutions to a linear seventh-order Volterra-type IDE of the form:

$$x^{(7)}(t) + \sum_{k=0}^{6} \left[a_k(t)x^{(k)}(t) + \int_{t_0}^{t} Q_k(t,\tau)x^{(k)}(\tau)d\tau \right] = f(t), t \geq t_0. \qquad (1)$$

To solve this problem, a methodology is proposed based on the following ideas: the non-standard method of reducing equations to a system [1-4], the method of squaring equations [5, p. 28], Volterra's equation transformation method [6, pp. 194-217], the method of integral inequalities [7], the application of the generalized Sylvester's criterion [8, p. 137], and the Lyusternik-Sobolev lemma [9, pp. 393-394; 3]

In the seventh-order IDE (1), we will make the following non-standard substitution:

$$x^{(5)}(t) + \sum_{k=0}^{4} q_k x^{(k)}(t) = W(t)y(t), \qquad (2)$$

where $q_k (k = 0,1,2,3,4)$ - are certain auxiliary parameters, $0 < W(t)$ is some weight function, $y(t)$ is a new unknown function.

Note that substitution (2) is an analogue of non-standard substitutions found in works [1-4].

Differentiating substitution (2), we obtain:

$$x^{(6)}(t) + q_4 x^{(5)}(t) + q_3 x^{(4)}(t) + q_2 x'''(t) + q_1 x''(t) + q_0 x'(t)$$
$$= W'(t)y(t) + W(t)y'(t),$$

from which, due to (2), we obtain

$$x^{(6)}(t) = (q_4^2 - q_3)x^{(4)}(t) + (q_3 q_4 - q_2)x'''(t) + (q_2 q_4 - q_1)x''(t) +$$

$$+(q_1q_4-q_0)x'(t)+q_0q_4x(t)+W_*(t)y(t)+W(t)y'(t), \quad (3)$$

where $W_*(t)\equiv W'(t)-q_4W(t)$.

Furthermore, from (3) and (2), it follows

$$x^{(7)}(t)=(q_4^2-q_3)x^{(5)}(t)+(q_3q_4-q_2)x^{(4)}(t)+(q_2q_4-q_1)x'''(t)+$$
$$+(q_1q_4-q_0)x''(t)+q_0q_4x'(t)+W_*'(t)y(t)+W_*(t)y'(t)+W'(t)y'(t)+$$
$$+W(t)y''(t)=(q_4^2-q_3)[W(t)y(t)-q_4x^{(4)}(t)-q_3x'''(t)-q_2x''(t)-$$
$$-q_1x'(t)-q_0x(t)]+(q_3q_4-q_2)x^{(4)}(t)+(q_2q_4-q_1)x'''(t)+$$
$$+(q_1q_4-q_0)x''(t)+q_0q_4x'(t)+W_*'(t)y(t)+W_*(t)y'(t)+W'(t)y'(t)+$$
$$+W(t)y''(t)=(2q_3q_4-q_2-q_4^3)x^{(4)}(t)+(q_3^2-q_3q_4^2+q_2q_4-q_1)x'''(t)+$$
$$+(q_2q_3-q_4^2q_2+q_1q_4-q_0)x''(t)+(q_1q_3-q_4^2q_1+q_0q_4)x'(t)+$$
$$+(q_0q_3-q_4^2q_0)x(t)+[W_*'(t)+(q_4^2-q_3)W(t)]y(t)+$$
$$+[W_*(t)+W'(t)]y'(t)+W(t)y''(t). \quad (4)$$

Now, substitute (2)-(4) into the IDE (1):

$$(2q_3q_4-q_2-q_4^3)x^{(4)}(t)+(q_3^2-q_3q_4^2+q_2q_4-q_1)x'''(t)+$$
$$+(q_2q_3-q_4^2q_2+q_1q_4-q_0)x''(t)+(q_1q_3-q_4^2q_1+q_0q_4)x'(t)+$$
$$+(q_0q_3-q_4^2q_0)x(t)+[W_*'(t)+(q_4^2-q_3)W(t)]y(t)+$$
$$+[W_*(t)+W'(t)]y'(t)+W(t)y''(t)+a_6(t)[(q_4^2-q_3)x^{(4)}(t)+$$
$$+(q_3q_4-q_2)x'''(t)+(q_2q_4-q_1)x''(t)+(q_1q_4-q_0)x'(t)+q_0q_4x(t)+$$
$$+W_*(t)y(t)+W(t)y'(t)]+a_5(t)[-q_4x^{(4)}(t)-q_3x'''(t)-q_2x''(t)-$$
$$-q_1x'(t)-q_0x(t)+W(t)y(t)]+a_4(t)x^{(4)}(t)+a_3(t)x'''(t)+a_2(t)x''(t)+$$
$$+a_1(t)x'(t)+a_0(t)x(t)+\int_{t_0}^{t}\{Q_0(t,\tau)x(\tau)+Q_1(t,\tau)x'(\tau)+Q_2(t,\tau)x''(\tau)+$$
$$+Q_3(t,\tau)x'''(\tau)+Q_4(t,\tau)x^{(4)}(\tau)+Q_5(t,\tau)[-q_4x^{(4)}(\tau)-q_3x'''(\tau)-$$
$$-q_2x''(\tau)-q_1x'(\tau)-q_0x(\tau)+W(\tau)y(\tau)]+Q_6(t,\tau)[(q_4^2-q_3)x^{(4)}(\tau)+$$
$$+(q_3q_4-q_2)x'''(\tau)+(q_2q_4-q_1)x''(\tau)+(q_1q_4-q_0)x'(\tau)+q_0q_4x(\tau)+$$
$$+W_*(\tau)y(\tau)+W(\tau)y'(\tau)]x^{(6)}(\tau)\}d\tau=f(t), t\geq t_0. \quad (5)$$

Let's introduce the notations:

$b_6(t)\equiv a_6(t)-q_4+2W'(t)(W(t))^{-1}$ (coefficient $y'(t)$);

$b_5(t)\equiv a_5(t)+a_6(t)W_*(t)(W(t))^{-1}+q_4^2-q_3+W_*'(t)(W(t))^{-1}$

(coefficient $y(t)$);

$b_4(t) \equiv [a_4(t) - q_4 a_5(t) + (q_4^2 - q_3)a_6(t) + 2q_3 q_4 - q_2 - q_4^3](W(t))^{-1}$
(coefficient $x^{(4)}(t)$);
$b_3(t) \equiv [a_3(t) - q_3 a_5(t) + (q_3 q_4 - q_2)a_6(t) + q_3^2 - q_3 q_4^2 + q_2 q_4 - q_1](W(t))^{-1}$
(coefficient $x'''(t)$);
$b_2(t) \equiv [a_2(t) - q_2 a_5(t) + (q_2 q_4 - q_1)a_6(t) + q_2 q_3 - q_4^2 q_2 + q_1 q_4 - q_0](W(t))^{-1}$
(coefficient $x''(t)$);
$b_1(t) \equiv [a_1(t) - q_1 a_5(t) + (q_1 q_4 - q_0)a_6(t) + q_1 q_3 - q_4^2 q_1 + q_0 q_4](W(t))^{-1}$
(coefficient $x'(t)$);
$b_0(t) \equiv [a_0(t) - q_0 a_5(t) + q_0 q_4 a_6(t) + q_0 q_3 - q_4^2 q_0](W(t))^{-1}$
(coefficient $x(t)$);
$P_0(t,\tau) \equiv (W(t))^{-1}[Q_0(t,\tau) - q_0 Q_5(t,\tau) + q_0 q_4 Q_6(t,\tau)]$ (kernel with $x(\tau)$);
$P_1(t,\tau) \equiv (W(t))^{-1}[Q_1(t,\tau) - q_1 Q_5(t,\tau) + (q_1 q_4 - q_0)Q_6(t,\tau)]$
(kernel with $x'(\tau)$);
$P_2(t,\tau) \equiv (W(t))^{-1}[Q_2(t,\tau) - q_2 Q_5(t,\tau) + (q_2 q_4 - q_1)Q_6(t,\tau)]$
(kernel with $x''(\tau)$);
$P_3(t,\tau) \equiv (W(t))^{-1}[Q_3(t,\tau) - q_3 Q_5(t,\tau) + (q_3 q_4 - q_2)Q_6(t,\tau)]$
(kernel with $x'''(\tau)$);
$P_4(t,\tau) \equiv (W(t))^{-1}[Q_4(t,\tau) - q_4 Q_5(t,\tau) + (q_4^2 - q_3)Q_6(t,\tau)]$
(kernel with $x^{(4)}(\tau)$);
$P_5(t,\tau) \equiv (W(t))^{-1}[Q_5(t,\tau)W(\tau) + Q_6(t,\tau)W_*(\tau)]$ (kernel with $y(\tau)$);
$K(t,\tau) \equiv (W(t))^{-1}Q_6(t,\tau)W(\tau)$ (kernel with $y'(\tau)$);
$F(t) \equiv (W(t))^{-1}f(t)$ (new free term).

Dividing both sides of relation (5) by $W(t)$ considering the introduced notations, and combining the obtained, we get a second-order IDE for $y(t)$ with DE (2) we obtain the following system:

$$\begin{cases} x^{(5)}(t) + \sum_{k=0}^{4} q_k x^{(k)}(t) = W(t)y(t), \\ y''(t) + b_6(t)y'(t) + b_5(t)y(t) + \sum_{k=0}^{4} b_k(t)x^{(k)}(t) + \\ + \int_{t_0}^{t} [\sum_{k=0}^{4} P_k(t,\tau)x^{(k)}(\tau) + P_5(t,\tau)y(\tau) + K(t,\tau)y'(\tau)]d\tau = F(t), t \geq t_0, \end{cases} \tag{6}$$

equivalent to the given seventh-order IDE (1).

To finalize the solution to the problem stated above, it remains to further develop the methods for investigating the asymptotic stability of solutions to linear fourth and fifth-order Volterra-type integro-differential equations on the half-axis as outlined in [3, 4]. it remains to show that for any solution $(x(t), y(t))$ first, square the first equation of system (6), then integrate over the interval from t_0 until t, then multiply the second equation of system (6) by $y'(t)$, integrate over the interval from t_0 until t, for the kernel $K(t,\tau)$ and the free term $F(t)$ apply the method of cutting functions, and finally, sum up the identities obtained for both equations. Then we will have the final identity for system (6). To this identity, we apply the method of integral inequalities from [7]. In this process, we use the generalized Sylvester's criterion, the Cauchy-Bunyakovsky inequality, and finally, the Lyusternik-Sobolev lemma.

The results of the investigation of system (6) will be presented in another work by the author.

Noted that method of investigation of asymptotic stability from [3] was used for IDE of fifth order in article [10].

The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 21, 2023

Revised: December 22, 2023

Accepted: January 15, 2024

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MSC 70F99

HEXAGONAL REGULARIZATION PHENOMENON FOR SYSTEM OF DIFFERENTIAL EQUATIONS DESCRIBING REPELLING POINTS ON A SQUARE

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The central part of arrangement of many repelling pointwise electrical charges in very viscous media within a square described by the system of ordinary differential equations with discontinuous right hand parts forms a regular hexagonal lattice. The constant of numerosity is about 100.

Keywords: mathematical model, repelling, ordinary differential equation, system of equations, phenomenon, constant of numerosity

Оң жактары үзгүл болгон кадимки дифференциалдык теңдемелер системасы менен сүрөттөлгөн чарчынын ичинде өтө илээшкектүү чөйрөдө көптөгөн түртүүчү чекиттик электр заряддарынын жайгашуусунун борбордук бөлүгү туура алты бурчтуу торду түзөт. Көпчө туруктуусу болжол менен 100.

Урунттуу сөздөр: математикалык модел, түртүшүү, кадимки дифференциалдык теңдеме, теңдемелер системасы, кубулуш, көпчө туруктуусу.

Центральная часть расположения большого количества отталкивающихся точечных электрических зарядов в очень вязкой среде внутри квадрата, описываемого системой обыкновенных дифференциальных уравнений с разрывными правыми частями, образует правильную шестиугольную решетку. Константа множественности – около 100.

Ключевые слова: математическая модель, отталкивание, обыкновенное дифференциальное уравнение, система уравнений, явление, константа множественности

Introduction

Some observations, numerical experiments since [1], also [2] yielded

Hypothesis. A random process for sufficiently large number of components in a compact media propelled by outer energy may be self-organizing.

We consider distributions of continuous electrical (similar, repelling) charges on conductors. Such items were considered in many papers. Also, theorems on uniqueness of such distributions were proven. For instance, continuous charges on surfaces were considered in [3], ones on lines were done in [4]. We found that such distributions have other properties, including absence of uniqueness. For example, two charges have three stable arrangements within a regular triangle.

We [5] put a general problem to investigate arrangements of discrete electrical charges, considered motion of a charge under sum of all repelling forces from other charges within the interior of a domain and along its boundary. We [6] proved existence of solution of task for two moving charges on an axis; in [7] we also did existence of solution for one immovable charge and one moving charge on a half-axis. We [8] also proved existence of solution of a similar task for two immovable charges and one moving charge on a segment. We [9] considered an arbitrary number of charges on a segment.

Denote $\boldsymbol{R}:=(-\infty,\infty)$, $\boldsymbol{R}_+:=[0,\infty)$, $\boldsymbol{S}_2:=[0,1]\times[0,1]$. Let the time $t\in\boldsymbol{R}_+$.

1. Description of the problem

We consider $n\geq 2$ points moving on the square $\boldsymbol{S}_2$. Denote their coordinates as $z_i(t)=\{x_i(t), y_i(t)\}$, $i=0..n$. Their initial positions (all are distinct) are given:

$$\{x_i(0), y_i(0)\}=\{x_{i0}, y_{i0}\},\ i=0..n. \tag{1}$$

They are mutually repelling by the Coulomb law.

Suppose that the media of $\boldsymbol{S}_2$ is very viscous. Then forces of inertia can be neglected and a force pushes a body in the media as well as it is immobile each moment. Also, suppose that the velocity of a body in the media is proportional to the force. Denote

$$F_k =$$

$$=\sum_{i\neq k}\{x_k(t)-x_i(t), y_k(t)-y_i(t)\}\left(\left(x_k(t)-x_i(t)\right)^2+\left(y_k(t)-y_i(t)\right)^2\right)^{-3/2}.$$

We obtain the following system of ordinary differential equations of the first order (the coefficient is 1) with initial conditions (1):

$$\{x_k'(t), y_k'(t)\}=\begin{cases} F_k \ if\ \{x_k(t), y_k(t)\}\ is\ in\ the\ interior\ of\ S_2; \\ \quad projection\ of\ F_k\ on\ the\ side\ of\ S_2 \\ \quad if\ \{x_k(t), y_k(t)\}\ is\ in\ the\ side\ of\ S_2; \\ 0\ if\ \{x_k(t), y_k(t)\}\ is\ in\ the\ corner\ of\ S_2; \end{cases} \quad t\in R_+ \tag{2}$$

2. System of difference equations

We wrote the following program in pascal, corresponding to (1)-(2): sides of $\boldsymbol{S_2}$ are 700 pixels; #N means number of points; initial values are chosen randomly:

```
PROGRAM tagaeva_square;
USES CRT, graph;
var a,b,hxy,vx,vy,dx,dy,dxy,dxy1,hxy1,z,z2,xj,yj,dxy2,dxyd,xsw,ysw: double;
i,j,nxy,it,nt,np,ihand,n_time,ik: longint;
var drv, mode,f,n: integer;
x,y:array[1..800] of double;
xn,yn:array[1..800] of integer;
begin {main}
drv:=0; mode:=VgaHi;
InitGraph(drv,mode,'c:\tp\bgi');
randomize;
SetTextStyle(0,0,2);
OutTextXY(30,20,'Tagaeva, 2023. Repelling #N +charges on +square');
OutTextXY(100,40,'(Wait a little)');
z:=700.; z2:=z/2.0; a:=25.; b:=0.01;
np:=10; {nt:=500*n_time;} hxy:=0.5; hxy1:=hxy;
{INPUT}
nt:=1000; nxy:=#N;
for ik:=1 to nxy do begin  x[ik]:=z*(random*0.8+0.1);
y[ik]:=z*(random*0.7+0.1);
xn[ik]:=round(x[ik]); yn[ik]:=round(y[ik]);
Setcolor(green); circle(xn[ik]+80,yn[ik]+70,2);
                    end;
   for it:=0 to nt do
          begin {it} if it>np then hxy:=2.0*hxy1;
  if it>2*np then hxy:=4.0*hxy1;
          for i:=1 to nxy do
       begin {i=ix}
  vx:=a/x[i]-a/(z-x[i]); vy:=a/y[i]-a/(z-y[i]);
  for j:=1 to nxy do
```

```
begin if j<>i then begin
xj:=x[j]; yj:=y[j];
dxy2:=sqr(x[i]-xj)+sqr(y[i]-yj)+1.;
dxy1:=z/(dxy2*sqrt(dxy2)); {if dxy1<sqr(z)/nxy*0.5 then} begin
dx:=(x[i]-xj)*dxy1; dy:=(y[i]-yj)*dxy1;
vx:=vx+dx; vy:=vy+dy; end;
end; end;
 x[i]:=x[i]+vx*hxy; if x[i]>z*(1.-b) then x[i]:=z*(1-b);
 if x[i]<b*z then x[i]:=b*z;
 y[i]:=y[i]+vy*hxy; if y[i]>z*(1.-b) then y[i]:=z*(1.-b);
 if y[i]<b*z then y[i]:=b*z;
      end {i=ix};
{SW} xsw:=x[1]; ysw:=y[1]; for i:=1 to nxy-1 do
      begin x[i]:=x[i+1]; y[i]:=y[i+1] end;
      x[nxy]:=xsw; y[nxy]:=ysw;  {end SW}
for ik:=1 to nxy do begin xn[ik]:=round(x[ik]);
yn[ik]:=round(y[ik]) end;
{for ik:=1 to nxy do write(xn[ik]:5,yn[ik]:5); writeln;
      readln;}
        end {it};
Setcolor(white);
repeat
for ik:=1 to nxy do begin
{circle(xn[ik]+80,yn[ik]+70,6);} circle(xn[ik]+80,yn[ik]+70,3);
circle(xn[ik]+80,yn[ik]+70,2) end;
delay(100);
until keypressed; end.
```

3. Experiments with the system of difference equations

The more is #N, the more points are situated on sides of $\boldsymbol{S_2}$. It corresponds to the known fact that all charges are situated at the surface of the charged conductor. Since #N>100 there arise regular hexagonal lattices (each charge has each charge has six neighbors at approximately equal distances in the central part of $\boldsymbol{S_2}$.

4. Conclusion

The fourth example of implementation of effect of numerosity in irgöö-type processes [9] is found in the paper. It also is an example of the law of the transformation of quantity into quality. This phenomenon can be connected with the real processes of crystallization.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 17, 2023

Revised: December 22, 2023

Accepted: January 15, 2024

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NON-OSCILLATING SOLUTIONS OF VECTOR-MATRIX DYNAMICAL SYSTEMS WITH DOMINANCE

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Supra, the author proved existence of positive solutions of initial value problems for dynamical systems: scalar differential equations with small delay of argument and corresponding difference equations. In the paper conditions for existence of solutions without zeros for 2-vector differential equations with small delay of argument and corresponding systems of difference equations are found.

Keywords: dynamical system, solution, differential equation with delay, difference equation, initial value problem.

Мурда автор төмөнкүдөй динамикалык системалар үчүн баштапкы маселелердин оң чыгарылыштарынын жашоосун далилдеди: скалярдык, аргументинин кечигүүсү кичине болгон дифференциалдык теңдемелер менен дал келген айырма теңдемелери. Бул макалада 2-вектордук аргументинин кечигүүсү кичине болгон дифференциалдык теңдемелер менен дал келген айырма теңдемелери системалары үчүн нөлдөрү жок чыгарылыштарынын жашоосунун шарттары табылды.

Урунттуу сөздөр: динамикалык система, чыгарылыш, кечигүүчү аргументтүү дифференциалдык теңдеме, айырма теңдемеси, баштапкы маселе.

Ранее автор доказала существование положительных решений начальных задач для динамических систем: скалярных дифференциальных уравнений с малым запаздыванием аргумента и соответствующих разностных уравнений. В статье найдены условия существования решений без нулей для 2-векторных дифференциальных уравнений с малым запаздыванием аргумента и соответствующих систем разностных уравнений.

Ключевые слова: динамическая система, решение, дифференциальное уравнение с запаздывающим аргументом, разностное уравнение, начальная задача.

Keywords: finite-difference equation; differential equation with retarded argument; system of equations; special solution; initial problem; asymptotic.

1. Introduction

Formerly, in order to conduct the in-depth study of differential equations with delay, the author proposed the method of splitting the solution space reducing such equations to the systems of operator-difference equations. Using this method, the author assumed new conditions, i.e. the absolute domains for coefficients sufficient for the existence of special (slowly changing) solutions, and proved the presence of approximating and asymptotically approximating properties in them, as well as the asymptotic one-dimensional space of solutions of the initial problems for linear scalar differential equations with small delay of argument and the corresponding difference equation systems (special solutions correspond, to the solutions with a slowly changing first component and a relatively small second component). For the purposes of the single-point representation of the obtained results and other data related to the theory of dynamic systems (the distance between the solution values tends to zero alongside the unlimited increase in argument), throughout this research paper the author uses the concept of the asymptotic equivalence of solutions for dynamic systems, as it was introduced by the author in their previous papers.

In the paper conditions for existence of solutions without zeros for 2-vector differential equations with small delay of argument and corresponding systems of difference equations are found.

In order to provide solution of the initial problem for a linear differential equation with a limited retarded argument, some sources, including the book [1] and research papers [2] and [3], discuss the conditions when there exists such one-dimensional sub-space of solutions (referred to as special), so that any solution tends to increase its argument towards one of the special solutions. The review of such academic sources is given in the research paper [4].

The book [5], it is illustrated that using the differential equations with small argument retardation and the technique of splitting the solution space, it is possible to enable their transformation into the systems of operator-difference equations with preservation of their specifics. Thus, the new conditions ensuring the existence of special (slowly changing) solutions are obtained, their possession of approximating and asymptotically approximating properties is defined, as well as the asymptotic one-dimensional solution space of initial problems for linear scalar differential equations with small argument retardation and corresponding difference system of equations (special solutions correspond to the solutions with a slowly changing first component, and a relatively small second component).

In [6] existence of positive solutions of initial value problems for dynamical systems: scalar differential equations with small delay of argument and corresponding difference equations is proven.

Section 2 introduces the required definitions and denotations.

Section 3 presents the theorem on existence of special solutions of systems of difference equations.

Section 4 presents the newly found conditions for 2-vector delay-differential equations.

2. Denotations and definitions

Denote: $\boldsymbol{N}_0=\{0,1,2,3,\ldots\}$, $\boldsymbol{N}=\{1,2,3,\ldots\}$, $\boldsymbol{R}=(-\infty,\infty)$,$\boldsymbol{R}_+=[0,\infty)$,$\boldsymbol{R}_{++}=(0,\infty)$, I_n - $n\times n$ as identity matrix, $n\in\boldsymbol{N}$.

$C^{m(k)}D$ is the space of functions $u: D\to\boldsymbol{R}^m$ continuous with the derivatives till the order k; D is the domain in $\boldsymbol{R}$, $0\in D, m\in\boldsymbol{N}, k\in\boldsymbol{N}_0$; $C^{*m(k)}D$ is the sub-space of functions meeting the condition $u(0)=0\in\boldsymbol{R}^m$. $m=1$ and $k=0$ shall be neglected.

For the purposes of the uniform representation of problems with continuous and discrete time, it shall be assumed that the argument of the desired function t belongs to quite an ordered set Λ, possessing its smallest element (herein referred to as 0), but still not possessing any largest element. Normally, $\Lambda=\boldsymbol{R}_+$ or $\Lambda=\boldsymbol{N}_0$ are employed.

This paper discusses the initial problems only. Assuming that the initial problem always has some solution, then the solution is the only and global one, and, thus, it covers the entire set Λ; so, then the space of solutions of any particular dynamic system with the entry condition φ may be regarded as the operator $W(t,\varphi):\Lambda\times\Phi\to Z$, with Φ being the topological space of the initial conditions, and Z being the topological space of the solution values. In the case of $\Lambda=\boldsymbol{R}_+$, let us assume that $W(t,\varphi)$ is continuous with respect to t.

D e f i n i t i o n 1. If the operator $W(t,\varphi)$ is linear and $(\forall t\in\Lambda)(W(t,\varphi_0)\neq 0)$ then the solution $W(t,\varphi_0)$ is said to be non-oscillating.

D e f i n i t i o n 2. If the space Z can be split: $Z=Z_1\oplus Z_2$ in such a way that Z_2-components of a solution has a "slight effect" on Z_1-components then the equation is said to be "with dominance" (of Z_1 on Z_2).

3. System of difference equations with dominance

Consider the system of difference equations

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} 1+a_{11,n} & a_{12,n} & a_{13,n} \\ a_{21,n} & 1+a_{11,n} & a_{23,n} \\ a_{31,n} & a_{32,n} & a_{33,n} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}, n = 0,1,2,\ldots \qquad (1)$$

with the initial condition

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} x_{0,0} \\ y_{0,0} \\ z_{0,0} \end{pmatrix} \qquad (2)$$

where $x_n, y_n \in R$ and z_n are elements of a Banach space B, $a_{ij,n}$ are corresponding numbers of R or linear operators. Denote $d:=sup\{\|a_{ij,n}\|: i,j,n\}$.

Define the norm *max* in the space $R^2\times B$.

T h e o r e m 1. If $d< 3-\sqrt{8}=0.17157\ldots$

then there exist such numbers $w,\ \eta_- \in R_{++}$ that the assertions

$$\{x_{0,0}, y_{0,0}\}\neq 0,\ ||z_{0,0}||\leq w||\{x_{0,0}, y_{0,0}\}|| \qquad (3)$$

imply that the solution $\{X_n, Y_n, Z_n\}$ of the IVP (1)-(2) meets the condition

$$(\forall n\in N)(\ ||\{X_n, Y_n\}|| \geq \eta_-^{\,n}||\{X_0, Y_0\}||>0;\ ||Z_n||\leq w||\{X_n, Y_n\}||). \qquad (4)$$

P r o o f. Let $||\{x_{0,0}, x_{0,0}\}||=1$.

For brevity of writing, we will consider pass from $n=0$ to $n=1$.

Without loss of generality, $X_0=1$, $|Y_0|\leq 1$. Estimate

$X_1\geq 1\cdot X_0 - |a_{11}|\, X_0 - |a_{12}|\cdot|Y_0| - ||a_{13}||\cdot|\,|Z_0\,||\geq 1\cdot 1 - d\cdot 1 - d\cdot 1 - d\cdot w = 1-(2+w)d.$

Hence, $||\{X_1, Y_1\}||\geq (1-(2+w)d)\ ||\{X_0, Y_0\}||$.

If $(1-(2+w)d)\geq\eta_-$ then $||\{X_1, Y_1\}||\geq \eta_-\ ||\{X_0, Y_0\}||$.

Estimate

$||Z_1\,||\leq |a_{31}|\, X_0 + |a_{32}|\cdot|Y_0| + ||a_{33}||\cdot|\,|Z_0\,||\leq d\cdot 1 + d\cdot 1 + d\cdot w = (2+w)d.$

If $(1-(2+w)d)\geq\eta_-$ and $(2+w)d\leq w\eta_-$ then

$||Z_1\,||\leq w\eta_- \leq w||\{X_0, Y_0\}||$.

Consider the system $((1-(2+w)d)=\eta_-)\wedge((2+w)d=w\eta_-)$.

Hence, $(1-(2+w)d)w=(2+w)d$.

We obtain the equation $d\cdot w^2+(3d-1)w+2d=0$.

The discriminant $D=(3d-1)^2-8d^2=d^2-6d+1=1-d(6-d)>1-(3-\sqrt{8})(6-(3-\sqrt{8}))=$ $=1-(3-\sqrt{8})(3+\sqrt{8})=0$.

Let $w^*:=(1-3d-\sqrt{(d^2-6d+1)})/(2d)$.

We have $(1-3d)^2-(d^2-6d+1)=1-6d+9d^2-d^2+6d-1=8d^2>0$. Hence, $w^*>0$.

Let $\eta_-^*:=1-(2+w^*)d$.

The inequality $1-(2+w^*)d>0$ is equivalent to

$1-2d-(1-3d-\sqrt{D})/2>0;\ 2-4d-1+3d+\sqrt{D}>0;\ 1-d+\sqrt{D}>0.$

Moreover, the inequality $1-(2+w^*)d>\sqrt{2}-1$ (exact estimation) can be proven:

$2-\sqrt{2}>(2+w^*)d$ is equivalent to

$2-\sqrt{2}>2d+(1-3d-\sqrt{D})/2;\ 4-2\sqrt{2}>4d+1-3d-\sqrt{D};$

$3-d>2\sqrt{2}-\sqrt{D};\ d<3-\sqrt{8}+\sqrt{D}.$

Theorem is proven.

T h e o r e m 2. If $d\le 0.166$ then (3) imply $(\forall n\in N)(\{X_n, Y_n, Z_n\}\neq 0)$

P r o o f. Direct calculation yields $w^*(0.166)<0.98<1$. By Theorem 1,

$||\{X_n, Y_n, Z_n\}||=||\{X_n, Y_n, 0\}+\{0, 0, Z_n\}||\ge||\{X_n, Y_n, 0\}||-||\{0, 0, Z_n\}||=$

$=||\{X_n, Y_n\}||-||Z_n||\ge||\{X_n, Y_n\}||-w||\{X_n, Y_n\}||\ge 0.02||\{X_n, Y_n\}||>0.$

4. 2-vector delay-differential equation

Let us consider an equation:

$$w'(t)=P(t)w(t-h)\ (h\in\boldsymbol{R}_{++}),\ t\in\boldsymbol{R}_+, \tag{5}$$

where $w(t)$ is the unknown 2-vector-function, and $P(t)$ is the given bounded continuous 2×2-matrix-function respectively, with the initial condition:

$$w(t)=\varphi(t),\ t\in[-h,0], \tag{6}$$

where $\varphi(t)$ is the given continuous 2-vector-function.

It is well-known that the initial value problem (5)-(6) is equivalent to the following system of equations for functions $U_n(t)\in C^{2(1)}[-h,0]$, $n\in\boldsymbol{N}_0$:

$$U_0(t)=S_0\big(\varphi(\cdot)\big)(t):=\varphi(0)+\int_{-h}^{t}P(s+h)\varphi(s)\,ds;$$

$$U_{n+1}(t)=S_n\big(U_n(\cdot)\big)(t):=U_n(0)+\int_{-h}^{t}P(s+nh+h)U_n(s)\,ds, n\in\boldsymbol{N}_0. \quad (7)$$

In [7], the author proposed the following method: let us consider the space $C^{2(1)}[-h,0]=\boldsymbol{R}^2\times C^{2(1)}[-h,0]$ as Cartesian product of space of functions-constants equal to $\boldsymbol{R}^2$, and the space $\Omega=C^{*2(1)}[-h,0]$, namely $U_n(t)=x_n+y_n(t)$.

Let us denote that $P_n(\tau)=P(\tau+nh+h)$.

Consequently, the shift operators equal to the value of h in (7) shall be given in the following way:

$$u_{n+1}+z_{n+1}(t)=S_n\big(u_n+z_n(\cdot)\big)(t)=u_n+z_n(0)+\int_{-h}^{t}P_n(s)\big(u_n+z_n(s)\big)\,ds=$$

$$=u_n+\int_{-h}^{0}P_n(s)\big(u_n+z_n(s)\big)\,ds+\int_{0}^{t}P_n(s)\big(u_n+z_n(s)\big)\,ds=$$

$$=\left(\int_{-h}^{0}P_n(s)\,ds+I_2\right)u_n+\int_{-h}^{0}P_n(s)z_n(s)\,ds+\int_{0}^{t}P_n(s)\,ds\,u_n+ \quad (8)$$

$$+\int_{0}^{t}P_n(s)z_n(s)\,ds.$$

Let us denote (a 2×2-matrix) $a_n:=\int_{-h}^{0}P_n(s)\,ds+I_2$ (vector functional) $b_nz(\cdot):=\int_{-h}^{0}P_n(s)z(s)\,ds$, (2×2-matrix-function) $\mathrm{c}_n(t):=\int_{0}^{t}P_n(s)\,ds$,

(operator of the function) $d_nz(\cdot)(t):=\int_{0}^{t}P_n(s)z(s)\,ds$. (9)

Then, the system in (8) is modified as follows:

$$u_{n+1}=a_nu_n+b_nz_n(\cdot),$$

$$z_{n+1}(t)=\mathrm{c}_n(t)u_n+d_nz_n(\cdot)(t), n\in N_0. \quad (10)$$

Presenting $P_n(t)=\begin{pmatrix}p_{11,n}(t) & p_{12,n}(t)\\ p_{21,n}(t) & p_{22,n}(t)\end{pmatrix}$ and $u_n=\begin{pmatrix}x_n\\ y_n\end{pmatrix}$ we obtain a system of type (1):

$$\begin{pmatrix}x_{n+1}\\ y_{n+1}\end{pmatrix}=\begin{pmatrix}x_n\\ y_n\end{pmatrix}+\int_{-h}^{0}P_n(s)\,ds\begin{pmatrix}x_n\\ y_n\end{pmatrix}+\int_{-h}^{0}P_n(s)z_n(s)\,ds;$$

$$z_{n+1}(t)=\int_{0}^{t}P_n(s)\,ds\begin{pmatrix}x_n\\ y_n\end{pmatrix}+\int_{0}^{t}P_n(s)z_n(s)\,ds.$$

As a consequence of Theorem 2:

T h e o r e m 3. If 1) $\Delta:=max\{sup\{|p_{ij,n}(t)|:t\in R_+\}:i,j=1,2\}h$ is sufficiently small;

2) $\varphi(0)\neq 0$ and $sup\{\|\varphi(t)-\varphi(0)\|: t\in[-h,0]\}$ is sufficiently small then the solution of the initial value problem (7)-(8) is non-oscillating.

Conclusion

A new class of matrix-functions "with dominance" is distinguished. Some elements of the main diagonal are close to one, others are small in norm. Such matrix-functions provide category of dynamical systems (difference equations and equivalent delay-differential equations) with non-oscillating solutions.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 2, 2023

Revised: December 8, 2023

Accepted: January 15, 2024

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NONLINEAR MODEL OF UNEVEN ECONOMIC GROWTH OF REGIONS OF THE KYRGYZ REPUBLIC

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During the years of independence, the unevenness (asymmetry) of economic development of the regions of the Kyrgyz Republic has increased, the gap between regions in the most important indicators of regional production, income levels and poverty, and the quality of life of the population is increasing. Smoothing out economic asymmetry is one of the most important problems of public administration. We have proposed a nonlinear model based on production functions such as VES -functions (VES -variable elasticity substitution production function).

Keywords: uneven development, nonlinear model, region, VES-production function.

Эгемендүүлүк жылдарында Кыргыз Республикасынын региондорунун экономикалык өнүгүүсүнүн бирдей эместиги (ассиметриялуулугу) өстү, региондук өндүрүштүн, кирешелердин жана жакырчылыктын деңгээлинин, калктын жашоо сапатынын эң маанилүү көрсөткүчтөрү боюнча региондордун ортосундагы ажырым өсүп жатат. Экономикалык асимметрияны текшилөө мамлекеттик башкаруунун маанилүү проблемаларынын бири болуп саналат. Биз VES-функциялары сыяктуу өндүрүш функцияларына негизделген сызыктуу эмес моделди сунуш кылдык.

Урунттуу сөздөр: тегиз эмес өнүгүү, сызыктуу эмес модель, аймак, VES - өндүрүштүк функция.

В годы независимости неравномерность (асиметрия) экономического развития регионов КР возросла, увеличивается разрыв между регионами по важнейшим показателям регионального производства, уровня доходов и бедности, качества жизнь населения. Сглаживание экономической асимметрии является одним из важнейших проблем государственного управления. Нами предложена нелинейная модель на базе производственных функций типа $VES -$функций ($VES -$ variable elasticity substitution production function).

Ключевые слова: неравномерность развития, нелинейная модель, регион, $VES -$ производственная функция.

Introduction

Solving the problems of sustainable economic growth of the regions of the Kyrgyz Republic in new conditions requires the development of appropriate forecasting and analytical tools that would allow the authorities of the country's region to carry out scientifically based forecasts of the region's development, taking into account the requirements of annual forecasts of the socio-economic development of the country as a whole. The core of such tools should be an economic and mathematical model of economic development in the region, which would make it possible in the medium and long term to assess the limits of growth of gross output and gross regional product depending on investment efforts, effective demand, demographic factors, environmental behavior, as well as others. aspects of sustainable growth. The factor of socio-economic asymmetry in regional development significantly complicates the process of modeling economic growth and makes the conclusions obtained in the fundamental theories and models of economic growth untenable.

In macroeconomic studies of the structure of the economy at the regional level, one can use the single-sector model of R. Solow. In this model, the regional economic system is considered as a single unstructured whole, producing one universal product that can be both consumed and invested. The three-sector model reflects the reproduction process in more detail. The basic model when considering the structure of the economy by sector in general and employment in particular is the theory of three sectors, developed by K. Clark, and over time supplemented by D. Bell, E. Toffler and W. Rostow. This model is still relevant today, but requires new high-quality content. Its essence lies in changing the share of the primary, secondary, tertiary sectors in gross domestic production.

Three-sector model

To develop a three-sector model, we will take the Solow model as the base model. As an optimality criterion, we consider the maximum production of consumer goods per employee in the production sector.

The production subsystem of the economy can be divided into three sectors. The first sector is the material (zero) sector, which produces objects of labor. This sector includes the mining industry, electric power industry, agricultural production, construction materials industry, freight transport, office communications, and trade. The second sector is the capital-creating sector (the first) – the sector produces means of labor: machines, equipment, industrial buildings, etc. The third sector is the consumer (second) sector, which produces consumer goods. This sector includes civil construction, agricultural processing (light and food industries), passenger transport, civil communications, and trade in consumer goods.

Let us assume that general production assets (GPA) are assigned to each sector. Investment and labor can move freely between sectors. Then you can accept the following conditions:

- the technological structure is considered constant and is specified using nonlinear-homogeneous neoclassical production functions $X_i = F(K_i, L_i) \quad i = 0,1,2$.

- the total number of employees L (in the production sector) changes with a constant growth rate v. Then v in discrete time has the form (t-year number)

- $v = \frac{L(t+1)-L(0)}{L(t)}$

upon transition to continuous time, it takes the form of a differential equation

$\frac{dL}{dt} = vL,\ L(0) = L^0,$ its solution is $L = L^0 e^{vt}$.

- there is no investment lag.
- the wear coefficients of GPA μ_i and direct material costs α_i sectors are constant.
- the economy is closed, i.e. foreign trade is not considered.

From assumptions (3, 4) it follows that the change in the general income of the i-th sector over the year consists of two parts: depreciation ($-K_i$) and growth due to gross capital investments (I_i), that is, $K_i(t+1) - K_i(t) = -\mu_i K_i + I_i(t), \quad i = 0,1,2$ or in continuous form: $K_i(t+\Delta t) - K_i(t) = [-\mu_i K_i(t) + I_i(t)]\Delta t$ at $\Delta t \to 0$.

We obtain a differential equation for the GPA sectors:

$$\frac{dK_i}{dt} = -\mu_i K_i + I_i, K_i(0) = K_i^0,$$

GPA and the number of employees in sectors (K_i ,L_i) are instantaneous indicators, i.e. their value can be measured at any point in time. Sector output and investment (X_i, I_i) are flow type indicators, i.e. their values accumulate over the year starting at time t.

Three-sector economic model:

$L = L^0 e^{vt}$ is the number of employees;

$L_0 + L_1 + L_2 = L$ is distribution of employees by industry;

$\frac{dK_i}{dt} = -\mu_i K_i + I_i, K_i(0) = K_i^0$ is dynamics of funds by sector;

$X_i = F(K_i, L_i) \quad i = 0,1,2$ is output by sector;

$X_1 = I_0 + I_1 + I_2$ – distribution of products of the capital-creating sector;

$X_2 = \alpha_0 X_0 + \alpha_1 X_1 + \alpha_2 X_2$ is distribution of products of the material sector.

The model is dynamic and includes linear dynamic elements:

$$\frac{dK_i}{dt} = -\mu_i K_i + I_i, \qquad \frac{dL}{dt} = vL.$$

It is nonlinear, since sector outputs are specified by nonlinear production functions $X_i = F(K_i, L_i)$ and is multiphase, since its state is represented by three variables K_0, K_1, K_2 interconnected through balances. The emergence of the economic system appears in the balance sheets, i.e. the presence of such general properties that are not inherent in the individual elements that make it up: each sector produces as much product as other sectors and consumers need and for as long as the resources are sufficient.

The endogenous (determined by the model) variables are the GPA and sector outputs. Exogenous (specified from outside the model) variables are the growth rate of the employed v, the depreciation coefficients of GPA, the coefficients of direct material costs, the initial value of the number of employees, and their initial distribution.

Management is carried out by distributing labor ($L_0 + L_1 + L_2 = L$) and investment ($X_1 = I_0 + I_1 + I_2$) resources.

When constructing the PF, certain difficulties arose that are characteristic of the economy of Kyrgyzstan. The use of the neoclassical *CES* production function for analysis does not always allow us to assert that the elasticity values of labor substitution with capital σ are constant, since the cost of fixed and working capital strongly depends on the volume of investment, which is associated with capital and determines its dynamics. It should also be noted that the adoption of additional assumptions justifying the possibility of using *CES* production functions of these types in the long run reduces the accuracy of the resulting estimates.

Recently, in studies of the place of *CES*-production functions, *VES*-functions (*VES*-variable elasticity substitution production function) have been used, which take into account changes in the elasticity of labor substitution with capital σ in a certain period. Various options are used to represent this production function analytically.

Calculations are made based on statistics from 2010 to 2022 [2]. X_0 is production material costs. X_1 is the indicator "Accumulation" minus "Production of consumer goods". X_2 is "Non-productive consumption". Numbers K_i is were determined according to the indicators "Fixed production assets by industry". Numbers L_i were determined by the indicators "Distribution of the population employed in the economy by industry".

The resulting function looks as:

$$X_0 = 13{,}24e^{0{,}08t}K_0^{0{,}39}L_0^{0{,}61}e^{-1{,}12k}$$

$$X_1 = 17{,}32e^{0{,}13t}K_1^{0{,}56}L_1^{0{,}44}e^{-2{,}32k}$$

$$X_2 = 32{,}11e^{0{,}17t}K_0^{0{,}47}L_0^{0{,}53}e^{-2{,}17k}$$

Value $\sigma = 1 + \frac{\alpha k}{[b-\alpha k]^2 - b}$,

where $\alpha = 1{,}24$; $b = 0{,}27$.

Approximation characteristics: $s^2 = 0{,}32; dw = 1{,}37;\ v = 3{,}094;\ R^2 = 0{,}864$ show that the obtained parameter estimates are statistically significant.

Analysis of the obtained parameter values shows that the constructed production function of the *VES*-function type by sector makes it possible to obtain a more accurate approximation from 65 to 85% of the values of the final product of the economic system and the modified production function $g(k)$ to the corresponding initial data. The elasticity coefficients of output for fixed assets in three sectors show a downward trend. The elasticity coefficient for labor resources increases slightly. And total elasticity shows that output increases more slowly than resource inputs.

The values of the standard deviation for the constructed production function of the *VES*-function type are less than for the previously developed *CES*-functions. This allows us to conclude that the proposed *VES* function provides a more "stable" approximation of the calculated values of Y and the modified production function $g(k)$.

It can be noted that the proposed and implemented algorithm for constructing a δ-homogeneous *VES*-production function that meets the requirements for neoclassical production functions allows the specified function to construct with a sufficiently high accuracy of approximation of the parameters characterizing the functioning of the economic system.

To solve socio-economic problems and for the optimal distribution of available resources, the state needs to make adjustments using various mechanisms for regulating the distribution of resources or through direct participation in the economic process. The task of improving the well-being of the population should be a priority, so managing structural changes requires special research. Managing structural imbalances makes it possible to increase labor productivity and capital productivity.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 8, 2023

Revised: December 12, 2023

Accepted: January 15, 2024

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ISSUES OF NATURAL RESOURCE MANAGEMENT AND AGRICULTURAL FORECASTING IN THE KYRGYZ REPUBLIC

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The purpose of the paper is the theoretical and methodological justification of the socio-economic and environmental aspects of the rational use of natural resources of Kyrgyzstan, the development of practical recommendations. In the article based on the analysis and evaluation of the rational use of natural resources identified factors associated with climate change, degradation of agricultural land, pastures, increasing population, increased foreign integration, the degree of food security, the level of food consumption, the state of export-import operations, and the impact of these factors on poverty and public health.

The subject and object of the paper is a set of organizational and economic problems associated with the rational use of natural resources. The authors focus on developing specific proposals for the economic and rational use of natural resources. The forecast until 2035 shows that for the growth of agricultural products, a set of measures is necessary related to the effective management of natural resources, technical equipment, and financing of agriculture in terms of improving the variety of seeds of agricultural crops and breeding breeds in agriculture.

Keywords. Ecology, social sector, agricultural sector, foreign trade, food security, quality, standards, water, pastures, optimization, modernization, innovation, training, persons with disabilities, resources, export potential, forecast.

Макаланын максаты – Кыргызстандын жаратылыш ресурстарын сарамжалдуу пайдаланууну жана айыл чарба продукциясын өндүрүүнү болжолдоону теориялык жана методологиялык жактан негиздеп, практикалык сунуштарды иштеп чыгуу. Жаратылыш ресурстарын сарамжалдуу пайдаланууну талдоонун жана баалоонун негизинде макалада климаттын өзгөрүшүнө, айыл чарба жерлеринин, жайыттардын деградациясына, калктын санынын өсүшүнө, тышкы интеграциянын күчөшүнө, азык-түлүк коопсуздугунун даражасына, азык-түлүк керектөө деңгээлине, экспорттук-импорттук операциялардын абалы жана бул факторлордун жакырчылыкка жана калктын ден соолугуна тийгизген таасири.

Макаланын предмети жана объекти болуп жаратылыш ресурстарын сарамжалдуу пайдалануу менен байланышкан уюштуруучулук жана экономикалык көйгөйлөрдүн жыйындысы саналат. Авторлор жаратылыш ресурстарын унөмдүү жана рационалдуу пайдалануу боюнча конкреттүү сунуштарды иштеп чыгууга көңүл бурушкан. 2035-жылга чейин жүргүзүлгөн математикалык моделдөө айыл чарба өндүрүшүнүн өсүшү үчүн жаратылыш ресурстарын натыйжалуу башкарууга, техникалык жактан жабдууга жана айыл чарбасын каржылоого байланыштуу иш-чаралардын комплекстүү аткарууну талап кылынарын көрсөтүп турат.

Урунттуу сөздөр. Экология, социалдык сектор, агрардык сектор, тышкы соода, азык-түлүк коопсуздугу, сапат, стандарттар, суу, жайыт, оптималдаштыруу, модернизация, инновация, кадрларды даярдоо, мүмкүнчүлүгү чектелген адамдар, ресурстар, экспорттук потенциал, болжомол.

Целью статьи является теоретическое и методологическое обоснование рационального использования природных ресурсов и прогнозирование производства сельскохозяйственной продукции Кыргызстана, разработка практических рекомендаций. В статье на основе анализа и оценки рационального использования природных ресурсов выявлены факторы, связанные с изменением климата, деградацией сельскохозяйственных угодий, пастбищ, ростом численности населения, усилением внешней интеграции, степенью продовольственной безопасности, уровнем потребления продовольствия, состояние экспортно-импортных операций и влияние этих факторов на бедность, и здоровье населения.

Предметом и объектом статьи является совокупность организационно-экономических проблем, связанных с рациональным использованием природных ресурсов. Авторы акцентируют внимание на разработке конкретных предложений по экономному и рациональному использованию природных ресурсов. Проведенное математическое моделирование до 2035 года показывает, что для роста производства сельскохозяйственной продукции необходим комплекс мер, связанных с эффективным управлением природными ресурсами, техническим оснащением и финансированием сельского хозяйства в части улучшения разнообразия семян сельскохозяйственных культур и селекционных пород в сельское хозяйство.

Ключевые слова. Экология, социальный сектор, аграрный сектор, внешняя торговля, продовольственная безопасность, качество, стандарты, вода, пастбища, оптимизация, модернизация, инновации, обучение, люди с ограниченными возможностями, ресурсы, экспортный потенциал, прогноз.

At present, the environmental, social and economic development of society is already evidently occurring with a gross imbalance and imbalance in the biosphere. Strengthening of foreign trade relations puts requirements of rational use of natural resources, ensuring food and environmental safety, social responsibility, as well as ensuring competitiveness of food products exported to the world market on a new level. At the same time, the level of methodologies and practical developments and mechanisms for the rational use of natural resources developed to date, taking into account the environmental, social and economic situation is not yet fully able to effectively address environmental issues and the use of natural resources of the country.

With expansion of integration of the country into the world economic system, diversity of social production the problem of food security of the country and production of ecologically clean agrarian production corresponding to international quality and standards becomes more and more acute.

In this aspect, research aimed at further development of crop and livestock sectors, more effective use of land and water resources of Kyrgyzstan are of great relevance.

The subject and object of the paper is the totality of resource potential related to the development of agricultural production. The authors focus on the development of the state program "Prospects of rational use of natural resources of agricultural production in conditions of integration of Kyrgyzstan".

Currently, state regulation of the processes of rational use of food resources and protection of land and water and biosphere should be based on the application of organizational and economic mechanisms related to food resources, population and foreign trade.

Of particular importance are studies on the development and implementation of soil-protective and innovative activities, cultivation of crops and livestock and poultry, taking into account different soil and climate and location of the branches of agricultural production.

At the same time the assessment of realities and prospects plays in solving the problems of satisfying the population's demand for food products in terms of quality, ensuring food security of the country, improving the socio-economic and environmental situation of society, as well as transportation, trade, service, customs control, etc.

In order to successfully ensure the socio-economic and environmental aspects of the rational use of natural resources of agricultural production, it is necessary to consistently and timely identify trends and determine the prospects of its development. With the rational use of natural resources of agrarian production is closely related, the state of the social sphere, the environmental well-being of sectors of the economy.

As a result of the study theoretical and methodological foundations of the essence and content of the rational use of natural resources of agricultural production in terms of foreign trade integration are substantiated, the author's point of view on the definition of the main factors is given and the prospects of agricultural production are determined. The use of theoretical and practical recommendations can allow creating conditions for increasing the potential of agricultural production, increasing the volume of export-oriented agricultural commodity products, improving foreign trade turnover, increasing control over the observance of environmental purity of produced goods, improving the welfare of society, reducing poverty and migration outflows of the rural population.

The methodological basis of the study were the scientific works of domestic and foreign scientists on the interaction of society with the natural environment and rational use, the classification of natural resources, the problems of production, evaluation and optimization of agricultural production, economics of nature management and government regulation reflecting the objective regularities of this process. Wide use was made of normative-methodological developments, statistical data and literary sources and Internet sources.

In the process of research methods of system analysis, economic-statistical, monographic, comparative methods, etc. were used.

Many aspects of ecological-economic and social direction were addressed in the activities of international organizations such as the UN, WHO, GEF, FAO, etc., within the framework of relevant programs, conventions, declarations, as well as the reports of conferences, seminars, symposiums, trainings and discussions.

An important condition for solving socio-economic and environmental aspects in the rational use of natural resources of agrarian production under conditions of strengthening foreign trade relations, is to accelerate the pace of innovation and optimization of the development of agricultural sectors, transfer the economy of agricultural production to the intensive way, the rational use of productive capacity and labor resources, saving material and other resources and ensuring food security of the country and the build-up of the ob

The study showed that the uneven denationalization and privatization of state property, the distribution of natural and productive resources, the lack of methodological approaches, taking into account the resource potential and socio-economic situation of the country, as well as the inconsistency of agricultural reform did not ensure stable growth of the agricultural economy and poverty reduction in rural areas. As a result of the agrarian reform, the number of small-scale peasant (farmer) farms sharply increased, and the number of large-scale commodity farms significantly decreased. In Kyrgyzstan, from 1991 to the present transition to market relations, the growth rate of gross domestic product to the previous year decreased, if in 1991 it was 92.1%, in 2021 it was 91.4%. The share of commodity production of agricultural products from the level of gross domestic production has sharply decreased from 35.3% in 1991 to 13.5% in 2021 [2].

Insufficient growth rates of gross domestic product, relatively low crop yields and animal productivity, high population growth rate of Kyrgyzstan does not fully allow to ensure food production in the size that fully meets the growing needs of the population.

Further sustainable growth of the agricultural economy of Kyrgyzstan is closely linked to the further rational use of natural resources of agricultural production, ensuring food and environmental security, increasing the rate of production of export-oriented agricultural products and improving the social condition of the population.

Kyrgyzstan has significant resource and economic potential for economic and social development. The total area of land in Kyrgyzstan is 19,994.9 thousand hectares, including 10,607.7 thousand hectares of agricultural land, of which 1,287.4 thousand hectares of arable land and 9,174.3 thousand hectares of hayfields and pastures [6].

However, the transformation and use of agricultural lands takes place against the background of the destruction of ecology, the balance of the biosphere, the accelerated development of anthropogenic processes, such as soil erosion, reduction of humus content, waterlogging, degradation, pollution of productive lands, unreasonable land allocation for non-agricultural needs, the continued growth of lands unhappy with irrigation and meliorative conditions. This has a direct negative impact on the development of agricultural production and improvement of the efficiency of the agrarian sector of the economy. At the same time, Kyrgyzstan has not yet developed environmental insurance of the use of natural resources of the agrarian sector of the economy. We support the opinion of the authors Naminov K.A., Yablunovsky M.Y., Nadbitov N.K. that "... environmental insurance as an element of environmental and economic innovation development system can act as a financial tool for implementation of environmental policy of the enterprise, as well as a tool to strengthen environmental control, which ultimately will promote sustainable economic development and ensure the implementation of the economic, environmental, social policy of government agencies" [3].

Yes ecological insurance can provide reduction of risk events in the use of natural resources, especially the agrarian sector of the economy.

Since land and water resources are the main content of natural resources, and they are limited in scale and volume. this in turn raises the problem of rational resource use, in particular the agricultural sector. In Kyrgyzstan, the agricultural sector of the economy is the main source of food supply for the population and the creation of jobs not only for the villagers, but also in the industrial centers of the republic that process agricultural raw materials. However, the volume of production and sales of agricultural products and marketability remains low. Thus, in 2019, the turnover of grain, tobacco, oilseeds, potatoes, slaughtered animals and poultry, raw milk and wool (by physical weight) ranges from 59.8 to 82.0 percent.

Of these, about 721 thousand hectares are occupied by arable land subject to water erosion. One of the extremely negative factors contributing to water erosion is the presence of terrain slopes. There are about 614 thousand hectares of such lands out of the total arable area [6].

One of the reasons is that in recent years there has been a decrease in natural fertility, soil washing out and gully formation, soil degradation due to erosion, salinization and technical awarding, and measures for creation of field belts, plantations in gullies, on river banks and water bodies, foreseen for wind and water erosion have tendency to decrease due to lack of financial means and establishment of farms (peasant farms) 452.3 thousand farms [2].

Virtually most of the existing peasant (farm) households do not participate in the prevention of environmental violations. As a result of irrational use of natural resources of Kyrgyzstan, lack of organizational and economic mechanisms, specialization of organization of production and market of produced agricultural products, produced agricultural products are often lost and spoiled at the place of production, and in many cases do not reach consumers because of low volume, low quality, lack of necessary market information base, infrastructure and trade and logistics centers.

At present, one of the most important strategic tasks is the rational use of water resources in the Kyrgyz Republic, which is a priority in the national economy. This is associated primarily with an increase in population and reduction of agricultural land, a consistent decline in water resources, in particular irrigation water.

In the "Screenshot from CABAR.asia conference" Prof. Hojimukhammad Umarov, PhD in Economics from Tajikistan notes that, "In Tajikistan, water scarcity and drought issues are less manifest, that "We need to consider Central Asia as a whole, from glaciers to deserts, because a significant part of the flow disappears in these deserts. We need to create a joint commission, include geologists and hydrologists, make calculations and see how much water we lose to infiltration, to evaporation in those numerous reservoirs. Besides, in the countries located below, air temperature is much higher in summer" [4]. In our opinion, it is possible to agree partially that this question should be considered jointly by the commission, to include there geologists and hydrologists. In our opinion, during consideration besides participation of geologists and hydrologists, but other experienced specialists, experts and it is necessary to consider world experience of distribution, payment for using water supply, as well as geographical location, social economic condition of states and ecological consequences of countries participating in using water and glaciers.

In 5 years, Central Asia will enter a state of chronic, constant water shortage. This forecast was voiced on November 16 by the chief economist of the Eurasian Development Bank, Evgeny Vinokurov, at a round table on sustainable development in Almaty [5].

Table 1. Forecast of the main indicators of growth of gross agricultural output in the Kyrgyz Republic until 2035.

	Trend equation	Approximation	Determination coefficient R2	2032 Forecast
Gross product in agriculture (million soms)	$y = 3260*t^2-18950*t+207490$	A=7.09	R2=0.86	1132700
Volume of investment (Total)	$y =35220*x^{0.0505}$	A=8.89	R2=0.6	40978
Number of tractors	y = -407.81 / x + 20755	A=1.17	R2=0.54	20734
combines	y = 5* x + 2354	A=0.9	R2=0.93	2454
Number of patent applications	$y = 212.85*0.827^x$	A=12	R2=0.52	4.8
Investments in agricul-ture (inclu-ding pro-cessing) (million soms)	y = 32575* ln(t) + 250867	A=12	R2=0.52	34845
Issued loa-ns for agri-culture from all sources	y = 2572* ln(t) - 262	A=15	R2=0.52	7444

As can be seen from the table, 5 variables are correlated with the main indicator, with gross agricultural output, R2 for all values above 0.50, respectively, the selected indicators are significant. The forecast until 2035 shows that for the growth of agricultural products, a set of measures is necessary related to the effective management of natural resources, technical equipment, and financing of agriculture in terms of improving the variety of seeds of agricultural crops and breeding breeds in agriculture.

According to the land cadastre, there are 1170.4 thousand hectares of saline lands, including 398.6 thousand hectares of slightly saline, 399.1 thousand hectares of medium saline, 301.1 thousand hectares of strongly saline and 70.8 thousand hectares of saline lands. Along with saline lands, the area of waterlogged lands increases due to inactivity of collector-drainage networks. According to data from the land cadastre, the area of land subject to water and wind erosion on the territory of Kyrgyzstan is about 5 million hectares or 45.7% of the total area of agricultural land [8].

The current practice of small-scale farming leads to the destruction of soil fertility due to non-compliance with agro-meliorative technology. Breach of environmental disadvantage in the use of land resources was considered by the Government of Kyrgyzstan. [7] noted that, the fertility of arable land is decreasing every year. It is noted that, "...the main reasons for deterioration of irrigated lands are insufficient natural drainability of the territory, the initial absence or destruction of collector-drainage network, large losses of irrigation water during filtration in irrigation canals, non-normalized irrigation regime".

At the same time, the non-observance of agricultural practices and crop rotation by numerous small peasant (private) farms has led to the destruction of soil fertility. Salinization and waterlogging of irrigated agriculture takes place. The ameliorative condition of irrigated agriculture has significantly deteriorated, and the level of state support for water production on irrigated lands has decreased. Formation of a large layer of small economic entities-water users has led to disorganization of irrigation systems operation, destruction of on-farm network. Soil fertility decreases due to the removal of nutrients by plants, failure to apply fertilizers also leads to a decrease in their fertility.

In recent years, the development of agriculture based on the use of the most fertile soils of the southwest (Osh, Jalal-Abad and Batken provinces) has been sharply constrained by the practical reduction of irrigation water resources, at the same time the scale of pasture degradation has increased due to the lack of science-based approaches to pasture resources use, which leads to destruction of the ecological state and balance of the biosphere.

Due to the low educational level of managers and employees of peasant (private) farms, mineral fertilizers are used without taking into account the content of humus in the soil, as well as in gross violation of sanitary norms. In some farms, pesticides and toxic chemicals are stored in warehouses that do not meet typical standards. Irrational or deliberate use of pesticides and toxic chemicals in agricultural production leads to accumulation of their residual amounts in soil, water and agricultural products. It is known that agricultural food products obtained with gross violations of the norms of pesticides, toxic chemicals and mineral fertilizers in field farming, the use of banned drugs and vaccines in animal husbandry can affect public health.

According to [6], the application of mineral fertilizers for individual crops in 2019 amounted to 291.9 in terms of active substance, while this figure in 2015 was - 333.7 thousand quintals. Application of organic fertilizers decreased compared to 2015 by 157.6 thousand tons and amounted to 258.0 thousand tons. There is no accurate accounting of the application of mineral fertilizers, pesticides, pesticides, drugs, vaccines by peasant (private) farms. Consequently, reducing the level of organic fertilizers can affect the reduction of productivity of fields, and the excessive use of mineral fertilizers directly affects the quality and environmental cleanliness of agricultural products and the health of the population.

In Kyrgyzstan in 2019 a total of 1,553.4 thousand people were registered as sick, some of them were diagnosed with infectious and parasitic diseases - 93, injuries and poisonings - 87.4 thousand people, complications of pregnancy, childbirth and postnatal period - 50.5 thousand people, diseases of blood, blood forming organs and some violations involving immune mechanism - 50.6 thousand people, congenital anomalies (malformations of development) - 6.3 thousand people. It should be noted, the total number of persons with disabilities first recognized with disabilities has increased from 2015 to 2019 from 10.9 to 11.0 thousand people. Overall, from 2000 to 2018 has doubled, amounting to 186.7 thousand people, respectively, in 2018 [6].

At present, it is necessary to further expand the sown areas, it is necessary to optimize their structure in favor of high-yield crop varieties that require higher rates of fertilizers, moisture and a high level of protection against weeds and plant pests. Of particular importance are intensification factors that increase the productivity of cultivated plants by increasing the genetic potential by breeding methods and optimization of environmental conditions. Simultaneously with strengthening of the fodder base it is necessary to intensify animal breeding on the basis of improving the structure, modernization of production processes, qualitative composition of cattle and poultry, improving of selection and breeding work.

An important role in the development of sectors of agriculture creating modern material and technical conditions, modernization of agricultural production belongs to the banking sector, as well as various production services in the field of rural construction, water, land reclamation, irrigation works, electrification, mechanization and chemicalization of production, repair and industrial-technical service (tech-service) of agricultural machinery and equipment. A special place is occupied by personnel training, clustering of agricultural sectors, digitalization, especially providing them with investment support, creation of transport, road and economic, trade and logistics infrastructure and services.

To achieve the intended goal, we consider expedient the coordinated and voluntary enlargement of farms (peasant farms) on the basis of one agricultural cooperative or association not less than 400-500 hectares of irrigated arable land, to ensure and observe the agrotechnical measures for cultivation of crops and the content of livestock breeding. Especially the observance of crop rotation requirements in the cultivation of all crops, regardless of species and varieties from a scientific point of view is mandatory. In this regard, in the branch of agriculture it is necessary to fully master the science-based systems of farming and livestock breeding, the introduction of innovative technology. It is necessary to introduce proven high-yield crop varieties, integrated mechanization of cultivation and harvesting, rational use of organic and mineral fertilizers, increased use of biological means of crop protection from weeds, pests and diseases, timely varietal change and variety renewal based on industrial organization of seed production in the field. Achieve rational use of irrigation water through transformation of less intensive lands into more intensive ones, namely conversion of part of rainfed cropland and low productive pastures into irrigated cropland, perennial plantations, cultivated pastures and irrigated hayfields.

In [1] the authors note that "...the fundamental principle of state regulation of foreign economic activity is the principle of excluding unwarranted intervention of the state and its bodies in this activity, causing damage to its participants in the economy of the whole country. The economic policy of the state should be directed on perfection of a normative-legal base of regulation of foreign economic activity and strengthening of the control over foreign economic operations". We agree that "Economic policy of the state should be directed on perfection of normative-legal base of regulation of foreign economic activity and strengthening of the control over foreign economic operations", really development of new normative-legal acts on rational use of natural resources providing quality of made production, to increase export potential of the country and to provide receipt of foreign currency means to the state treasury.

As noted above, during the years of reform in the agricultural sector of the economy through the unequal denationalization and privatization of state property growth rates of gross domestic product decreased, significantly reduced the share of commodity production of agricultural products from the level of gross domestic product. All this contributed to the emergence of poverty, an increase in the number of people with disabilities, poor families, children left without parental care and other socially vulnerable citizens caused by low incomes and unemployment of a significant part, especially the increased migration outflow of the population in rural areas.

The issues of socio-economic and environmental problems of rational use of natural resources of agricultural production in the context of integration has a multi-stage nature. This is achieved through intergovernmental agreement, the adoption of regulatory and legal documents, through the optimization and rationalization of country specialization, improvement of the Customs Code of Kyrgyzstan, as well as through the expansion of its import substitution of food products in the context of international integration.

The use of theoretical and practical recommendations can allow to create conditions for building the capacity of agricultural products, increasing the volume of export-oriented agricultural commodity products, improving external turnover, strengthening control over the environmental purity of the products produced, improving the welfare of society, reducing poverty and migration outflows of the rural population.

Timely regulation of the rational use of natural resources can ensure the stable growth of the national economy, increase labor employment, improve the living standards of the rural population, reduce poverty, reduce labor migration, and most importantly will continuously and consistently provide industry with raw materials.

In our opinion, the development of the agrarian sector of the economy is achieved through the enlargement of small peasant (farm) farms, shops for processing agricultural products and sewing of consumer goods and their modernization, creating conditions for the introduction of clustering of production, the country can produce brand products of Kyrgyzstan (environmentally friendly), which can become competitive in the world market.

It is necessary to develop a state program "Prospects for the rational use of natural resources of agricultural production in the context of integration of Kyrgyzstan", taking into account the geographical location (lack of access to the sea) and the availability of resource potential of the agricultural sector of the economy, which should provide for the further effective development of the national economy. The essence of which is the intensification and specialization of production with the rational use of natural and economic resources, the introduction of innovation in agriculture.

All this requires the development of mutually beneficial contracts and agreements between neighboring states and the EAEU states. It is necessary to take into account the availability and reserves of natural resources, the economic potential of Kyrgyzstan in particular the development of industries, agriculture, financial, social sectors, and transport, road and trade and logistics infrastructure and services. In our opinion, the development of road maps, development of infrastructure of economic sectors and digitalization of it, i.e. only by creating the appropriate infrastructure, determining the prospects for development of economic sectors, making the road map and appropriate state support can provide a stable growth of the agricultural economy of the country.

Through the modernization and timely use of innovative technology in agricultural production can achieve full satisfaction of the needs of the population in the food. For this purpose, we consider it necessary, to enlarge small shops for processing agricultural products, to increase the number of modernized large processing enterprises, to introduce waste-free technology everywhere, to increase the production of meat and dairy products, production of soft drinks, mineral water, various fruit juices, ice cream and other food, to provide the population with high quality food products.

Allocation in the State Budget of Kyrgyzstan a separate line "Protection of flora and fauna objects and their habitat", including agricultural land and irrigation water will contribute to the development of agricultural production and ecology. Financial support in the form of insurance by the state can ensure the reduction of risk events in the use of natural resources, especially the agricultural sector of the economy.

In order to ensure the quality of agricultural production, to strengthen the laboratory base for assessing the quality of agricultural products and feed. To improve the methodological basis for the training of specialists and staff. Systematically conduct training on the rational use of natural resources in agricultural production, production technology, compliance with international standards of quality of goods produced agricultural products, compliance with regulations, environmental and sanitary norms, as well as the rules of sale and control of goods produced agricultural products in the foreign market.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 14, 2023

Revised: December 12, 2023

Accepted: January 15, 2024

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91G05

EXAMPLES OF THE APPLICATION OF PARABOLIC INTERPOLATION OF MORTALITY TABLES FOR FRACTIONAL AGES

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In this article we will consider the use of parabolic interpolation of mortality tables for fractional ages, and its application in problems of actuarial mathematics. Let's consider ways to use parabolic interpolation to solve problems related to life insurance and determining net premiums.

Key words: actuarial mathematics, mortality table, interpolation of mortality tables, mortality tables for fractional ages, insurance.

Бул макалада бөлчөк жаштар үчүн өлүм таблицасын параболалык интерполяциялоо, жана аларды актуардык математикасын маселесинде колдонуусу каралган. Параболалык интерполяциялоонун өмүр камсыздоодо жана нетто премияларды аныктоодо колдонулушу көрсөтүлгөн.

Урунттуу сөздөр: актуардык математика, өлүм таблицасы, өлүм таблицасын интерполяциялоо, бөлчөк жаш үчүн өлүм таблицалары, өмүрдү камсыздандыруу.

В данной статье мы рассмотрим применение параболической интерполяции таблиц смертности для дробных возрастов, и его применение в задачах актуарной математики. Рассмотрены способы применения параболической интерполяция для решения задач, связанных со страхованием жизни и определением нетто премий.

Ключевые слова: актуарная математика, таблицы смертности, интерполяция таблиц смертности, таблицы смертности для дробных возрастов, страхование жизни.

1. Introduction

Mortality tables provided by the National Statistical Committee of the Kyrgyz Republic [1] play an important role in demographic studies, insurance, pensions and other areas. But situations often arise where it is necessary to determine mortality for fractional ages; in such cases, interpolation methods are used.

Here are the main designations and definitions.

${}_tp_x$ is the probability that a person aged x years will live to age $(x+t)$.

${}_tq_x$ is the probability that a person aged x will die before the age of $(x+t)$.

$s(x)$ is a survival function.

v is a discount factor.

Definition 1. An interpolation formula for determining the survival function for fractional ages determined by the formula

$$s^2(x+t) = (1-t)s^2(x) + ts^2(x+1), \tag{1}$$

where $0 \le t < 1, x \in Z^+$, is said to be the parabolic interpolation.

In the works [2], [3] by the first author, it was shown how to determine various indicators of mortality tables for the case of parabolic interpolation.

But, it should be noted that there are a large number of interpolation methods, which are given in works [4]-[7] that describe examples of the use of mortality tables.

We will show the application of the method of parabolic interpolation of mortality tables for fractional ages.

2. Application of parabolic interpolation

$s(x)$ is known from the KR mortality table [1], for integer values of x. Parabolic interpolation is a decreasing function. Moreover, the parabolic interpolation of the function has an upward convexity, which suggests that parabolic interpolation is the only interpolation currently used that has an upward convexity. From the graph of the survival function of the data taken for 2023, one can notice that there is just an upward convexity.

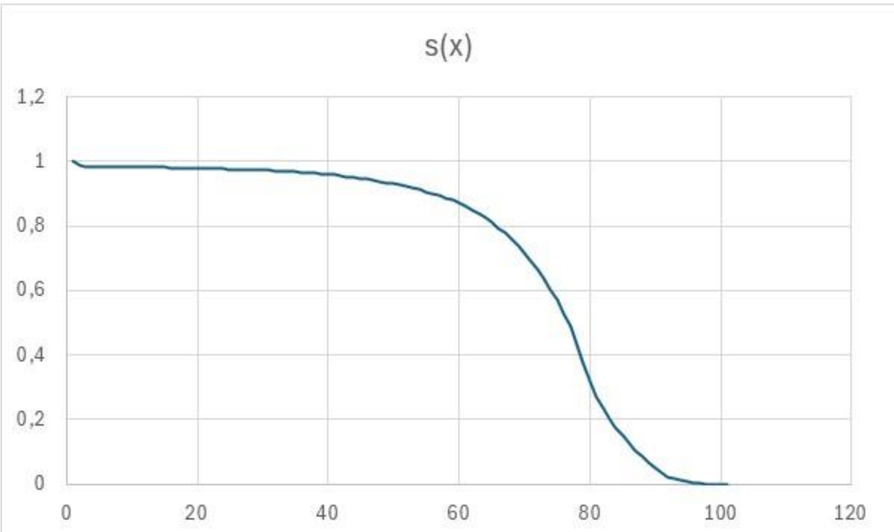

figure 1. Survival function graph.

Based on the data from [1], we will plot the function p_x

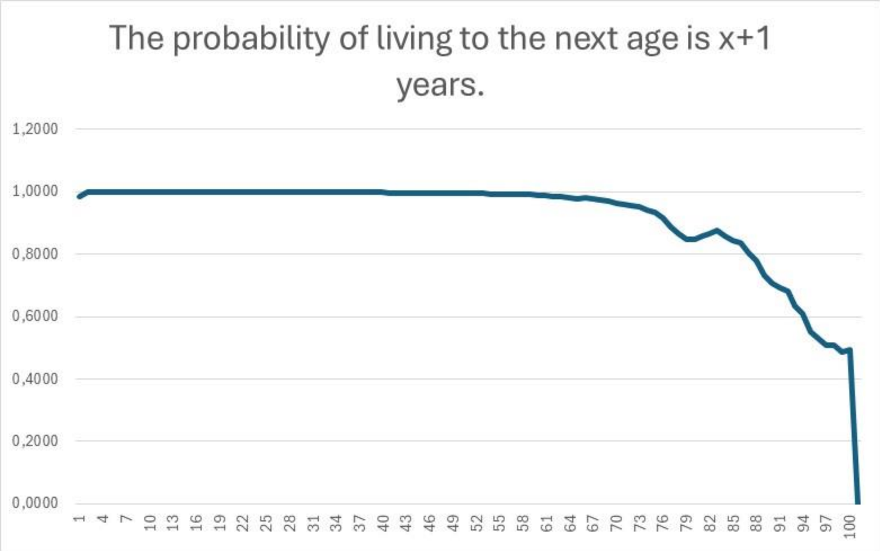

Figure 2. Probability of surviving to $(x+1)$ years.

Figure 2 shows that the probability of survival is low in the first year of life ($p_0 = 98.6\%$), which corresponds to high infant mortality under one year, reaching 1.5%.

Using parabolic interpolation formulas, we will solve several tasks.

Task 1. Calculate a) $s\left(18+\frac{1}{12}\right)$, b) ${}_{\frac{2}{3}}p_{35}$, c) ${}_{\frac{1}{4}}q_{45}$.

Solution. From the mortality table [1] we know

$s(18) = 0{,}97918$; $s(19) = 0{,}97864$;

$p_{35} = 0{,}9984$; $p_{45} = 0{,}9971$.

a) $s\left(18+\frac{1}{12}\right) = \sqrt{s^2(18) - \frac{1}{12}\left(s^2(18) - s^2(19)\right)} = \sqrt{0{,}95871} = 0{,}97913.$

b) ${}_{\frac{2}{3}}p_{35} = \sqrt{1-\frac{2}{3}(1-p_{35}^2)} = \sqrt{0{,}99782} = 0{,}99891.$

c) ${}_{\frac{1}{4}}q_{45} = 1 - \sqrt{1-\frac{1}{4}(1-p_{45}^2)} = 1-\sqrt{0{,}9985} = 0{,}00072.$

Task 2. Calculate the net premium of an insurance policy in the amount of 250 000 for survival to 65 years 3 months, for a person aged 25 years 6 months. If the market interest rate is 8.5% per annum.

Solution. Let us calculate the probability of survival of a person aged $25\frac{1}{2}$ years to

$65\frac{1}{4}$ years.

$$ {}_{39\frac{3}{4}}p_{25\frac{1}{2}} = \frac{s\left(65\frac{1}{4}\right)}{s\left(25\frac{1}{2}\right)} = \frac{0{,}78582}{0{,}97474} = 0{,}80618 $$

Now let's calculate the net Premium

$$ A = P \cdot v^{39\frac{3}{4}} \cdot {}_{39\frac{3}{4}}p_{25\frac{1}{2}} = 250000 \cdot \left(\frac{1}{1{,}085}\right)^{39\frac{3}{4}} \cdot 0{,}80618 = 7\,871{,}18 $$

Answer: 7871,18.

3. Conclusion

In this work, we observe how the parabolic interpolation method [2]-[3] can be used to solve problems of actuarial mathematics and when calculating indicators for life insurance problems.

Also, this interpolation method can be used in training specialists in the field of actuarial mathematics, insurance and statistics.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 21, 2023

Revised: December 8, 2023

Accepted: January 15, 2024

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MSC 74E25

MATHEMATICAL MODEL FOR TEXTURE PRESENTATION OF COLORS

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The authors propose developing of list of colors with corresponding mathematical descriptions of relief textures. Such textures can be implemented on 3D-printers. They can be used for preparing “colored” pictures and production of stickers on corresponding things for blind men.

Keywords: color, texture, mathematical model, blind

Авторлор рельефтик текстуралардын тиешелүү математикалык сүрөттөлүшү менен түстөрдүн тизмесин иштеп чыгууну сунуш кылышат. Мындай текстураларды 3D-принтерлерде ишке ашырууга болот. Аларды сокурлар үчүн «түстүү» сүрөттөрдү даярдоо жана тиешелүү нерселерге стикерлерди жасоо үчүн колдонсо болот.

Урунттуу сөздөр: түс, текстура, математикалык модель, сокур

Авторы предлагают разработать перечень цветов с соответствующими математическими описаниями рельефных текстур. Такие текстуры можно реализовать на 3D-принтерах. Их можно использовать для изготовления «цветных» картинок и изготовления наклеек на соответствующие вещи для слепых.

Ключевые слова: цвет, текстура, математическая модель, слепой

Introduction

There exist various devices informing persons on seeing colors, for instance [1]. We put problem to promote immediate sensation of imitation of color.

We consider texture as filling areas with repeating elements. Relief texture for blind people can be implemented on 3D-printers [2].

A review of the literature shows that there is no standard for the texture representation of colors.

None information on this subject was found in Internet. For example, [3] proposes textured walk-way for blind people: stripes and dots colored yellow for other people.

[4] describes many relief images in details but does not mention colors.

We propose to develop such standard and to apply it.

Project of mathematical model for colors

1) White color (*the universal color for background*): large dots O O O O O
O O O O

2) Black color (*absence of colors; sky in night*): little dots · · · · ·

3) Grey color (*between white color and black color*): medium dots • • • • •

4) Red color (*danger, attention, blood; a color being closer to red causes immediate sensation of warmth on skin*): barbed triangles ∆ ∆ ∆ ∆ ∆ ∆

5) Yellow color (*sunshine*): squares

6) Green color (*plants, leaves*): oblique crosses × × × × ×

7) Blue color (*sky without clouds*): signs of heart

8) Violet color (*close to blue color*): rhombs ◊ ◊ ◊ ◊ ◊

9) Brown color (*indefinite color*): asterisks * * * * *

Our proposals

Modern computer technology makes it possible to achieve the goal:

To enable all blind people in Kyrgyzstan to feel the colors of some household items and all the colors in specially made paintings and to use the names of colors in communication.

1) Develop a standard for the textural image of colors;

2) Develop software for a 3D-printer;

3) Provide all blind people in Kyrgyzstan with a set of stickers and pictures with appropriate instructions for use by relatives;

4) Provide organizations where blind people work with printers with such software.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 27, 2023

Revised: December 8, 2023

Accepted: January 15, 2024

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MSC 18C05

BASIC COMPLEXES IN MATHEMATICAL CATEGORIES, EFFECTS AND PHENOMENA

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Supra, the notion of category of equations was introduced by the author with assistance of the notion “predicate” on the base of the principle of preservation of solution while transformations. History of mathematics is presented as successful treating of phenomena arising both in mathematics and in life; phenomena as consequences of effects is proposed. A notion of a basic complex of notions and relations as a core of any branch of mathematics instead of equivalent systems of axioms is introduced. Both well-known and found and distinguished by the author and her colleagues effects and phenomena are surveyed.

Keywords: category, phenomenon, effect, axiom, computer, experiment.

Мурда автор, өзгөртүүлөрдө чыгарылышты сактоо принцибинин негизинде “предикат” тү-шүнүгүнүн жардамы менен теңдемелердин категориясынын түшүнүгүн киргизген. Матема-тиканын тарыхы математикада да, турмушта да пайда болгон кубулуштарды удаа кароо катары берилген; таасирлердин натыйжасы катары көрүнүштөр сунушталат. Аксиомалар-дын эквиваленттүү системаларынын ордуна математиканын кандайдыр бир тармагынын өзөгү катары түшүнүктөрдүн жана мамилелердин негизги комплекси жөнүндөгү түшүнүк киргизилет. Белгилүү да, автор жана анын кесиптештери тарабынан табылган жана айыр-маланган эффекттер, кубулуштар изилденет.

Урунттуу сөздөр: категория, кубулуш, эффект, аксиома, компьютер, тажрыйба.

Ранее автором было введено понятие категории уравнений с помощью понятия “предикат” на основе принципа сохранения решения при преобразованиях. История математики представлена как последовательное исследование явлений, возникающих как в математике, так и в жизни; явления как последствия эффектов. Вводится понятие базового комплекса понятий и отношений как ядра любой отрасли математики вместо эквивалентных систем аксиом. Представлен обзор как известных, так и обнаруженных и выделенных автором и ее коллегами эффектов и явлений.

Ключевые слова: категория, явление, эффект, аксиома, компьютер, эксперимент.

1. Introduction

Investigations in the category of topological spaces in Kyrgyzstan were initiated by A.A.Borubaev [1]. We [2] introduced the notion of the category of equations was with assistance of the notion “predicate” on the base of the principle of preservation of solution while transformations; elements of the category of equations and its subcategories were constructed on the base of well-known categories.

We proposed framework definitions of effect and phenomenon in mathematics [3]. We proposed to consider phenomena as consequences of effects.

Below we present some items of its history is as successful treating of phenomena arising in mathematics and life. We introduce a notion of a basic complex of notions and relations as a core of any branch of mathematics instead of equivalent systems of axioms.

Methodic of systematic search of new effects and phenomena by means of computer experiments is proposed.

Section 2 contains survey of known categories.

Section 3 contains some items of history of mathematics.

Section 4 presents a framework definition of a basic complex.

We distinguished "effect of analyticity“; we separated "effect of self-organizing" and "synergetic effect"; we found "irgöö effect" and "effect of numerosity" within the general effect of self-organizing; we and other authors found some new phenomena as consequences of these effects.

Examples of basic complexes, both well-known and these effects and phenomena as their consequences are given in Section 5.

Some items of this paper were delivered at the VII World Congress of Turkic World Mathematicians at city of Turkestan, October 2023.

2. Survey of known categories

2.1) The basic category *Set* of sets. *Ob(Set)* are sets; *Mor(Set)* are functions.

2.2) The category *Func* of functions (synonyms in various branches of mathematics: maps, operators, transformations). It is mentioned in publications but we did not find its formal description. *Ob(Func) = Mor(Set), Mor(Func)* are trans-formations of functions.

2.3) The category *Top* of topological spaces. *Ob(Top)* are topological spaces, *Mor(Top)* are continuous functions.

We proposed 2.4) Category of equations *Equa*. *Ob(Equa)* contains tuples {Non-empty sets X, Y, predicate $P(x)$ on X, transformation $B: X \to Y$}.

If $(\exists x \in X)(P(x) \wedge (y=B(x)))$ then $y \in Y$ is said to be a solution of the equation $\{X, Y, P, B\}$. Particularly, if B is the identity operator I, then we obtain the equation *"P(x)"* only. *Mor(Equa)* are such transformations of tuples $\{X, Y, P, B\}$ that solutions (or their absence) preserve.

2.5) We also defined some subcategories of *Equa:* the category *Equa-Func* of equations for functions; The category *Equa-Par* of equations with parameters; the category *Equa-Func-Par* of equations for functions with parameters.

3. Approaches to history of mathematics

There are a lot of surveys of the history of mathematics and its branches from various standpoints. Some publications state facts without critics and generalizations.

M.Kline since [4] presents mathematics as a series of attempts to strictly describe and uniformly substantiate arising facts, ideas...

A.A.Borubaev since [5] presents developing of topology as successful implementation of idea of continuity in systems of axioms.

A.A.Borubaev, P.S.Pankov [6] described developing of the idea of space.

A.A.Chekeev noted that in actual investigations in any branch of mathematics they use not any system of axioms but a set of interrelated "well-known, evident" concepts, statements and facts. Unfortunately, he did not publish it in time.

4. Framework definitions

Notions "effect" and "phenomenon" were used widely including mathematics but we had not could find any definitions for them. We proposed framework definition for "phenomenon" and noted that in history of mathematics, firstly phenomena and paradoxes appeared in mathematics itself or in life and further researchers tried to explain them or to create any theory. We also noted that various phenomena were not logically connected but had same origin ("effect").

Consider a theorem in general as an implication of conditions $A \Rightarrow B$, or, more concrete, if there is a general class X of objects x and $A \subset X$, $B \subset X$ then $A \subset B$ or $(x \in A) \Rightarrow (x \in B)$. To prove sufficiency of A for B one is to construct an example without both A and B. An (interesting, nonpresumable, single) way of violating B is said to be a *phenomenon*.

If P is a property (or some properties) of elements $x \in X$ having a property E such that a logical proof $(E \wedge C) \Rightarrow P$ (where C is any additional condition) is too complicated and the property P was discovered not by a logical way but by meeting paradoxes, by experiments in physics and chemistry or by computational experiments in mathematics then E is said to be an *effect*.

Careful analysis of Euclid's Elements in XVIII century demonstrated that they were not an axiomatic system of geometry, many theorems were proven by "common sense". A complete axiomatic system was developed at the end of XIX century but it turned out to be too complicated and is not used actually.

In the end of XIX century it turned out that Cantor's theory of sets is controversial. To "complete" this theory, various proposals have been made including attempts to overcome "saying about itself" such as Russell's theory of definite descriptions. But they were too complicated, were critiqued and their multitude itself testifies absence of generally accepted theory of sets.

The notion of a real number is not axiomatic actually. Definitions by Dedekind, by decimal system, by continuous fractions involve actual infinity and cannot be applied immediately. Many authors of text-books prefer to list necessary properties.

Topological space may be defined through open sets and dually through closed sets; by neighborhoods and dually by "exteriors", etc.

Mathematics of origami have interesting results but does not have any system of axioms.

To get around such difficulties the notion “category” was introduced. But it contains non-formal notion “class”.

Hence, some branches of mathematics have not such generally accepted systems.

By developing A.A.Chekeev's idea we introduce the notion of a basic complex of notions and relations as an actual core of any branch of mathematics instead of equivalent systems of axioms.

5. Some basic complexes and effects in mathematics

5.1. Effect of infinity. Probably, the notion of infinity in science and first paradox on this the-me was discovered by Zenon, V century B.C.: Achilles and Turtle. The phenomenon "A part of an infinite object can be equivalent to itself" was discovered by Galileo, XVI century; hints in the notion "homoeomeria" (similar-to-part) were made by Anaxagorus, V c. B.C. This phenomenon refuted the intuitively evident statement fixed as Euclid's fifth axiom "the whole is greater than the part". The further development of mathematics was significantly connected with attempts to overcome such paradoxes. At last, P.Cohen (1963) demonstrated that (by our term) a suitable basic complex involving infinity is impossible. As "positive" consequence of homoeomeria we mention the phenomenon of fractals (B. Mandelbrot, 1975).

5.2. Continuity and smoothness. The idea of continuity yields "some tasks for functions have solutions for continuous ones". Its initial version was proved by K. Weierstrass. Consequences of these properties were obtained in various branches of mathematics. Most of them are "predicable, obvious" such as "superposition of continuous/smooth functions is continuous/smooth". Hence, they are not "effects" in our terminology. Because of uncertainty of notions "real number" and "continuity" there were numerous attempts to derive any kind of smoothness from continuity in XVIII-XIX centuries. They were stopped by Weierstrass's example (1895) of a continuous function that is differentiable nowhere. Such hopes resumed for "analyticity" (see below).

5.3. Effect of singular perturbations. To explain the physical phenomenon of boundary layer in flow of a slightly viscous fluid near a solid boundary L.Prandtl (1905) discovered the effect of singular perturbations. One of mathematical implementations: the effect of parameter $0 < \varepsilon << 1$ by the highest derivative in a differential or integro-differential equation. Some authors discovered phenomena of interior boundary layer. We searched consequences of the effect of singular perturbations systematically, and we found the phenomenon of singular (lengthening as ε tends to 0) cycle, the phenomenon of the deepening boundary layer and other ones.

5.4. Effect of analyticity: some tasks being non-correct for continuous and even infinitely differentiable functions became correct for analytical ones, on the base of following results (phenomena): K.S.Alybaev: existence of level lines for singularly perturbed ordinary differential equations; P.S.Pankov and T.M.Imanaliev: convergence of the grid method for partial differential equations out of the domain covered by characteristics; L.Askar kyzy, under our leadership: correctness of the final value problem for the heat equation; correctness of some integral equations of the first kind; correctness of the initial value problem for the Laplace elliptical equation. These results were firstly groped by numerical experiments.

5.5. Effect of higher dimensions. It is well-known that passing from 1D (numbers) to 2D (2×2-matrices) yielded many phenomena: non-commutability of multiplication; zero product of non-zero factors etc. Passing to next dimensions yields possibilities of new combinations of eigenvalues of matrices. Passing from 2D to 3D in the theory of non-linear differential equations yielded an unexpected phenomenon of strange attractors. Passing from 3D to 4D P.S.Pankov and S.B.Tagaeva discovered and implemented "mechanical strange attractor" as a round bowl with odd number of symmetrically arranged bulbs on the edge and a little ball moving within the bowl under gravitation.

5.6. Effect of self-organizing. We found the first mention of concrete implementation of it in literature: irgöö (in Kyrgyzstan, some centuries ago): "random vibration of “*many*” balls of different sizes of same material in a wide symmetrical vessel yields migration of the biggest one to the center of their surface." We implemented computer simulation [7]. Further, this phenomenon, named “Brazil nut effect (BNE), was re-discovered in other regions. Probably, the first publication was [8], implementation (independent of ours) was [9].

The second one was "Rayleigh-Bénard convection cells" (1900). We proposed to use this term: “irgöö effect” for all self-organizing processes with finite number of components; we called the necessity of “*many*” as “numerosity effect”, we did an approximate lower boundary for “*many*” providing a phenomenon as “numerosity constant”. We propose to use the term “synergetic effect” for self-organizing processes in continuous media.

6. Conclusion

We hope that this paper would promote discovering new effects and phenomena in various branches of mathematics and its applications by computational experiments and other methods, and numerosity constants would be found for any real and virtual processes, that there exist many specific 4D-phenomena to be discovered.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 27, 2023

Revised: December 25, 2023

Accepted: January 15, 2024

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COMPUTER PROGRAMS FOR MATHEMATICAL COMPETITIONS AND EXAMINATIONS

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The Institute participates in conducting mathematical competitions for Youngers, employees of the Institute teach Youngers to write original computer programs for examinations in mathematics. Both well-known and elaborated in the Institute definitions, methods and techniques for this purpose are described in the paper.

Keywords: computer, program, mathematics, competition, examination

Институт студенттер үчүн математикалык мелдештерди өткөрүүгө катышат, Институттун кызматкерлери студенттерди математика боюнча сынактарды үчүн оригиналдуулугу бар компьютердик программаларды жазууга үйрөтүп жүрөт. Бул максат үчүн белгилүү да, Институтта ойдон чыгарылган да аныктоолор, усулдар жана ыкмалар макалада жазылат.

Урунттуу сөздөр: компьютер, программа, математика, мелдеш, сынак

Институт участвует в проведении математических соревнований для студентов, сотрудники Института учат студентов писать оригинальные компьютерные программы для экзаменов по математике. В статье описаны как широко известные, так и разработанные в Институте определения, методы и приемы.

Ключевые слова: компьютер, программа, математика, соревнование, экзамен

Introduction

The Institute participates in conducting mathematical competitions for Youngers, employees of the Institute teach Youngers to write original computer programs for examinations in mathematics. Both well-known and elaborated in the Institute definitions, methods and techniques to develop complex multimedia interactive soft to teach and to check understanding, knowledge and skills in mathematics, real knowledge of languages are described in the paper. Various types of tasks: interactive, “measuring imagery”, “Socrates-type”, “guess to correct”, “from outside”, “answer as action”, necessary-collective are described.

Denote jury or teacher as Elder, contestant or examinee as Younger, competition or examination as Testing.

We hope that implementation of such soft would be an interesting and useful training for Youngers in programming. They will be able to check their work immediately, with their friends, younger sisters and brothers.

Some items of this paper were delivered at the VII World Congress of Turkic World Mathematicians at city of Turkestan, October 2023.

Remark. Nowadays Youngers use artificial intellect during Testing and it is almost impossible to prevent it. Hence, more complex tasks must be checked by authors through Artificial Intellect. If Artificial Intellect can solve a task then it cannot be given.

1. Desired features of softs (briefly)

1.1. Testing and Study modes (Soft is a kind of text-book).

1.2. Several languages.

1.3. Tuning by Elder for various Testings.

1.4. Availability of Tasks.

1.5. Generativity: Task does not exist before Testing.

1.6. Uniqueness: all Youngers are given different Tasks.

1.7. Complete secrecy: nobody including authors of Tasks and Elders know right answers until end of Testing.

1.8. Various types of Tasks and presentation of Tasks with multimedia.

1.9. Younger’s respond in various forms (not “multiple choice”).

2. Desired features of softs - explanations

2.1. Soft must have Testing and Study modes. In Study mode Younger inputs initial data for a task, solves a task and compares their result with the true one. i.e. Soft is a kind of text-book too.

2.2. Texts in Soft must be written in several languages (with Younger's option to switch in Study mode: Soft is a kind of dictionary). This will promote Bologna process and wide using of Soft.

2.3. Setting option for Elder: select a language; select a scope of tasks for Testing from the general list of tasks; set time; is it possible to retry if the answer is wrong?; are marks shown immediately (current examinations) or after Testing? Replenishing with new tasks, to involve creative teachers and Youngers into mutual work.

Forming a file with Younger's name, marks, date, time and other information in ciphered form. Remote participation in Testing (on-line or off-line with sending this file to Elder.

2.4. Availability of Tasks. We introduced the term *Independent learning* [1].

If possible: do not require special knowledge; do not use special terms and denotations (or give simple examples with them) although the condition of Task becomes longer; reduce dependence on language (graphical form is preferable); admit partial and approximate answers (with corresponding reduction of score).

Example 1. NOT "Find the sum of arithmetic sequence with first term 3, common difference 2 and last term 301"

BUT "Find the sum 3+5+7+…+301 (4+6+8+…+14=54)".

Example 2. NOT "Find $lim\{7k/(k+6) \mid k\to\infty\}$"

BUT "Given $a_1=5/13;\ a_2=6/15;\ a_3=7/17$ … Find a_{100} with accuracy 0.001".

Example 3. NOT "Calculate $\int_0^5 x^2dx$" BUT "Calculate the area under …

(graph with formula $y=x^2$) exactly or approximately".

2.5. Parameterized tasks or Generativity. Task does not exist before Testing and is generated randomly (not only numerical data, content of task may vary too, under demand " of same difficulty level").

General Example 4. min / max; increasing function / decreasing function;

rectangular / equilateral trapezium;

various types of equations ($x^2+px=q,\ x^2+q=px,\ x+p=q/x$).

As a consequence,

2.6. Uniqueness: all Youngers are given sufficiently different Tasks, and

2.7. Complete secrecy is guaranteed.

Socrates’ method [2] provides the following

Hypothesis. By using suitable tasks and questions, proposing to con-duct experiments the teacher can drive a Younger (or, more effectively, a group of Youngers) at independent discovery of many facts of elementary mathematics, of discrete mathematics, some facts of analysis and of physics.

Such tasks will be marked*. They also provide Availability.

2.8. Various types of Tasks (text, drawing, picture, short videoclip, sound (earphones are necessary)). The following subitems overlap.

2.8.1. Common type of tasks. Condition: text; answer: integer number. There are some techniques to reduce various mathematical objects to definite integer numbers including ”corrected reality”: ”*Let the free fall constant on surface of the Earth be (exactly) 9.8 meter/second*^{2} *; ”let all atomic weights be integer numbers”; ”let* $\pi = 22/7$*”.*

2.8.2. Interactive tasks: ”if you find answer in less than 10 attempts you will get full score” including binary or more effective search.

2.8.3. [1] ”Give approximate answer X in short time (20 seconds)” including tasks on measuring imagery or ”intellectual eye measurer”: a task with exact or almost exact answer X_0 (positive real number or other continuous object) is given, if $| X / X_0 - 1 | < 0.1$ then the mark is A+.

2.8.4. ”Form the task having the given solution (give numbers in condition of a task)”.

2.8.5. Tasks ”from outside” with real or virtual objects to check skills too. Example 5. The real object (a sheet with many details) must be printed before Testing. But number of (random) combinations of details is too large to be solved before Testing.

2.8.6. Algebra. ”Solve in short time (10 seconds)”.

Example 5. $99023456 + 2000 - 99023455=$

$9902342 - 990232 * 999236=$

2.8.7. “Guess to correct”.

Example 6. 890+56=834.

Example 7*. GAHTGAHTGAWTGAHT; UTVMIXIMWTU

2.8.8. Real counting. Given ”slightly organized” pseudo-graphic.

Example 8. >>**|**|||********<***** **** * ****

”How many asterisks are here? Write in binary system”.

2.9. Younger’s respond (integer number(s), ”real” number(s) with prescribed accuracy, word, “Point-and-Click”, “Draw-and-Drop” actions by computer mouse).

3. Examples of arithmetical and algebraic tasks

All tasks below are supposed to be parameterized. Both traditional and our tasks are presented.

For independent discovery:

Task 1. Sequence (2.6*).

a) ”Count the sum 1 + 3 + 5 + ... + 17”.

b) ”Count the sum 5 + 7 + 9 + ... + 39”.

c) ”Count the sum 100 + 97 + 94 + 91 + ... + 4”

Version for collective solution:

Task 2. Task for the second teammate:

”The sum 100 + 97 + 94 + 91 + ... < the first teammate’s respond > .

Find the last member of the sequence”.

Task 3. Inverse equation (2.6*).

a) ”AX + B = 105; X = 26/5. Find any natural numbers A and B.”

b) ”AX(X + B) = D; X = 9 + √ 5. Find any integer numbers A≠ 0, B, D.”

Version for collective solution:

Task 4. Task for the second teammate with numbers given by the first teammate:

a) ”< A > X+ < B >= 1050. Round X up to integer.

” b) ”< A > X(X + B) =< D >.

Round up the greater root of the equation to integer.”

For an examination in a language:

Task 5. Zoology. ”How many legs(paws) have a calf, a bee, two kittens, a spider, a snake, a worm and three geese together?” Younger’s ignorance of at least one of these words will be detected.

4. Conclusion

We hope that these methods and examples would make competitions in mathematics more interesting and useful.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 10, 2023

Revised: December 5, 2023

Accepted: January 15, 2024

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MSC 54D05

CONTROLLED MOTION AND VISIBILITY OF STRETCHED OBJECTS IN VIRTUAL SPACES

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Controlled motion in virtual topological spaces was proposed in Kyrgyzstan. Such spaces were called and defined as kinematical ones. Algorithms of controlled motion of points in kinematical spaces were implemented to reach any point. The second au-thor proposed controlled motion of stretched objects in kinematical spaces to reach points. Controlled motion to see any marked point in kinematical spaces is proposed and considered in the paper.

Key words: topological space, kinematical space, controlled motion, visibility.

Кыргызстанда элестетилген топологиялык мейкиндиктерде башкарылуучу кыймылдоо сунуш кылынган. Мындай мейкиндиктер кинематикалык мей-киндиктер деп аталган жана аныкталган. Кинематикалык мейкиндиктерде чекитке жетүү үчүн чекитти жылдыруунун алгоритмдери ишке ашырылган. Автор кинематикалык мейкиндиктерде чекиттерге жетүү үчүн созулган объ-ектилердин башкарылуучу кыймылын сунуш кылган. Макалада белгиленген чекитти көрүү үчүн башкарылуучу кыймыл сунуш кылынат жана каралат.

Урунттуу сөздөр: топологиялык мейкиндик, кинематикалык мейкиндик, башкарылуучу кыймылдоо, көрүнүүчүлүк.

В Кыргызстане было предложено движение в виртуальных топологичес-ких пространствах. Такие пространства были названы и определены, как кине-матические. Были реализованы алгоритмы управляемого движения точек в кинематических пространствах для достижения какой-либо точки. Второй автор предложил управляемое движение протяженных объектов в кинематических пространствах для достижения точек. В статье предложено и рассмотрено управляемое движение, чтобы увидеть отмеченные точки.

Ключевые слова: топологическое пространство, кинематическое пространство, управляемое движение, видимость.

1. Introduction

Controlled motion in virtual topological spaces was proposed in Kyrgyzstan [1], [2]. Such spaces were called and defined as kinematical ones. Definition of a kinematical space M (the metric ρ_K is the minimal time of passing between points) with controlled motion of points was introduced. Algorithms of controlled motion of points in kinematical spaces were implemented to reach any point. The second author proposed controlled motion of stretched objects in kinematical spaces to reach points [3].

There are many papers on visibility in Euclidean space and its subspaces, for example [4-8], but the only work on visibility in a non-Euclidean space [9] was found. The author considers a hyperbolic plane with analog of square grid and identify visibility with existence of a geodesic line. But it is suitable only for this task.

The report [10] on visibility took a bronze medal but was never published. Controlled motion to see any marked point in kinematical spaces is proposed and considered in the paper.

Section 2 contains definitions.

Section 3 is brief description of [10]: Given the Riemann surface of the equation $w^2 + (z-a)(z-b)=0$. There is Person in a round car of a given radius r, $2r < |a-b|$ and a point object in this space. How much time does Person need to get to see the object? The theorem to detect such this minimal time by coordinates of an initial point is proven.

Section 4 contains definitions of visibility and tasks.

2. Definitions for motion of points with bounded velocity

As a codifying the ancient idea of controlled motion with bounded velocity, the notion of kinematical space was introduced [1].

D e f i n i t i o n 1. A computer program is said to be a **presentation** of a computer kinematical space if:

P1) there is an (infinite) metrical space X of points and a set X_1 of display-presentable points being sufficiently dense in X;

P2) the user can pass from any point x_1 in X_1 to any other point x_2 by a sequence of adjacent points in X_1 by their will;

P3) the minimal time to reach x_2 from x_1 is (approximately) equal of the minimal time to reach x_2 from x_1.

The space X is said to be a **kinematic space**; the space X_1 is said to be a **computer kinematic space**; this minimal time is said to be the **kinematical distance** ρ_X between x_1 and x_2; a sequence of adjacent points is said to be a **route**.

Considered the practical task. Let there be a thing and obstacles. It is necessary to move the thing to another place. Is possible? If yes then in what minimal time it can be done?

D e f i n i t i o n 2. There is a family P of connected subsets of the set X (we will call them **passes**); each pass has the positive **length (time)** and a family Q of connected subsets of the set X (we will call them **things**).

(It means that a thing moves along a pass).

The space X is said to be a **generalized kinematic space**.

(G1) For each $x \in p \in P$ there exists such $q \in Q$ that $x \in q$.

(G2) For each $x_1 \neq x_2 \in X$ there exists such pass $p \in P$ that $x_1, x_2 \in P$ and the set of lengths of such p is bounded with a positive number below; this infimum is said to be the **generalized kinematical distance** ρ_X between points x_1 and x_2.

(G3) For each $q_1 \neq q_2 \in Q$ there exists such pass $p \in P$ that $q_1, q_2 \in P$ and the set of lengths of such p is bounded with a positive number below; this infimum is said to be the **generalized kinematical distance** ρ_X between things p_1 and p_2.

(G4) If $x_1, x_2 \in p_1$ and $x_2, x_3 \in p_2$ then there exists such pass $p_3 \in P$ that $x_1, x_2, x_3 \in p_3$ and $length(p_3) \leq length(p_1) + length(p_2)$.

If (G5) For each $x_1 \neq x_2 \in X$ there exists such pass $p_{12} \in P$ that $length(p_{12}) = \rho_X (x_1, x_2)$ then the generalized kinematical space X is said to be **flat** (with respect to P).

3. Visibility over Riemann surface

Without loss of generality, let $a= -1, b=1, r<1$, and the equation is

$w = \sqrt{(z^2-1)}, w= \sqrt{((z-1)(z+1))}.$

The Riemann surface for this equation has two branch points: $A=(-1;0), B=(1;0).$

Let the speed of Person be 1. “Ray (P)Q” means a ray going from Q in the direction opposite to P. Denote the circle of radius r with center at point B as OB.

This space can be constructed as follows. Let's take two planes (assume that we are on the first of them, at point M, located at a distance of no less than r from point B, in the quadrant IV).

The plane on which Person is located will be called "this sheet," and the half of the sheet (quadrants I and II or quadrants III and IV) on which Person is located will be called "this half." Accordingly, we will say "that sheet" and "that half".

Let's make a cut on each sheet along the segment AB and glue the near side of this sheet with the far side of that sheet, and also glue the near side of that sheet with the far side of this sheet.

If M does not lie on the OX axis, then we will call the far region that part of that half of this sheet, bounded by ray $(M)A$, segment AB and ray $(M)B$, which Person does not see.

So, from each point Person can see the entire half of this sheet, that half of this sheet without the far region, and the far region of that sheet.

From a point on the OX axis inside the segment AB, half of one sheet and the other half of another sheet are visible. From a point on the OX axis outside the segment AB, this entire sheet is visible and the other one is not visible.

If point M is not on the OX axis $(M \notin OX)$, then the solution to the problem is given by either point M and the corresponding point on another sheet, or two points on the OX axis, and the angle of movement from one to the other relative to one point B should be equal to 2π.

We get the following trajectories: $(M \in OX)$

T1) From the point M (as the first point) along a tangent to OB, around OB and to the same segment on OX where M is located (the new point N will be on another sheet).

$(M \notin OX)$

T2) From point M (as the first point) along a tangent to OB, around OB and along a tangent back to the second point M', corresponding to M (it will be on another sheet).

T3) From point M along a tangent to OB, to the intersection with the OX axis (this will be the first point K), then along the tangent until OB touches, around OB to the point N, lying on the same section of the OX axis (this will be the second point).

We obtain the following formulas. Let the point $M=(x,y)$, P and Q (if moving counterclockwise) be the points of tangency of the tangents drawn from M to OB,

The distances $MB=d=\sqrt{(x-1)^2+y^2}$, $MP=MQ=h=\sqrt{d^2-r^2}$, $t=\cos^{-1}(r/d)$. The lengths of the trajectories: $T_1=MP+arc\ PN=h+r(2\pi-t)$; $T_2=MP+arc\ PQ+QM=2h+r(2\pi-2t)$; $T_3=MP+arc\ PN$.

Denote $x_1=|x-1|$. (The distance between Person and the line $x=1$, on the which point B lies).

If $x_1<r$, then both points of tangency are closer to the M than the OX axis, otherwise one of them is closer and the other one is farther.

By symmetry, assume that $x\geq 1$, then we need to move to the point P. Denote $s=\angle MBX=\cos^{-1}(x_1/d)$, then $\angle PBX=t-s$. $T_3=h+r(2\pi-(t-s))$.

T h e o r e m 1. If the starting point lies on the OX axis, then the minimum time to see the object in any case is $h+r(2\pi-t)$, otherwise it is equal to
$\min\{2h+r(2\pi-2t), h+r(2\pi-(t-s))\}$.

4. Definitions and tasks on visibility

D e f i n i t i o n 3. If there exists an open set G in a kinematical 2D-space S isometric to an open set G' in $\boldsymbol{R}^2$ and a segment $A'B'\subset G'$ then for the preimages the point B is seen from the point A.

T a s k 1. Initial position of Person is given. What is the minimal time to inspect all the space G?

T a s k 2. What is the minimal time to see the given point?

T a s k 3. What is the minimal time to see all boundary of a given set in S?

We hope that implementation of these tasks would yield new properties of kinematical spaces.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 7, 2023

Revised: December 4, 2023

Accepted: January 15, 2024

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MSC 34A26, 91F20

INTERACTIVE MATHEMATICAL TASKS WITHOUT WORDS

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In frames of developing general methods for independent interactive computer presentation of various objects, interactive mathematical tasks without any conventional denotations (words, digits) are classified and various examples are proposed. Especially, such tasks are intended for children. A philosophical problem is put: what notions can be presented in such a way?
Keywords: independent learning, mathematical task, interactive task, computer contest

Компьютерде көз карандысыз интерактивдик чагылдыруу үчүн жалпы усулдугун өнүктүрүүнүн ичинде, шарттуу белгилерди (сөздөр, сандар) колдонбостон интерактивдүү математикалык маселелер классификацияланды жана ар түрдүү маселелер сунушталды. Атап айтканда, мындай маселелер балдар үчүн арналган. Коюлган философиялык көйгөй: дагы кандай түшүнүктөрдү ушундай жол менен көрсөтүүгө болот?
Урунттуу сөздөр: көз карандысыз үйрөнүү, математикалык маселе, интерактивдүү маселе, компьютердик мелдеш

В рамках разработки общей методики для независимого интерактивного компьютерного представления различных объектов, классифицированы интерактивные математические задачи без использования условных обозначений (слов, цифр) и предложены различные примеры. В особенности, такие задачи предназначены для детей. Поставлена философская проблема: какие понятия можно представить таким образом?
Ключевые слова: независимое обучение, математическая задача, интерактивная задача, компьютерный конкурс

1. Introduction

Certainly, there are many differences in language, terminology and traditions in teaching worldwide. There are two ways: to elaborate any unified world-wide system of conventions (which existed in Europe on the basis of Latin language in the Middle ages, exists on the base of Arabian language in some countries and is built on the basis of English now) or to develop less conventional teaching methods (it is the subject of this paper).

Remark. There are also elaborated many international denotations. Ancient are Yin-Yang, Lion or Eagle (power in all meanings), schematic human shapes, Red color (prohibited, dangerous …), White color (peace), Snake with something (medicine), Anker (related to maritime affairs). Further, Wings (related to air), Red Cross and Crescent (medicine), Shot glass (fragile), Green (related to ecology) appeared. There are Oblique segment (prohibited) and other road signs, numerous computer icons, pictograms. For example, the cheerful smiley and the sad smiley can be used in programming of tasks in Section 3.

Thus, while learning the essence of any subject one has to get to know the system of symbols (notations) and special terms traditionally used in it. This causes the following disadvantages:

- pupils and even some teachers confuse the content of a subject with its form, they do not recognize real life applications of its items and cannot apply their skills and knowledge;
- many persons of good ability are frightened of such a system and do not try to learn at all;
- if the state (official) language of teaching or symbols used are well known by any pupils, then those pupils have a great advantage over others.

Therefore, we suggested developing Independent Learning [1]: ways of teaching and assessment which make minimal use of any media system not related to the content of the subject. The purpose of this paper is distinguishing of tasks without any conventional signs. We list well-known tasks (in our interpretation), ones published earlier by us and some new ones.

2. Classification of tasks

We will consider "Drag-and-Drop" tasks only. The user is to choose one of objects from "List of movable objects" and move it to the spot.

2.1. "Detect analogous, similar one." If such relation is transitive then a new notion arises as an element of a quotient space of objects.

2.2. "Continue the sequence".

2.3. "Complete puzzle".

3. Examples of tasks

We will use the only conventional denotation: "light green" as a purpose, a spot for purpose (we will denote such spot as (?)).

Firstly, solutions for tasks will be evident; further, the user will know this deno-tation. Begin with tasks for children.

Three versions to input natural numbers:

heap with “ones”; the user moves one-by-one from the heap to (?);

heap with “ones” and “twos” and “fives” …;

list of numbers including the right answer.

Task 1. Natural numbers. Target: 3..7 similar objects (^^^^^ (?))

List: ***   **** ***** ******

Task 2. Sequences. Traditional (2 7 12 17 (?)), (1 3 9 27 (?)) uses conventional signs (digits). We propose *** ***** ******* (?)

The following tasks use one or two scales in balance of in tilt (***=***) and natural numbers as sums of weights on scales. Firstly, the green (right) cup of the scales is up. When the scales are in balance, an approving beep sounds.

Task 3. Introducing to scales. *****=(?)

Task 4. Subtraction or solving the equation $a=b+x$: ******=****(?)

Task 5. Multiplication. Cat (or Apple) =***; Cat,Cat=(?)

Task 6. Division. Cat, Cat =********; Cat =(?)

Task 7. Linear equation. Cat*** =********; Cat =(?)

Cat,Cat* =*********; Cat =(?)

Task 8. Length. Purpose is a slightly broken line.

List: various segments, the length of one of them is close to the length of the purpose.

Task 9. Area. Purpose is a convex polyhedron or an oval.

List: various squares, the area of one of them is close to the area of the purpose.

Task 10. “Simplest, basic figures”. Circles, squares, rectangles, rhombs …; one of them (or some of them) is (are) without piece(s).

List: piece(s). The user is to “complete” the figure(s) with the piece(s).

Task 11. Geometrical transformations (rotation, homothety, stretching …). Example: Purpose: A (?)

List: P, Q, A, T or P, ∀, ∃, T

Task 12. Symmetry-1. Example: Purpose: T (?)

List: N, H, Z, E

Task 13. Symmetry-2. Examples: Purposes

WTFkTWTFkTWTFkTWTFT(?)TWTFkT (transitive symmetry)

WTH||TWT||H(?)W (mirror symmetry)

IHHX##IN(?)N
IOI
NONI##XHHI (rotational symmetry)

List: in these examples: shapes of letters.

Certainly, such tasks must be made of geometrical figures.

Task 14. Labyrinth. There are movable object (Avatar), Purpose and Labyrinth among them.

Task 15. Mechanics. Asymmetrical scales.

Example:(******)-3″-(=)-6″-(?).

Remarks. We took some ideas from [2]. Authors [2] asked children how did they invent such answers. But we advise not to ask because it is non-verbal way of thinking. On the contrary, let both computer and you as a teacher say after proper answers: *Congratulate! You have mastered the notion … !*

4. Conclusion

We hope that such contests including on-line ones would be interesting and useful for children and pupils. Also, invention and implementation of such tasks would be useful for IT-students and put philosophical problems: what notions can be presented in such a way? How to involve AI to solving such tasks?

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 13, 2023
Revised: December 5, 2023
Accepted: January 15, 2024

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MSC 34A26, 91F20

COMPLEX OF MATHEMATICAL MODELS FOR NOTIONS OF NATURAL LANGUAGES

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In previous works, in frames of developing methods for independent interactive computer presentation of natural languages, mathematical models for some notions were elaborated. They are based on “sequences of shifts”, conditions on sets and functional relations. Using of such models in complex is proposed in the paper. Examples in Kyrgyz and English languages are given.
Keywords: language, computer presentation, mathematical model, independent presentation

Мурунку эмгектерде, табигый тилдерди компьютерде көз карандысыз интерактивдик чагылдыруу үчүн жалпы усулдугун өнүктүрүүнүн ичинде кээ бир түшүнүк үчүн математикалык моделдер курулду. Алар «жылдыруулардын удаалаштылыктары», топтомдор үчүн шарттар жана функционалдык өз ара байланыштар аркылуу курулду. Макалада мындай моделдерди комплексте колдонуу сунуш кылынат. Кыргызча жана англисче мисалдар берилген.
Урунттуу сөздөр: *тил, компьютердик чагылдыруу, математикалык модель, көз карандысыз чагылдыруу*

В предыдущих работах, в рамках разработки общей методики независимого интерактивного компьютерного представления естественных языков были построены математические модели для некоторых понятий. Они основаны на «последовательностях сдвигов», условиях на множества и функциональных соотношениях. В статье предложено комплексное использование таких моделей. Даны примеры на кыргызском и английском языках.
Ключевые слова: *язык, компьютерное представление, математическая модель, независимое представление*

1. Introduction

At all times, travelers were forced to learn languages from inhabitants which did not know the traveller’s native language. As it is known, M.Berlitz (1878) was the first to develop such a method, “the immersive teaching method”. Further such method was improved by J.Asher (1965) as “Total physical response”.

Winograd T. (1972) proposed giving commands to a robot with such words as "table", "box", "block", "pyramid", "ball", "grasp", "moveto", "ungrasp".

Using these ideas, the following framework definition was proposed [2]: If a computer presentation does not depend on the user's knowledge and skills on similar objects then it is *independent*. Such presentations are more effective because the user can learn a language inductively, and they begin thinking in it, without translation in mind.

General methods for interactive computer presentations of natural languages were being developed [1-7] and mathematical models for some notions were elaborated. They are based on "sequences of shifts", conditions on sets and functional relations.

Using of such models in complex is proposed in the paper. Examples in Kyrgyz and English languages are given.

2. Definitions for independent presentation of notions

Definition 1. Let any *notion* (word of a language) be given. If an algorithm acting at a computer: generates (randomly) a sufficiently large amount of instances covering all essential aspects of the *notion* to the user, gives a command involving this notion in each situation, perceives the user's actions and performs their results clearly on a display, detects whether a result fits the command, then such algorithm is said to be a *computer interactive presentation* of the *notion.*

Certainly, commands are to contain other words too. But these words must not give any definitions or explanations of the notion.

Definition 2. If low energetic outer influences can cause sufficiently various reactions and changing of the inner state of the object (by means of inner energy of the object or of outer energy entering into object besides of such influences) at any time then such (permanently unstable) object is an *affectable object*, or a *subject*, and such outer influences are *commands.* A system of commands such that any subject can achieve desired efficiently various consequences from other one is a *language*.

The following definitions [5-7] are improved by us.

Definition 3. Mathematical models of 1-$(n+m)$-complexity consist of fixed $\{F_i|i=1..n\}$ and movable $\{M_j|j=1..m\}$ sets with capacity of controlled parallel shifts and temporal sequence of conditions of types $(M_j \subset F_i)$, $(M_j \cap F_i = \varnothing)$, $(M_j \cap F_i \neq \varnothing)$.

Mathematical models of 2-complexity also include capacity of controlled rotations of movable sets.

Definition 4. A transformable object is a (varying) set or union of sets, some points of it are connected with *functional relations*.

Let $S\subset D(display) \subset R^2$ be initial position of the object; $S(t)$ be the position and shape of the object at time t.

Let S and $S(t)$ be defined by points $z_1, z_2, \ldots, z_k \in S$ and their images correspondingly. Then $S(t)$ is completely defined as follows: $S_k(t, Z_1(t), Z_2(t), \ldots, Z_k(t))$. Values $z_1, z_2, \ldots, z_k$ are to be connected with functional relations.

Definition 5. The complexity of a mathematical model consists of: the maximal value of k of all movable and transformable objects, of n and m, as in Definition 3.

Computer interactive presentations are built on the base of mathematical models.

Hypothesis. Any notion has a minimalistic mathematical model (a version of minimal number of *entities* in Occam's sense).

3. Examples of mathematical models

Example 3.1. The verb ТАП-FIND. The environment is: 5..7 lids and the box. There is the (hidden) apple under one of the lids.

The mathematical models of a lid (movable), of a (hidden) apple (movable) and of a box (fixed) are involved.

The verb can be present either as a secondary one:

Буйрук: «Капкактарды жылдырып, алманы ТААП, аны үкөккө кой!»

Order: "Move the lids, FIND the apple and put it into the box!"

(primary notions are underlined)

or as a goal-setting verb:

Буйрук: «Алманы ТААП, аны үкөккө кой!»

Order: "FIND the apple and put it into the box!"

(the user is to guess what actions must be done).

The complexity is 1-(8+1).

Example 3.2. The verb БҮКТӨ-FOLD. The environment is: the unrolled kinematical chain of three links and the (little) box.

The mathematical models of a kinematical chain (transformable) and of a box (fixed) are involved.

As there is no common word for such an object, we will call it a “thing”.

Буйрук: «Нерсени БҮКТӨП, аны үкөккө кой!»

Order: “FOLD the thing and put it into the box!”

The complexity is 3-(1+1).

Example 3.3. The verb СИММЕТРИЯЛА-SYMMETRIZE.

Буйрук: «СИММЕТРИЯЛА!»

Order: “SYMMETRIZE!”

Version a). The environment is: an “almost symmetrical” thing and the piece.

The user is to move the piece to make the thing to be symmetrical.

■■■ ■∩ *■■■* ■	*∩■∩ * ■∩■■■■∪■ ∪■∪

The mathematical models of a fixed object and of a movable object are involved.

The complexity is 1-(1+1).

Version b). The environment is a symmetrical thing and two segments connected to the thing in symmetrical points but arranged not symmetrically.

■■■ ∩∩∩ \■■■_	_∩■∩ ■∩■■■■∪■ ∪■∪\

The mathematical model of a transformable object is involved.

The user is to turn one of the segments to make the image to be symmetrical.

The complexity is 2-(2+1).

4. Conclusion

We hope that such construction of software would improve existing softs on independent presentations of foundations of languages (Kyrgyz, English) and the complex examination of Kyrgyz language.

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The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic

Received: November 20, 2023

Revised: December 15, 2023

Accepted: January 15, 2024

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CONTENTS

Geometry and Topology

1. A. A. Borubaev, G. O. Namazova, M.K. Chamashev. Generalization of the main principles of functional analysis 3

bdykaimov I.Z. Some properties of the mappings, preserving the index of completeness towards the image 14

.M. Asanbekov. On quasi-identities of certain finite modular lattice 24

4. Kanetov B.E., Kanetova D.E., Altybaev N.I. u-Uniformly čech complete spaces 32

anetova D.E. Index of sequential completeness of uniform spaces and uniformly continuous mappings 40

Differential Equations, Dynamical System and Optimal Control

. Abiev. On dynamics of the normalized ricci flow on the wallach spaces 50

7. Asanov A., Asanova K.A. Boundedness of solutions of a class of linear differential equations of the second order on the semiaxis 58

8. Asanov A., Kadenova Z.A., Koshmurzaev I.B. One class of fredholm linear integral equations of the third kind on a semi-axis 64

9. Asanov A., Asanov R. A. On a class of nonlinear fredholm integral equations of the third kind on the segment 70

10. Baizakov A.B., Bekturova A.T. On reducing the analytical system of volterra integral equations with a regular singular point to a linear form 76

skandarov S. On the method of studying the asymptotic stability of solutions of a linear volterra integro-differential equation of the seventh order on the semi-axis 84

agaeva S.B. Hexagonal regularization phenomenon for system of differential equations describing repelling points on a square 90

heentaeva Zh. K. Non-oscillating solutions of vector-matrix dynamical systems with dominance 96

Mathematical and instrumental methods of economics

iibosunov B.I., Choroev K., Kydyrmaeva S.S. Nonlinear model of uneven economic growth of regions of the Kyrgyz Republic 104

15. Dzhumabaev K., Toktorov K. K., Jumabaev A. K. Issues of natural resource management and agricultural forecasting in the Kyrgyz Republic 111

16. Nazarbaev F.T., Bakberdieva M.B. Examples of the application of parabolic interpolation of mortality tables for fractional ages 126

Mathematical Modelling

17.Bekeshev D.D., Pankov P.S. Mathematical model for texture presentation of colors 131

enenbaeva G.M. Basic complexes in mathematical categories, effects and phenomena 134

Mathematics education

ankov P.S. Computer programs for mathematical competitions and examinations 142

20.Pankov P.S., Zhoraev A.H. Controlled motion and visibility of stretched objects in virtual spaces 148

ankov P., Bayalieva E. Interactive mathematical tasks without words 155

22.Pankov P., Karagulov A., Baktybekov A. Complex of mathematical models for notions of natural languages 161

ISSN 1694-8173

9 771694 817007