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Dedicated to Academician A. A. Borubaev on occasion of his 75th birthday

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BORUBAEV Altai Asylkanovich

A cademician, doctor of physical and mathematical sciences, professor

Dedicated to Academician A. A. Borubaev on occasion of his 75th birthday

Borubaev Altai Asylkanovich is Academician of National Academy of Sciences of Kyrgyz Republic, prominent statesman and public figure, Laureate of the Interstate Award “Stars of the Commonwealth” of CIS, two-time winner of the State Prize of the Kyrgyz Republic in Science and Technology, Honored Worker of science of Kyrgyz Republic, State Councilor of the Kyrgyz Republic of the 1st Class, Cavalier of the Order of "Manas" of the 3rd Class and of the Order of the "Commonwealth" of the CIS Interparliamentary Assembly, the director of Institute of Mathematics of National Academy of Sciences of Kyrgyz Republic.

Altai Borubaev is a Honored Worker of Science of the Kyrgyz Republic, two-time winner of the State Prize in Science and Technology, Doctor of Physical and Mathematical Sciences, Professor, the author of more than 150 scientific and methodological works, Honorary Professor of Moscow State University named after Lomonosov, President of the Mathematical Society of the Turkic world. Academician of National Academy of Sciences of the Kyrgyz Republic Borubaev Altai Asylkanovich is an outstanding scientist of Kyrgyzstan with a worldwide reputation.

His activities are aimed at the development of science and education of the Republic. After graduation from secondary school with a medal, A.A. Borubaev enrolled in the Mechanical- Mathematical Faculty of the Kyrgyz State University and graduated with honors in 1972. Subsequently, A.A. Borubaev pursued postgraduate and doctoral studies at the Lomonosov Moscow State University, where he was the first in Central Asia to successfully defend his Candidate (1977) and Doctoral (1991) dissertations in "Geometry and Topology." In 1981, the young scientist A.A. Borubaev was awarded the Lenin Komsomol Prize of Kyrgyzstan for his scientific works in topology.

In 1982-1984, young specialist A.A. Borubaev successfully completed a scientific internship at Charles University in Prague, Czechoslovakia, one of the world's oldest universities. Beginning his career at Kyrgyz State University in 1975, A.A. Borubaev advanced from lecturer to rector, got academic rank of professor in 1992 and was elected a corresponding member of National Academy of Sciences of the Kyrgyz Republic in 1993.

In 1992, by decree of the President of the Kyrgyz Republic, A.A. Borubaev was appointed First Deputy Minister of Education of the Kyrgyz Republic. On his initiative and under his leadership, the first law of sovereign Kyrgyzstan "On Education" was developed and adopted. New educational and scientific standards were introduced in the field of higher education.

In 1994, A.A. Borubaev was elected the rector of the I. Arabaev Kyrgyz State Pedagogical University. On his initiative, an educational, scientific, and production complex was established at this institution. It included a research institute in pedagogy, a pedagogical college, and dissertation councils. As a result, the Kyrgyz State Pedagogical University became one of the leading universities in the republic and a center for pedagogical research.

By decree of the President of the Kyrgyz Republic in 1995, he was awarded the title of "Honored Worker of science of the Kyrgyz Republic."

In 1998, A.A. Borubaev was elected the rector of the J. Balasagyn Kyrgyz National University. He secured salary increases for faculty, primarily through internal reserves. Under his leadership, the development of scientific and educational projects was strengthened, and international grants were won. Cooperation agreements were signed with leading universities abroad, including Lomonosov Moscow State University, the University of Marseille (France), the

University of Munich (Germany), the University of Turin (Italy), and Peking University (China).

Academician A. A. Borubaev succeeded in solving mathematical problems that had remained unsolvable for 40-50 years. He solved the problems of extending topological spaces posed by the American Mathematicians B. Banashevsky and M. Stone; the problem of embedding uniform spaces into Euclidean spaces posed by the Russian mathematician N. V. Smirnov, and the problem of constructing an absolute of uniform spaces posed by the Russian mathematician V. I. Ponomarev.

In 1998, A. A. Borubaev was awarded the State Prize of the Kyrgyz Republic in Science and Technology for a series of scientific works.

In 2000, at the suggestion of scientists, the Moscow International Observatory named a star after Academician A.A. Borubaev, "Altai," for outstanding scientific achievements. That same year, Academician A.A. Borubaev was awarded the rank of Honorary Professor of Lomonosov Moscow State University.

In 2000, A.A. Borubaev was unanimously elected as an Academician of the National Academy of Sciences of the Kyrgyz Republic.

In 2000, he was elected for the second time to the Assembly of People's Representatives of the Jogorku Kenesh of the Kyrgyz Republic (the first time was in 1998). The deputies elected him the chairperson of the Assembly of People's Representatives of the Jogorku Kenesh of the Kyrgyz Republic.

Under his leadership, the Assembly of People's Representatives adopted important laws and resolutions. Under A.A. Borubaev's personal leadership, the laws of the Kyrgyz Republic "On the National Academy of Sciences of the Kyrgyz Republic" and "On the Education of the Kyrgyz Republic" were drafted and adopted.

In 2001, A.A. Borubaev was awarded the Order of "Commonwealth" of the CIS Interparliamentary Assembly.

In 2002, A.A. Borubaev was awarded the State Prize of the Kyrgyz Republic in Science and Technology for the second time. That same year, the US Biographical Institute recognized A.A. Borubaev as "Person of the Year."

In 2003, he was awarded the gold medal of Charles University (Prague, Czech Republic). That same year, by decree of the President of the Russian Federation, A.A. Borubaev was awarded the jubilee medal "300th Anniversary of the City of St. Petersburg."

In 2005, by decree of the President of the Kyrgyz Republic, A.A. Borubaev was appointed Chairman of the National Attestation Commission of the Kyrgyz Republic. Under his leadership, new regulations for the awarding of academic degrees, academic titles, and the operation of dissertation councils were adopted. On his initiative, a highly effective 'Antiplagiat' (Anti-plagiarism) program was launched. The performance of the National Attestation Commission of the Kyrgyz Republic has significantly improved. The quality of awarding academic degrees and ranks has been elevated to a high level.

In 2005, the Mongolian Academy of Sciences awarded him an honorary doctorate degree.

In 2013, Academician A.A. Borubaev was elected the Vice-President of the National Academy of Sciences of the Kyrgyz Republic. From 2017 to the present, he has been the Director of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic. During this period, he has made a significant contribution to the development of both the National Academy of Sciences and the Institute of Mathematics. In 2024, he was awarded the jubilee medal “70 Years of the National Academy of Sciences of the Kyrgyz Republic”.

On the initiative of Academician A.A. Borubaev, the scientific journal “Reports of the National Academy of Sciences of the Kyrgyz Republic” was founded and has been published since 2014. Under his leadership, the first English-language scientific journal in Kyrgyzstan, “Herald of the Institute of Mathematics of the NAS KR", has been published regularly since 2018. In 2013, Academician A.A. Borubaev was awarded the "Socrates" Order by the International Socrates Committee for his outstanding contribution to world science. By the decision of the Committee, his name was included in the World Register of Outstanding Scientists of the 21st Century (London, UK). In the same year, he was awarded the Order of Peter the Great (Russia).

In 2017, A.A. Borubaev received the Order of the "Commonwealth" of the CIS Interparliamentary Assembly, and in 2020, he was awarded the Gold Medal of the World Intellectual Property Organization.

In 2021, Academician A.A. Borubaev made the first scientific discovery in mathematics in Central Asia, which was officially recognized by the "International Academy of Authors of Scientific Discoveries and Inventions" and the global scientific community.

Academician A.A. Borubaev founded the first scientific school of topology and geometry in Central Asia. He has mentored 8 Doctors of Sciences and more than 15 Candidates of Sciences; two of his disciples have been elected Corresponding Members of the National Academy of Sciences of the Kyrgyz Republic. His disciples, in turn, have begun producing their own Doctors of Sciences and have trained numerous Candidates of Sciences.

By decree of the President of the Kyrgyz Republic, A.A. Borubaev was awarded the Order of Manas, 3rd Class.In 2022, he was awarded the Jubilee Medal of the CIS Interparliamentary Assembly. In 2023, he was granted the rank of Honorary Professor of Qingdao University (China).

By decree of the President of the Kyrgyz Republic in 2023, he was awarded the jubilee medal "30 Years of the Constitution of the Kyrgyz Republic". In 2025, he was awarded the Jubilee Badge commemorating the 100th anniversary of the Kara-Kyrgyz Autonomous Oblast. Academician A. A. Borubaev is the author of 16 scientific monographs and books, and more than 130 scientific articles published in 15 countries. He has presented scientific papers at universities in 39 countries, including Lomonosov Moscow State University, Charles University (Prague), Sorbonne (France), Harvard (USA), Madrid (Spain), Pisa (Italy), Qingdao (China), Budapest (Hungary), Oxford (United Kingdom), Sofia (Bulgaria), Niš (Serbia), Bangalore (India), Sakarya (Turkey), and others. Under his leadership, more than 20 international scientific congresses and conferences have been held in the Republic.

Academician A.A. Borubaev is the President of the Union of Scientists of Kyrgyzstan, a member of the Presidium of the National Academy of Sciences of the Kyrgyz Republic, and the head of the Kyrgyz Mathematical Society. He initiated the creation of the Mathematical Society of the Turkic World, which encompasses 14 countries. He is the Honorary President of the Mathematical Society of the Turkic World, Chairman of the Kyrgyz-Qatari Friendship Society, and Chairman of the Dissertation Council for the Defense of Doctoral and Candidate's Dissertations in Mathematics. He is also a member of the editorial boards of several international scientific journals in mathematics.

In 2022, Academician A.A. Borubaev made a scientific discovery called "The permutability property of operations of the absolute, replenishment and expansion of a single space", confirmed by the International Academy of Authors of Scientific Discoveries and Inventions (09/23/2022, Diploma No. 74-C).). For a discovery in the field of mathematics, Academician A.A. Borubaev was awarded the Silver Medal of Honor. P. Kapitsa (Resolution of the Presidium of the Russian Academy of Natural Sciences No. 316 of 09/06/2022).The essence of scientific discovery is as follows:1) the previously unknown basic properties of the commutativity of the operation of the absolute and the most important extensions of the uniform space are established.

2) important organic connections are found between the key concepts of the theory of homogeneous and topological spaces.

MSC 05C20,05C38, 05C99, 54A20, 54C10, 54D99

CERTAIN TOPOLOGIES ON SIMPLE GRAPHS

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Dedicated to Academician A. A. Borubaev on occasion of his 75th birthday

The paper is an overview of some recently results on relationships between graph theory and topology. In particular, it contains results by the authors obtained in the last two-three about four topologies (the upper approximated G-topology, P-topology, cyclic topology, and extreme graphical topology) on the set of vertices of a simple undirected locally finite graph.

Keywords: simple graph, upper approximated topology, P-topology, cyclic topol- ogy, extreme topology.

Бул макалада графтар теориясы менен топологиянын ортосундагы байланышка байланыштуу акыркы жыйынтыктардын айрымдары каралат. Тактап айтканда, анда авторлор тарабынан акыркы эки-үч жыл ичинде жөнөкөй багытталбаган локалдык чектүү графтын чокулар топтомундагы төрт топология (жогорку жакындаштыруучу G-топологиясы, P-топологиясы, циклдик топология жана экстремалдык граф топологиясы) боюнча алынган жыйынтыктар келтирилген.

Урунттуу сөздөр: жөнөкөй граф, жогорку жакындаштыруучу топология, P-топологиясы, циклдик топология, экстремалдык топология.

Данная статья представляет собой обзор некоторых недавних результатов, касающихся взаимосвязи между теорией графов и топологией. В частности, она содержит результаты, полученные авторами за последние два-три года, о четырех топологиях (верхнеаппроксимированная G-топология, P-топология, циклическая топология и экстремальная графическая топология) на множестве вершин простого неориентированного локально конечного графа.

Ключевые слова: простой граф, верхне аппроксимированная топология, P-топология, циклическая топология, экстремальная топология.

1. Introduction

In the last few years several researchers introduced and investigated some classes of topologies on the vertex sets or the edge set of simple graphs. Some of the- ses topologies depend on the used neighborhood systems. In [6], Amiri et al. presented a graphical topological space on the vertex set of a graph whose topol- ogy is generated by the set of neighbourhoods of vertices in the graph. Sari and

Kopuzlu [36] used the ideas from the above mentioned paper to create new topologies on the vertex set of a simple undirected graph. Canoy and Nianga [30, 31] created certain topologies on the set of vertices via hop neighbourhoods. They also introduced a class of topologies via some operations. In [14, 15], Faten et al. used monophonic paths, the monophonic eccentric system, and the upper approximation system to define several topologies in graph theory. The notion of pathless directed topological space on the set of vertices of directed graphs was introduced and studied in [33, 34]. The paper [26] introduced the topologies, called compatible topologies, on the set of edges of a directed graph. Othman et al. [33] introduced the concept of pathless fields on the vertices set and pre- sented some applications on the human heart, and in [5] applications of graphical topologies in radar chart methods have been presented. Othman et al. [35] used open neighbourhood C-sets system to introduce class of topologies, called L2-topologies. Abu-Gdairi et al. [1], by using neighbourhood systems, introduced and investigated some topologies on graphs. Gamorez and Canoy [21] used the m-eccentric system to define some topologies on graphs. For the extreme neighborhood system, Ganesamoorthy and Jayanthi [23] defined the concept of extreme outer connected geodesic graphs in graph theory. Timmanaikar et. al. [41] introduced numerical representation method by using some topological indices and graphical properties. Atik et al. [9] introduced some rough approximation systems. Guler [24] used the notion of ideal collection to present and compare some approximations. Damag et al. [16] used monophonic paths and rough approximation system to construct the notion of rough directed topo- logical space and studied some its applications on the human circulatory and nervous systems by using this notion. In mathematical chemistry, Ahmed et. al.

[2] studied some topological properties on the corresponding graphs. Damag et al. [15] defined some topologies by using the monophonic eccentric system and employed this concept in studying the connectedness property for the diffusion and a form of the COVID-19. We advice the reader to consult also the papers [7, 8, 10, 12, 13, 17, 22, 25, 27, 28, 29, 37, 38, 43, 44] for more details about graphical topologies and their applications.

In this paper we give an overview of some recently obtained result by the authors The paper provides certain topologies on the set of vertices of a simple undirected graph: upper approximated G-topology, pathing topology, cyclic topology, and extreme graphical topology. Sections 3, 4, 5, 6 are written on the basis of the papers [5, 3, 4, 32], respectively.

2. Preliminaries

In this section we give necessary information from graph theory which will be used in the sequel. For undefined notions in graph theory, we refer the reader to [11, 18]. All graphs in this paper are assumed to be undirected and locally finite. Topological terminology follows the books [19, 20].

A graph G is a pair $(V(G), E(G))$ of a vertex set $V(G) \neq \emptyset$ and edge set $E(G) \neq \emptyset$. If the vertices $x, y \in V(G)$ are joined by an edge $\varsigma \in E(G)$ we say that x and y are *adjacent*, and write $V(\varsigma) = \{x, y\}$. The edge ς with $V(\varsigma) = \{x\}$ is called a loop. If $V(\varsigma) = V(\varsigma')$, then ς and ς' are called *multiple edges*. If a graph G is without loops and multiple edges, then it is called a *simple graph*. Gxy denotes the edge joining x and y. $\partial(x)$ denotes the *degree of* x that is defined as the number of points join x. A graph G with finite sets $V(G)$ and $E(G)$ is called finite, and if $\partial(x)$ is finite for all $x \in V(G)$, G is called locally finite.

A complete graph K_n, $n > 0$, (a simple graph with n vertices, where every pair of distinct vertices is connected by a unique edge), a complete bipartite graph $K_{n,m}$, n, $m > 0$, (a simple graph with two disjoint subsets of vertices, V_1 and V_2, containing n and m vertices, respectively, in which no edge has both end points in the same subset, and every vertex in V_1 is connected to every vertex in V_2), a cycle graph C_n, $n > 2$, (a simple graph with n vertices and n edges, where every vertex has degree 2) are fundamental concepts in graph theory and will be considered in this paper. A *friendship graph* F_n with $n > 1$ is a simple graph which can be constructed by joining n copies of the cycle graphs C_3 with a common vertex.

The open neighbourhood N_x of a vertex x is the set of vertices that adjacent with x. A *path* is a sequence of vertices connected by edges, with each vertex adjacent to the next in the sequence. A *connected graph* is a graph where there is a path between every pair of vertices. For any two vertices
$x, y \in V(G)$, $d(x, y)$ denotes the distance between points x and y which is the length of a shortest path between x and y in G. Any path of length $d(x, y)$ is called xy - *geodesic*. By $C[x, y]$ we mean the closed interval which consists of all vertices lying on some xy - geodesic of G. A subset $A \subseteq V(G)$ is called an *geodetic* in G if $C[A] = V(G)$, where $[A] = \cup_{x,y \in A} C[x, y]$.
A subset $A \subseteq V(G)$ is called an *outer connected geodetic set* if A is a geodetic set in G and either $A = V(G)$ or the subgraph induced by
$V(G) / A$ is a connected graph. The *outer connected geodetic number* of G is defined as the minimum cardinality of an outer connected geodetic set of G and is denoted by G_{goc}. A vertex $x \in V(G)$ is called an *extreme vertex* [42] of a graph G if the subgraph induced by N_x is complete. For any simple graph G, the set of all

extreme vertices is denoted by $V(G)_{ex}$, and the number of extreme vertices in G is called an *extreme order* of G and denoted by $|G|_{ex}$. If $|G|_{ex} = G_{goc}$, then the graph. G is called an *extreme outer connected geodesic* (EOCG) graph For any subset M of a nonempty set D and for any binary relation η on D, the lower approximation $\eta(M)$ and upper approximation $\eta(M)$ of a set M are defined by $\underline{\eta}(M) = \{x \in D : \eta_x \subseteq M\}$ and $\overline{\eta}(M) = \{x \in D : \eta_x \cap M \neq \emptyset\}$, respectively, where $\eta_x = \{y \in D : x\eta y\}$. For any subgraph Q of G, Atik et al., [9] defined the notions of lower $\overline{\eta}$

$\overline{N}_k(V(Q))$ and upper $N_k(V(Q))$ rough approximation k-systems given by

$\underline{N_k}(V(Q)) = \{x \in V(G) : N_k(x) \subseteq V(Q)\}$,

$\overline{N}_k(V(Q)) = V(Q) \cup \{x \in V(G) : N_k(x) \cap V(Q) \neq \emptyset.\}$.

3 Upper approximated G-topology

Let G be a simple graph. We firstly structure the neighborhood system for elements of $V(G)$. By using the notions of rough approximation j-neighborhood systems (which are introduced in [9]), for any vertex $x \in V(G)$, define the upper approximation neighborhood $\bar{x}$ of x as

$\bar{x} = N_x \cup \theta_x$,

where $\theta_x = \{y \in V(G) : N_x \cap N_y \neq 0\}$. By $V^{(\overline{G})}$ we mean the upper approxi-mation neighborhood system of a graph G, that is, $V^{(\overline{G})} = \{x : x \in V(G)\}$.

Theorem 3.1 *Let G be a simple graph. The family $\overline{U}_G = \{\overline{u}_x : x \in V(G)\}$ forms a base of a topology on $V(G)$, where $\bar{u}_x$ is the intersection of all upper approximation neighborhoods containing x.*

The topology whose base is described in the previous theorem is called the *upper approximated G-topology* and denoted by $\tau_{G(u)}$. The corresponding space is denoted by $(V(G), \tau_{G(u)})$.

We now describe the upper approximated G-topologies on classical graphs.

Theorem 3.2 *The following hold;*

(1) $(V(K_n), \tau_{kn(u)})$ *is the indiscrete space for all* $n > 0$;

(2) $(V(K_{n,m}), \tau_{km(u)})$ *is the indiscrete space for all* $n, m > 0$.

(3) $(V(C_n), \tau_{Cn(u)})$ *is the discrete space for all* $n > 5$.

Let us observe that the upper approximated G - topological space $(V(C_n), \tau_C n(u))$ of C_n for $n = 1, 2, 3, 4, 5$ is indiscrete.

The following theorem characterizes discrete upper approximated G-topologies.

Theorem 3.3 *Let G be a (locally finite) simple graph. Then for all $x, y \in V(G)$, $y \not\subseteq \bar{x}$* and *$x \not\subseteq \bar{y}$ if and only if $(V(G), \tau_{G(u)})$ is discrete.*

Recall that a topological space is called an Alexandroff space if the intersec- tion of any collection of open sets is open.

Theorem 3.4 *The upper approximated G-topological space of any (locally fi- nite) simple graph is an Alexandroff space.*

Theorem 3.5 *Let* G_1 *and* G_2 *be simple graphs without isolated vertices. If* G_1 *and* G_2 *are isomorphic, then the spaces* $(V(G_1), \tau_{G_1(u)})$ *and* $(V(G_2), \tau_{G_2(u)})$ *are homeomorphic.*

Note that the converse of this theorem need not be satisfied. The spaces $(V(K_{n,m}), \tau_{K\,n,m(u)})$ and $(V(K_{n+m}), \tau_{K_{n+m}(u)})$ are homeomorphic (being indis- crete of the same cardinality), while the graphs $K_{n,m}$ and K_{n+m} are not iso- morphic (since if $x \in V_1$ and $y \in V_2$, then x and y are joined in K_{n+m}, but not in $K_{n,m}$).

Theorem 3.6 *Let G be any simple graph without isolated vertices. If the upper approximated G-topological space* $(V(G), \tau_{G(u)})$ *is a connected space, then the graph G is connected.*

Notice that the cycle graph C_n is connected, but its upper approximated G-topological space $(V(C_n), \tau_{C_n(u)})$ is disconnected since it is discrete. On the other hand, the graphs K_n are $K_{n,m}$ are connected and have connected the corresponding upper approximated G-topological spaces.

For a simple graph G define $U_3(G)$ as the subset of $V(G)$ containing all vertices x with $|\bar{u}_x \leq 3|$, where $|\bar{u}_x|$ denotes the number of elements in $\bar{u}_x$. A simple graph G is called *upper connected* if the subgraph G^{U_3} of G, induced by $U_3(G)$, is connected.

Theorem 3.7 *Let G be an upper connected graph. Then the set D of all vertices in* $U_3(G)$ *with degree greater than one is dense in* $(U_3(G), \tau_{U_3(G)})$.

4 Pathing topology

As we mentioned a *path* in a graoh is a sequence of vertices connected by edges, with each vertex adjacent to the next in the sequence; the path which starts and ends at the same vertex is called a *closed path*. A vertex x in a graph G is called a *pathing vertex* if there is no closed path containing all elements of N_x. By $V_P(G)$ we denote the set of all pathing vertices in G.

For a vertex $x \in V(G)$, the set $K_P(x)$ of all pathing vertices adjacent with x is said to be the *open pathing neighborhood* of x; the *closed pathing neighborhood* $K_P[x]$ *of x is the set* $K_P[x] = K_P(x) \cup \{x\}$.

Definition 4.1 Let G be a graph. Consider the family

$$KV_P(G) = \{K_P[x]: x \in V(G)\}$$

The *P-topological space* of G is a pair $(V(G), \tau_{KV_P(G)})$, where $\tau_{KV}P(G)$ is the topology on $V(G)$ induced by the subbase $KV_P(G)$.

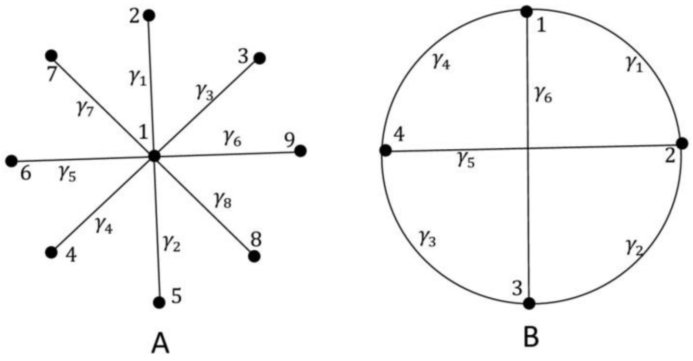

Figure 1:

Example 4.2 The graph G in Figure 1A has the discrete P-topology. The *P-topology* of the graph G in Figure 1B is indiscrete.

Theorem 4.3 *The following statements hold for the* P-*topology:*

(1) *The* P-*topology of the complete graph* K_n *is discrete for* $n = 1, 2$, *and indiscrete for all* $n > 2$;

(2) *The* P-*topology of a complete bipartite graph* $K_{n,m}$ *is discrete for* $n = m = 1$, $n = 2$ *and* $m = 1$, $n = m = 2$, *while for* $n, m \geq 2$, $n]neqm$, *I is of th3 form* $\{\emptyset, V(K_{n,m}), V_1, V_2\}$;

(3) *The* P-*topology of the cycle graph* C_n *is discrete for all* $n > 3$.

Theorem 4.4 *Let* G *be a simple graph. Then the space* $(V(G), \tau_{KV}P(G))$ *is discrete if and only if* $K_P[x] \not\subseteq K_P[y]$ *and* $K_P[y] \not\subseteq K_P[x]$ *for all* $x, y \in V(G)$.

Theorem 4.5 *Let* G *be a simple graph without isolated vertex. If* G *is a disconnected graph, then the P-topological space* $(V(G), \tau_{KV_P(G)})$ *is a disconnected space.*

It was shown in [3] that the converse of this theorem is not true in general.

5 Cyclic topology

This section is written n the basis of the paper [4] and contains some results on the C-topology on graphs.

A vertex x of a graph G is called a *C-vertex* if for each $y \in N_x$ there is a cycle subgraph C_ny of G such that $y \in V(C_ny)$. By $V_c(G)$ we denote the set of all C-vertices in G.

For a vertex $x \in V(G)$, we set $R_c(x) = \{y \in V_c(G) : y$ is adjacent with $x\}$

$$R_c[x] = R_c(x) \cup \{x\}.$$

$R_c(x)$ is called the *open C-neighborhood* of x, and $R_c[x]$ is called the *closed C-neighborhood* of x.

Definition 5.1 Let G be a graph. Define the family $RV_c(G)$ as follows:

$$RV_c(G) = \{R_c[x] : x \in V(G)\}.$$

Let $\tau_{RV}c(G)$ denote a topology on $V(G)$ generated by the subbase $RV_c(G$. The pair $(V(G), \tau_{RV_c G)})$ is called the *cyclic topological space* of G.

We begin with certain properties of cyclic topologies in important classes of graphs.

Theorem 5.2 *The cyclic topology satisfies:*

1. *The cyclic topology of the complete graphs K_1 and K_2 is discrete, and indiscrete for $n \geq 3$;*
2. *The cyclic topology of the complete bipartite graphs $K_{1,a}$, $K_{2,1}$ and $K_{2,2}$ is discrete, and for $n, m \geq 2$, $n \neq m$, it is of the form $\{\emptyset, V(K_{n,m}), V_1, V_2\}$, where $V(K_{n,m}) = V_1 \cup V_2$;*
3. *The cyclic topology of the cycle graph C_3 is indiscrete, and discrete on C_n for all $n \geq 4$.*

Theorem 5.3 *If simple graphs G_1 and G_2 without isolated vertices are iso-morphic, then the spaces $(V(G_1), \tau_{RV_c(G_1)})$ and $(V(G_2), \tau_{RV_c(G_2)})$ are homeo- morphic.*

Theorem 5.4 *If a simple graph G without isolated vertices is disconnected, then the cyclic topological space $(V(G), \tau_{RV_c(G)})$ is disconnected.*

The converse of this theorem is not true in general. In Figure 2 the simple graph G is connected, while the space $(V(G), \tau_{RV_c(G)})$ is discrete, thus discon- nected.

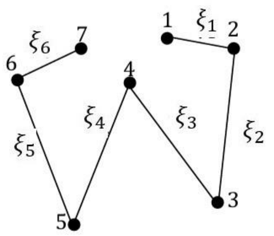

Figure 2: Disconnected space

Wa are going now to consider the cyclic topology on special class of graphs.

Let G be a simple graph. A vertex x in G adjacent to all the other vertices is called a *claw vertex*. A simple graph G is called a *claw graph* if the vertices of $V_c(G)$ are not claws and not isolated vertices.

Note that the graphic topological space $(V(G), \tau_{RV_c(G)})$ of any claw graph G is an Alexandroff space.

Theorem 5.5 *Let G be a claw graph such that $V_c(G)$ is a connected graph. Then the set*

$$D = \{x \in V_c(G) : \partial(x) > 1\}$$

is a dense set in the space $(V_c(G), \tau_{RV_c(G)})$.

6 Extreme topology

In this section we deal with extreme topology on undirected graphs. The sec- tionn is written on the basis of the paper [32].

A vertex x of a graph G is called an *extreme vertex* [42] if the subgraph induced by N_x is complete.

The set of all extreme vertices of a graph G is denoted by G_{ex}, and the number of extreme vertices in G is called an *extreme order* of G and denoted by $|G|_{ex}$.

In the class of simple graphs $G = (V(G), E(G))$, we use the extreme system and the rough approximation j-neighbourhood system to define new neighbourhood system for elements of $V(G)$, called an *extreme lower approximated neighbourhood system*. The relation ρ on $V(G)$ is defined by:

$x\rho y$ if $x \in N_y$ and x is an extreme vertex.

The *extreme neighborhood* $\rho_G^{ex}(x)$ of a vertex x in $V(G)$ is given by

$$\rho_G^{ex}(x) = \{x\} \cup \{y \in V(G): y\rho x\}$$

and

$$\rho_G^{ex}(x) = \left\{\rho_G^{ex}(x) \in V(G)\right\}$$

denotes the collection of all extreme neighborhoods of a graph G. The operator $\pi_G : V(G) \to P(V(G))$ is given by

$$\pi_G(x) = \theta_G(x) \cup \rho_G^{ex}(x),$$

where $P(V(G))$ is the power set of $V(G)$ and $\theta_G(x) = \{y \in V(G) : \rho^{ex}(x) \subset \rho^{ex}(y)\}$. The collection

$$\pi_G(V(G)) = \{\pi_G(x) : x \in V(G)\}$$

is called an *extreme lower approximation neighbourhood system* of G. For a subset $H \subseteq V(G)$ the extreme lower approximation neig hbourhood $\pi_G(H)$ of H is given by $\pi_G(H) = \theta_G(H) \cup \rho_G^{ex}(H)$, where $\theta_G(H) = \bigcup_{h\in H} \theta_G(h)$. That is, $\pi_G(H) = \bigcup_{h\in H} \pi_G(h)$.

Theorem 6.1 *In a simple graph G,* $B_G\{\pi_G^x : x \in V(G)\}$ *forms a base of*

some topology on $V(G)$, *where* π_G^x *is the intersection of all* $\pi_G(y)$ *involving* x.
[This means that the collection $\pi_G(V(G))$ *is a subbase for this topology.]*

The topology induced by B_G is denoted by τ_{B_G} and called the *extreme graph- ical topology* of G.
We are going now to describe the extreme graphical topology for some im- portant classes of graphs.
Theorem 6.2 *The extreme graphical topology satisfies the following*

(1) *The space* $(V(K_n), \tau_{B_{K_n}})$ *is indiscrete for every* $n > 0$*;*

(2) *The space* $(V(K_{n,m}), \tau_{B_{K_{n,m}}})$ *is discrete for every* $n, m > 1$*;*

(3) *The space* $(V(C_n), \tau_{B_{C_n}})$ *is discrete for all* $n > 3$, *while the space* $(V(C_n), \tau_{B_{C_n}})$ *is indiscrete;*

(4) *The space* $(V(P_K), \tau_{B_{P_m}})$ *for* $k > 4$, *where* $P_k : 1' \to 2' \to \cdots \to k'$, *has the base* $B_{P_K} = \{\{1',2'\},\{(k-1)', r\}, \{m'\}: 3 \le m \le k-2\}$*; the spaces* $(V(P_3), \tau_{B_{P_3}})$ *and* $(V(P_4), \tau_{B_{P_4}})$ *are quasi-discrete;*

(5) *The space* $(V(F_n), \tau_{B_{F_n}})$, $n > 1$ *has the base* $B_{F_n} = \{\{0\}, \{0, m', m''\} : 1 \le m \le n\}$, *where subgraphs* C_3 *in* F_n *are of the form* $\{0, m', m''\}$*;* $1 \le m \le n$*;*

(6) *The space* $(V(GF_n), \tau_{B_{GF_n}})$ *has the base* $B_{F_n} = \{\{m, m', m''\} : 1 \le m \le n\}$.

Here, GF_n denotes a generalized friendship graph: *generalized friendship graph* GF_n with $n > 3$ is a simple graph which can be constructed by attaching n copies of the cycle graphs C_3 to a cycle graph C_n.

Let us mention that the results obtained in this article ([32]) are applied to two chemical compounds introduced by Tareq et al. in[39, 40]. The graphs which represents the X-ray structures of these compounds both have extreme graphical topologies which are disconnected and non-discrete.

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A NEW APPROACH TO UNIVERSAL MEASURABILITY

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Dedicated to Academician A. A. Borubaev on occasion of his 75th birthday

V. Fedorchuk, A. Chizogidze, and T. Banakh in 2003 and V. Bogachev in 2024 posed the following questions: (i) is it true that $P_\tau(X)$ is C-embedded in $P_\sigma(X)$; (ii) Is it true that $P_R(X)$ is C-embedded in $P_R(\beta X)$ if and only if X is pseudocompact, where P_σ, P_τ, and P_R are the functors of probability σ-additive on the Baire σ-algebra, τ-additive, and Radon measures on the space X? The answers to these questions are negative. However, if instead of probability measures we consider the corresponding alternating measures M_σ, M_τ, and M_R, the situation changes. It is proved that (i) $M_\tau(X)$ is C-embedded in $M_\sigma(X)$; (ii) $M_R(X)$ is C-embedded in $M_R(\beta X)$ if and only if X is pseudocompact.

The question of C-embedding of measure spaces is an extension of the question of coincidence of measure spaces, which is a development of the classical concepts of universally measurable and universal measure zero sets. A general theorem is obtained, which leads to the mentioned results.

Key words: measure spaces, probability measures, Radon measures, universally measurable sets, universal measure zero sets.

2003-жылы В.Федорчук, А.Чизогидзе жана Т.Банач жана 2024-жылы В.Богачев төмөнкүдөй суроолорду беришкен: (i) $P_\tau(X)$ Сте $P_\sigma(X)$ киргизилген деген чынбы; (ii) эгерде X псевдокомпакт болсо, анда $P_R(X)$ Сте $P_R(\beta X)$ киргизилген деген чынбы, бул жерде P_σ, P_τ, жана P_R Байр σ-алгебрасында σ-кошумчасынын, τ-кошумчасынын жана X боюнча радондун ыктымалдык функциялары? Бул суроолорго жооптор терс. Бирок, эгерде ыктымалдык өлчөмдөрдүн ордуна тиешелүү кезектеги M_σ, M_τ жана M_R

өлчөмдөрүн карасак, кырдаал өзгөрөт. Далилденди: (i) $M_\tau(X)$ $M_\sigma(X)$ Сга камтылган; (ii) $M_R(X)$ эгерде псевдокомпакт болгондо гана $M_R(\beta X)$ Ске камтылган. Өлчөө мейкиндиктерин С-киргизүү маселеси чен мейкиндиктеринин дал келүүсү жөнүндөгү маселенин уландысы болуп саналат, ал нөлдүн универсалдуу өлчөнгөн жана универсалдуу топтомдорунун классикалык концепцияларынын өнүгүшү болуп эсептелинет. Айтылган натыйжаларга алып баруучу жалпы теорема алынат.

Урунттуу сөздөр: өлчөө мейкиндиктери, ыктымалдык чаралары, радондук чаралар, универсалдуу өлчөнгөн топтомдор, нөлдүк өлчөмдүн универсалдуу топтому.

В. Федорчук, А. Чизогидзе и Т. Банах в 2003 году и В. Богачев в 2024 году задали следующие вопросы: (i) верно ли, что $P_\tau(X)$ является С-вложенным в $P_\sigma(X)$; (ii) верно ли, что $P_R(X)$ является С-вложенным в $P_R(\beta X)$ тогда и только тогда, когда X является псевдокомпактным, где P_σ, P_τ, и P_R являются функторами вероятности σ-аддитивной на σ-алгебре Бэра, τ-аддитивной и радоновской мер на пространстве X? Ответы на эти вопросы отрицательны. Однако, если вместо вероятностных мер мы рассмотрим соответствующие знакопеременные меры M_σ, M_τ, и M_R, ситуация меняется. Доказано, что (i) $M_\tau(X)$ является С-вложенным в $M_\sigma(X)$; (ii) $M_R(X)$ является С-вложенным в $M_R(\beta X)$ тогда и только тогда, когда X является псевдокомпактным.

Вопрос о С-вложении пространств с мерой является расширением вопроса о совпадении пространств с мерой, который представляет собой развитие классических понятий универсально измеримых и универсальных множеств нулевой меры. Получена общая теорема, приводящая к указанным результатам.

Ключевые слова: пространства с мерой, вероятностные меры, меры Радона, универсально измеримые множества, универсальные множества нулевой меры.

1. Introduction

This paper considers various ways of extending the concept of universally measurable and universal measure zero sets from separable metrizable spaces to the class of Tychonoff spaces.

Let $X \subset R^1$ A set X is called universally measurable if X is μ 1In a slightly more general setting, we consider subsets of Polish spaces (=complete metric separable spaces). Preprint submitted to Elsevier December 24, 2025 measurable for any finite measure μ on the Borel sets of the line. A set X is called universal measure-zero if X has measure zero with respect to any continuous probability measure on

Borel sets. The classical extension of these concepts from the class of separable metrizable spaces to the class of Hausdorff spaces is Radon spaces and uni versally negligible spaces. These classes of spaces are inconvenient for appli cations, since not every compactum is a Radon space. Given Ritz's theorem, for applications in analysis, functional analysis, and topology, it is more convenient to consider measures on the Baire σ algebra of Tychonoff spaces. In [13], the concept of universally measurable spaces was extended from separable metrizable spaces to the class of Ty chonoff spaces. In this case, all Hausdorff compact spaces turned out to be universally measurable.

Further, we assume that all spaces are Tychonoff.

1.1. Measures on a Tychonoff Space

For a space X, we denote by $B(X)$ and $B_a(X)$ the σ-algebras of Borel and Baire sets of X.

For a compact space K, let $M(K)$ denote the dual of $C(K)$ with the weak∗ topology, where $C(K)$ is the Banach space of continuous functions on K with the sup-norm. By the fundamental theorem of Ritz, the functionals μ in $M(X)$ are identified with Borel regular measures on K. Denote $\cup(K)$ by the unit ball in $M(K)$, $M^+(K)$ by nonnegative functionals in $M(K)$, and $P(K)=\{\mu\in M^+(K):\mu(K)=1\}$ by probability measures on K.

For a Tychonoff space X, the most important measure spaces on X are: $M_a(X)$ by finitely adiative measures on $B_a(X)$; $M_\sigma(X)$ — count ably additive measures on $B_a(X)$; $M_\tau(X)$ — τ-additive measures on $B(X)$; $M_R(X)=M(X)$— Radon measures on $B(X)$; $M_\beta(X)$ — compactly sup ported measures on $B(X)$; $M_d(X)$ — discrete measures on $B(X)$. These mea sure spaces are identified with subsets of $M(\beta X)$: for $k\in\{d,\beta,R,\tau,\sigma,a\}$ and μ ∈ $M(\beta X)$, the inclusion $\mu\in M_k(X)$ holds if the condition (M_k) is satisfied:

(M_d) μ is concentrated on some countable subset X;

(M_β) μ is concentrated on some compact subset X;

(M_R) μ is concentrated on some σ-compact subset X;

(M_τ) $|\mu|(K)=0$ for all compact $K\subset\beta X/X$;

(M_σ) $|\mu|(K)=0$ for all compact $K\subset\beta X/X$ of type G_δ in βX;

(M_a) μ is arbitrary.

Note that

$$M_d(X)\subset M_R(X),\quad M_\beta(X)\subset M_R(X)\subset M_\tau(X)\subset M_\sigma(X)\subset M_a(X)=M(\beta X).$$

1.2. Generalization of universally measurable and universal measure zero Sets

For a set A ⊂ R, the classical notions of universally measurable and uni versal measure zero sets A are topological and are characterized as $M_\sigma(A) = M_R(A)$ and $M_\sigma(A) = M_d(A)$.

For $p,q \in \{a,\sigma,\tau,R,\beta,d\}$, the space X is called

- universally [p,q]-measurable if $M_p(X) = M_q(X)$;
- universally p-measurable if $M_p(X) = M_R(X)$;
- universally p-measure zero if $M_p(X) = M_d(X)$.

Section 2 of the paper [13] is devoted to universally [σ,τ]-measurable and universally σ-measurable spaces (which are called universally measurable in [13]).

Note 1. Let us formulate known results on this topic.

(i) K-Analytic (in particular, Souslin) spaces are universally σ-measurable [14, 7.14.125].

(ii) A space X is universally τ-measurable if and only if X is μ-measurable in βX for any measure μ ∈ P(βX) [14, Example 7.14.22].

(iii) A paracompact space X is universally [σ,τ]-measurable if and only if X does not contain an Ulam-measurable discrete closed subset (see [24, Theorem I.28] and [13, Section 2]). In particular, a Lindel¨of space is a universally [σ,τ]-measurable space [13, Corollary 2.2].

(iv) For a space X, the following conditions are equivalent [4, Theorem 1.22]: (a) X is pseudocompact;

(b) X is a universally [σ,a]-measurable space.

The following assertion is a strengthening of Remark 1(ii).

Theorem 1. For a space X, the following statements are equivalent:

(i) X is universally τ-measurable;

(ii) X is μ-measurable in βX for any measure μ ∈ P(βX);

(iii) there exists an embedding of space X into some space Y such that X is μ-measurable in Y for any measure μ ∈ P(βY);

(iii) for any embedding of space X into any space Y, X is μ-measurable in Y for any measure μ ∈ P(βY).

The Rudin–Knowles theorem [20, 18] states that for a compact space X, any Radon measure on X is discrete (i.e., $M_R(X) = M_d(X)$, which is equivalent to X being of universally R-measure zero) if and only if X is a sparse compact space. The Rudin–Knowles theorem implies the following assertion.

Proposition 1. A space X is of universally R-measure zero if and only if every compact subset of X is sparse.

Proposition 2. Let p ∈ {a,σ,τ}. A space X is universally of p-measure zero if and only if X is universally p-measurable and every compact subset of X is sparse.

1.3. Functional Generalizations of universally measurable and Universally Measure Zero Sets

Recall that a subset Y of X is C-embedded (C∗-embedded) in X if every (bounded) continuous function on Y extends to a continuous function on X.

We consider the subfunctors

$$M_p^+(X) = M_p(X) \cap M^+(\beta X), \quad U_p(X) = M_p(X) \cap U(\beta X), \quad U_p^+(X) = U_p(X) \cap M^+(X),$$
$$P_p(X) = M_h(X) \cap P(\beta X)$$

of non-negative measures, measures in norm not exceeding one, probability measures, respectively, of the functor $M_p(X)$. For p,q ∈ {a,σ,τ,R,β,d} and $Q \in \{M, M^+, U, U^+, P\}$, X is called

- *universally* $Q_{[p,q]}-$ *measurable* $Q^+_{[p,q]}-$ *measurable)* if $Q_p(X)$ is $C-$embedded (C^*-embedded) in $Q_q(X)$;
- *universally* Q_p- *measurable* (*Q_p^*-measurable)* if X is universally $Q_{[R,p]}-$ measurable ($Q^*_{[R,p]}-$measurable);
- *universally* Q_p- *measure* (*Q_p^*-measure) zero* if X is universally $Q_{[d,p]}-$ measurable ($Q^*_{[d,p]}-$measurable).

Using the introduced concepts, some problems in measure theory and topology (see [4, Question 1.21, Question 1.29], [13, Section 5, Problem 5.5], [3,2]) are special cases of the following general question.

Problem 1. For p,q ∈ {f,σ,τ,R,β,d} and $Q \in \{M, M^+, U, U^+, P\}$, de scribe universally $[a,b]$-measurable, universally $Q_{[a,b]}-$ measurable, and uni versally $Q^*_{[a,b]}-$ measurable spaces.

In [13, Section 5], the problem is posed: to describe universally $P_{[\sigma,\tau]}-$ measurable and universally $P_\sigma-$ measurable spaces. And the question arises [13, Problem 5.5]: is it true that every space X is universally $P_{[\sigma,\tau]}-$measurable? In [4], a negative answer is given to this question.

In [3, 2], universally P_a^*- measurable spaces are studied, and it is proved that if X is universally P_a^*-measurable space, then X and $P(X)$ are pseu docompact spaces. In [4, Theorem 1.9], a converse of Bogochev's theorem was obtained: a space X is universally P_a^* – measurable if and only if $P(X)$ is a pseudocompact space. The question was posed: is it true that if X is pseudocompact, then X is universally P_a^* – measurable? In [4, Theorem 1.9], a negative answer was given to this question (see Examples 1.13 and 1.15 of [4, Theorem 1.9]).

Note 2. Let us formulate known results on this topic.

(i) If X is a universally σ-measurable space and the finite powers X^n of X have a countable Souslin number (in particular, X is a Souslin space), then X^k is a universally P_σ – measurable space for any cardinal κ [13, Corollary 5.2].

(ii) Any $AE(0)$ – space is a universally P_σ – measurable space [13, Corollary 5.3].

(iii) For a space X, the following conditions are equivalent [4, Theorem 1.22]: (a) X is pseudocompact; (b) X is a universally [σ,a]-measurable space; (c) X is a universally $P^*_{[\sigma,a]}$ – measurable space.

(iii) For a space X, the following conditions are equivalent [4, Theorem 1.27]: (a) X is pseudo ω-bounded; (b) X is a universally $P^*_{[\beta,a]}$ – measurable space.

(iv) For a space X, the following conditions are equivalent [4, Theorem 1.9]: (a) $P(X)$ is pseudocompact; (b) X is a universally P_a^* – measurable space.

(v) For a space X with pregauge ω1, the following conditions are equiva lent [4, Theorem 1.28]: (a) $P_\tau(X)$ is pseudocompact; (b) X is a universally $P^*_{[\tau,a]}$ – measurable space.

1.4. Alternating Measures

Probability measure functors have the best categorical properties. Other measure functors are also important [19, 5, 8, 21, 6, 7, 22] and are more 5 convenient for some applications.

A subset M ofaspace X iscalled G_δ – dense if M intersects every nonempty G_δ – set (=the intersection of countably many open sets) in X.

A continuous mapping of compact spaces $f: X \to Y$ extends to a con tinuous mapping

$M(f): M(X) \to M(Y),\ M(f)(\mu)(h) = \mu(h \circ f)$ for $h \in C(Y)$,

A continuous mapping of spaces $f: X \to Y$ extends to a continuous mapping $\beta f : \beta X \to \beta Y$ of Stone–Cech extensions. Then

$M(\beta f)(Mp(X)) \subset M_p(Y),$

where $p \in \{a, \sigma, \tau, R, \beta, d\}$.

The following theorem is the main result of the paper.

Theorem 2. Let X be a space and $p,q \in \{a,\sigma,\tau,R,\beta,d\}$. The following conditions are equivalent:

(i) X is a universally $M_{[p,q]}-$ measurable space;

(ii) X is a universally $M^*_{[p,q]}-$measurable space;

(iii) $M_p(X)$ is $G_\delta-$ dense in $M_q(X)$;

(iv) For a measure $\mu \in M_q(X)$ and a sequence of functions $(h_n)_n \subset C(\beta X)$, there exists $\nu \in M_p(X)$ such that $\mu(h_n) = \nu(h_n)$ for all n:

(v)
$M(\beta f)(M_q(X)) \subset M_p(Y)$,

for any separable metric space Y *and continuous surjective mapping* $f: X \to Y$.

From this theorem we obtain the following corollaries.

Corollary 1. *If* X *is pseudocompact, then* X *is universally* M^*_a- *measurable.*

Corollary 2. *Any space is universally* $M_{[\tau,\sigma]}-$ *measurable.*

1.5. Probabilistic, Non-Negative, and Unit-Bounded Measures

There exists a pseudocompact locally compact space X that is neither universally $P^*_\sigma-$ measurable nor $P_{[\tau,\sigma]}-$ measurable [4, Example 1.15], which answers negatively the questions in [13, 3]. As Corollaries 1 and 2 show, the answers to these questions are affirmative if we consider alternating measures rather than probability measures.

The following assertion follows directly from the definitions. 6

Proposition 3. *Let* X *be a space, p,q* $\in$*{a,σ,τ,R,β,d}, and* $Q \in \{M, M^+, \bigcup, \bigcup^+, P\}$.
If X *is a universally* $Q_{p,q}-$ *measurable space, then* X *is a universally* $Q^*_{[p,q]}$
measurable space.

Theorem 3. *Let* X *be a space, p,q* $\in$*{a,σ,τ,R,β,d}, and* $Q \in \{M^+, \bigcup, P\}$.
If X *is a universally* $Q^*_{[p,q]}-$ *measurable space, then* X *is a universally* $M_{[p,q]}-$ *measurable space.*

Theorem 4. *Let* X *be a space and p,q* $\in$*{a,σ,τ,R,β,d}.*

(i) A space X *is a universally* $P_{[p,q]}-$*measurable space if and only if* X *is a universally* $M^+_{[p,q]}-$ *measurable space.*

(ii) A space X *is a universally* $P^+_{[p,q]}-$ *measurable space if and only if* X *is a universally* $M^{+^*}_{[p,q]}-$ *measurable space.*

Problem 2. Let p,q ∈ {f,σ,τ,R,β,d}. How are the following classes of spaces related?

• universally $P_{[p,q]}$ – measurable spaces;

• universally $P^{+}_{[p,q]}$ – measurable spaces;

• universally $U_{[p,q]}$ – measurable spaces;

• universally $U^{+}_{[p,q]}$ – measurable spaces;

• universally $M_{[p,q]}$ – measurable spaces.

2. Preliminary Information, Definitions, and Notation

Let ω = {0} ∪ N be the first countable ordinal, and ω_1 be the first uncountable ordinal. By space, we mean a Tychonoff space.

2.0.1. Definitions from Measure Theory

Let μ be a finite (i.e., μ(X) < ∞) σ - additive measure on the Borel σ algebra B(X) of X. Let $\mu = \mu^{+} - \mu^{-}$ be the Hahn–Jordan decomposition. The measure $|\mu| = \mu^{+} + \mu^{-}$ is called the total variation of μ. A measure μ is called *Radon* if for every Borel set B ⊂ X and for every ε > 0, there exists a compact $K \subset B$ such that $|\mu|(B\setminus K) < \varepsilon$. *The support suppμ* of a measure μ is the smallest closed set F ⊂ X for which $|\mu|(X \setminus F) = 0$. A measure μ is called *discrete* if |μ|(C) = 1 for some countable C ⊂ X. A measure μ 7 is called *continuous* if μ({x}) = 0 for every $x \in X$. A measure μ is called τ- *additive if*

$$|\mu|(\cup u) = \sup\{|\mu|(\cup v) : v \subset U, |v| < \omega\}$$

for any family U of open subsets of X.

For a non-negative measure μ, *we* denote

$\mu*(A) = \inf\{\mu(B) : A \subset B \in B(X)\}$, $\mu*(A) = \sup\{\mu(B) : A \supset B \in B(X)\}$

external and *internal* measures of a set $A \subset X$.

A countably additive measure μ on the σ-algebra $B(X)$ of Borel sets in X is called a *Borel* measure. We denote the set of probability Borel measures on X by $P_b(X)$. Let $\mu \in P_b(X)$. A probability measure μ is called a *Radon* measure if, for every Borel set $B \subset X$ and for every $\varepsilon > 0$, there exists a compact $K \subset B$ such that $\mu(B \setminus K) < \varepsilon$. We denote the set of probability τ -additive measures on X by $P_\tau(X)$. Note that $P(X) \subset P_\tau(X)$.

Let Y be a space and $X \subset Y$. The set $P_b(X)$ is naturally embedded in $P_b(Y)$, and a measure $\mu \in P_b(X)$ corresponds to a measure $\tilde{\mu} \in P_b(Y)$ such that $\tilde{\mu}(B) = \mu(B \cap Y)$ for $B \in B(Y)$. We assume that $P_b(X) \subset P_b(Y)$. If Y is compact, then $P(X) \subset P_\tau(X) \subset P(Y)$.

Proposition 4 ([1]). Let Y be a compact space and $X \subset Y$. Then w(X) = *w(P(X)) = w(* P_τ *(* X *))* and

$$P(X) = \{\mu \in P(Y) : \mu_*(X) = 1\},\ P_\tau(X) = \{\mu \in P(Y) : \mu^*(X) = 1\}.$$

The embedding of $P(X)$ *and* $P_\tau(X)$ *into* $P(Y)$ *is a topological embedding.*

Let X be a space. Then

$$M_k(X) = \{t\mu - qv : \mu, v \in P_k(X),\ 0 \le t,q\},$$

$$M_k(X) = \{t\mu : \mu \in Pk(X),\ 0 \le t\},$$

$$\cup_k(X) = \{t\mu + qv : \mu, v \in Pk(X),\ 0 \le t,q \le 1\}.$$

for $k \in \{d,\beta,R,\tau,\sigma,a\}$. The space $\cup_k(X)$ is the unit ball in $M_k(X)$. The spaces $\cup_k(X)$ and $M_k(X)$ for $k \in \{\beta,R,\tau\}$ were studied in the works [19,5,8,21,6,7,22].

2.0.2. Definitions from Topology

If X is a space and $M \subset X$, then we denote by $\overline{M}^X$ the closure of M in X. If the context makes it clear which X is meant, we write $\overline{M}$ instead of $\overline{M}^X$.

A space X is said to have a countable Suslin number if every disjoint family of open sets in X is at most countable.

A space X is called ω-bounded if M is compact for every countable M $\subset X$. Aspace X is called *totally countably compact* if for every infinite $L \subset X$, there exists a countable $M \subset L$ such that M is compact (see [25]).

A space X is *called pseudo-ω-bounded* if for every sequence $(\cup_n)_n$ of open nonempty open sets, there exists a compact subset $K \subset X$ such that $\cup_n \cap K \neq \emptyset$ for all n (see [17, 10, 16]).

A space X is called almost pseudo ω-bounded (almost pseudo-ω-boun ded) if, for any sequence (Un)n of open nonempty open sets, there exists a subsequence (nk)k and a compact subset $K \subset X$ such that $\cup_{n_k} \cap K \neq \emptyset$ for all k [10, Definition 3.8], [11, Definition 1.4.5].

A subset $G \subset X$ of X iscalleda G_δ - setifitis the intersection of countably many open sets.

A subset S $\subset X$ of X is called G_δ - dense in X if S intersects every nonempty G_δ - set. A space X is pseudocompact if and only if X is G_δ - dense in the Stone–Cech extension of βX.

A space X is called *realcompact* if X embeds closedly in the product of lines RA. A *Hewitt realcompactification* νX of X is a real complete extension of X such that X is dense in νX and X is *C*- embedded in νX (see [9, section

3.11]). The Hewitt extension νX is naturally embedded in the Stone–Cech extension

βX : p $\in \nu X \subset \beta X$ if and only if every G_δ- set $G \subset \beta X$ containing p intersects X. The space X is pseudocompact if and only if $\nu X = \beta X$. The space X is real complete if and only if $\nu X = X$ (see [?]) section 3.11] EngelkingBookRu.

A subset F $\subset X$ is called a null set if $F = f^{-1}(0)$ for some continuous function on X.

A space X is called perfectly κ - normal if the closure of every open set is a null set.

3. Proofs. Let us describe the main steps of the proof.

Proposition 5. *Let L_w be a locally convex linear space L with the weak topology. Then, for a dense subset M $\subset L_w$, the following conditions are equivalent:*

(1) M is C^*- embedded in L_w;

(2) M is C- embedded in L_w;

(3) M is G_δ- dense in L_w.

Proof. We show that L_w is perfectly κ-normal. Let $\hat{L}_w$ be the completion of L_w. Then L_w is linearly homeomorphic to a power of the line R^A [15, Corollary 3.12.43]. The space R^A is perfectly κ- normal [23]. Since L_w is dense in L_w, L_w is perfectly κ- normal. (ii) follows from (i) and Lemma 9.9.35, Theorem 6.1.5, Proposition 6.1.6, Theorem 6.1.7, Lemma 8.4.5, and Theorem 9.9.36 from [12]. *Proof of Theorem 2.* We assume that p $\neq$ q.

Proposition 5 implies the equivalence of items (i), (ii), and (iii) of Theorem 2.

(iv)⇔(v) is easily verified.

(iii)⇔(v) We show that $M_q(Y)$ has a countable pseudocharacter. If $q = a$, then $M_p(X) \subset M_\sigma(X)$ and $M_\sigma(X)$ are G_δ- dense in $M(\beta X)$. Consequently, X is pseudocompact. Then Y is a metric compact set and $M_q(Y) = M(Y)$ has a countable network. If $q \neq a$, then $M_q(Y)) \subset M_\tau(Y)$ has a countable network.

Since $M_p(X)$ is G_δ- dense in $M_q(X)$, it follows that $M(\beta f)(M_p(X))$ is G_δ- dense in $M(\beta f)(M_q(X)) \subset M_q(Y)$. Consequently,

$M(\beta f)(Mp(X)) = M(\beta f)(M_q(X)) \subset M_q(Y)$.

(iv)⇔(iii) Assume the contrary, $M_p\ (X)$ is not $G\delta$ - dense in $M_q(X)$. Let $G \subset M_q(X)\ M_p\ (X)$ be a non-empty zero set and $\mu \in G$. Then there will be sequence of functions $(h_n)_n \subset C(\beta X)$ such that $F = \bigcap_n h_n^{-1}(h_n(\mu)) \subset G$. Since $F \subset M_q(X) \setminus M_p(X)$, then condition (iv) is violated.

Theorem 4 follows from the fact that $P_q(X)$ is C-embedded (C^*-embedded) in $P_p\ (X)$ if and only if $M_q^+(X)$ is C- embedded (C^*- embedded) in $M_p^+(X)$.

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CONFORMAL FEDOSOV STRUCTURE IN PROJECTIVE SYMMETRIC AND PROJECTIVE RECURRENT EQUIAFFINE SPACES

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Dedicated to the 75th anniversary of the academician Altai Asylbekovich Borubaev

In this article, we prove that projective symmetric and projective recurrent equiaffine manifolds whose projective curvature tensor does not satisfy condition (7) do not admit conformal Fedosov structures. This result generalizes our earlier non-existence theorems for symmetric and recurrent (pseudo) -Riemannian spaces.

Key Words and Phrases: Conformal Fedosov struc-ture, projective recurrent space, projective symmetric space, equiaffine space.

Бул эмгекте проекциялык ийрилик тензору (7) шартты канааттандырбаган проекциялык симметриялуу жана проекциялык рекурренттүү эквиафиндик коллекторлор конформдык Федосов структураларына жол бербестигин далилдейбиз. Бул жыйынтык симметриялуу жана рекурренттик (псевдо) Римандык мейкиндиктер үчүн мурунку жок болуу теоремаларын жалпылайт.

Ачкыч сөздөр жана сөз айкаштары: конформдык Федосов структурасы, проекциялык рекурренттик мейкиндик, проекциялык симметриялык мейкиндик, эквиафиндик мейкиндик.

В этой статье мы доказываем, что проективные симметричные и проективные рекуррентные равноаффинные многообразия, проективный тензор кривизны которых не удовлетворяет условию (7), не допускают конформных структур Федосова. Этот результат обобщает наши более ранние теоремы о несуществовании для симметричных и рекуррентных (псевдо)-римановых пространств.

Ключевые слова и фразы: конформная структура Федосова, проективное рекуррентное пространство, проективное симметричное пространство, равноаффинное пространство.

1. Introduction

In this paper, we study projective symmetric and projective recur-rent equiaffine manifolds equipped with conformal Fedosov struc-tures. We prove that, apart from projective spaces whose projec-tive curvature tensor does not satisfy condition (7) do not admit conformal Fedosov structures. In particular, this generalizes the results obtained for symmetric and recurrent (pseudo-)Riemannian spaces [10].

2. Conformal Fedosov Spaces

For a conformally symplectic manifold [12], one considers a local equivalence class of symplectic forms, defined up to multiplication by a nonzero scalar. This setting naturally connects projective dif-ferential geometry with the theory of Fedosov manifolds and leads to the concept of a conformally Fedosov manifold.

More precisely, a conformally symplectic manifold M of dimen-sion $n \geq 4$ is equipped with a nondegenerate 2-form J for which there exists a closed 1-form a satisfying

$$dJ = 2a \wedge J. \tag{1}$$

is injective. Consequently, whenever (1) holds, the 1-form a is uniquely determined. The form a is referred to as the *Lee form* [7].

If dim $M \geq 6$, the identity $d^2J = 0$ further implies that

$da \wedge J = 0.$

In local coordinates, equation (1) can be written in the form

$\nabla_{[i}J_{jk]} = a_{[i}J_{jk]},$

where square brackets denote alternation over the enclosed indices without normalization.

A *projective structure* [14] on a manifold M is defined as an equivalence class of torsion-free affine connections. Two connections ∇ and $\hat{\nabla}$ are said to be projectively equivalent if there exists

a 1-form v_a such that

$\hat{\nabla}_a\varphi_b = \nabla_a\varphi_b - v_a\varphi_b - v_b\varphi_a.$

For an antisymmetric tensor J , this transformation law implies

$$\hat{\nabla}_{(a}J_{b)c} = \nabla_{(a}J_{b)c} - 3v_{(a}J_{b)c}.$$

In particular, if J is projectively invariant, the above condition reduces to

$$\nabla_{(a}J_{b)c} = \nabla_{(a}J_{b)c} - 3v_{(a}J_{b)c}.$$

for some 1-form β.

Combining this condition with (1) leads to the following defini-tion, cf. [4].

Definition 1. A *conformally Fedosov manifold* (M, [J], ∇]) is a conformally symplectic manifold (M, [J]) such that, for a skew-symmetric representative $J \in [J\ ]$, there exists a 1-form a with components a_i satisfying $\nabla_k J_{ij} = J_{i[j}a_{k]}$.

This nonlinear system is equivalent to the linear system

$$\nabla_k J^{ij} = a^i\delta_k^j - a^j\delta_k^i, \qquad (2)$$

where J^{ij} denote the components of the inverse matrix of (J_{ij}) and δ_j^i is the Kronecker delta. This formulation is preferable, asit is linear in the unknowns J^{ij} and a_i. By eliminating the 1-form a, one obtains a closed first-order linear system for J^{ij}, which is particularly well suited for further analysis.

The study of equations of this type has been addressed, for example, in [1, 2].

3. Integrability Conditions of the Fun-damental Equations and their Differ-ential Prolongations

Let (M, ∇) be an n-dimensional equiaffine manifold A_n. We denote covariant differentiation with respect to the affine connection ∇ by a comma " , ". Assume that $n \geq 4$ is even.

Suppose that there exists a conformal Fedosov structure J on the equiaffine space A_n satisfying equations (2).

We take the covariant derivative of (2) with respect to the vari-able x^l and then alternate the resulting expression with respect to the indices k and l. Using the Ricci identities for an affine connection, we obtain

$$J^{ij},kl - J^{ij},lk = -J^{\alpha j}\, R^i\alpha kl - J^{i\alpha}R^j\alpha kl,$$

where R_{ijk}^h are components of the Riemann curvature tensor, we

obtain the following *integrability condition* of equation (2)

$$-J^{\alpha j} R_{\alpha kl} - J^{i\alpha} R_{\alpha kl} = a_l^i \delta_k^j - a_l^j \delta_k^i - a_k^i \delta_l^j + a_k^j \delta_l^i, \tag{3}$$

where $a_k^i = a^i{}_{,k}$.

Covariantly differentiating this expression with respect to the variable x^m, we obtain

$$-J^{\alpha j}{}_{,m} R^i_{\alpha kl} - J^{i\alpha}{}_{,m} R^j_{\alpha kl} - J^{\alpha j} R^i_{\alpha kl,m} - J^{i\alpha} R^j_{\alpha kl,m} = a^j_{lm}\delta^i_k - a^j_{lm}\delta^i_k - a^i_{km}\delta^j_l + a^j_{km}\delta^i_l, \tag{4}$$

where $a^i{}_{km} = a^i{}_{,km}$.

Substituting the derivatives of J from (2) into (4), we obtain the following conditions:

$$-J^{\alpha j}{}_{,m} R^i_{\alpha kl} - J^{i\alpha}{}_{,m} R^j_{\alpha kl} - J^{\alpha j} R^i_{\alpha kl,m} - J^{i\alpha} R^j_{\alpha kl,m} = a^j_{lm}\delta^i_k - a^j_{lm}\delta^i_k - a^i_{km}\delta^j_l + a^j_{km}\delta^i_l,$$

where $a^i{}_{km} = a^i{}_{,km}$.

Substituting the derivatives of J from (2) into (4), we obtain the following conditions:

$$a^j R^i_{mkl} - a^i R^j_{mkl} - \delta^j_m a^\alpha R^i_{\alpha kl} + \delta^i_m a^\alpha R^j_{\alpha kl} - J^{\alpha j} R^i_{\alpha kl,m} - J^{i\alpha} R^j_{\alpha kl,m} =$$
$$= a^i_{lm}\delta^j_k - a^j_{lm}\delta^i_k - a^i_{km}\delta^j_l + a^j_{km}\delta^i_l.$$

These conditions are referred to as the *first differential prolongation* of the integrability conditions (3) for equations (2).

4. Projective symmetric and projective recurrent conformal Fedosov spaces

Symmetric and *recurrent* spaces are characterized by conditions on the first covariant derivative of the curvature tensor:

$$\nabla R = 0 \qquad \text{and} \qquad \nabla R = \varphi \otimes R,$$

where φ is a nonvanishing 1-form.

Symmetric and recurrent spaces play an important role in dif-ferential geometry and its applications. Symmetric spaces were independently introduced in the 1920s by P. A. Shirokov [11] and E′. Cartan [3]. Recurrent spaces were introduced by Ruse [13]. These spaces have been studied extensively; see, for example, [6] and [9, pp. 117].

The above classes of spaces have been generalized in various ways. One of them is *projectively symmetric* and *projectively re-current spaces* A_n, characterized by terms on the first derivative of projective curvature tensor:

$$\nabla W = 0 \qquad \text{and} \qquad \nabla W = \varphi \otimes W.$$

where φ is a non vanishing linear form.

As is known, (pseudo-)Riemannian projectively symmetric and projectively recurrent spaces are symmetric and recurrent, respec-tively, see [5], [9]. For spaces with affine connection (include equiaffine spaces) this is not true, as confirmed by examples constructed in the work [8].

Let us consider two classes of spaces with affine connection whose characteristics correspond to tensor conditions:

$$R^h_{ijk} = a^h \Omega_{ijk} + \delta^h_j \tilde{\omega}_{ik} - \delta^h_k \tilde{\omega}_{ij}, \tag{6}$$

$$W^h_{ijk} = a^h \Omega_{ijk} + \delta^h_j \omega_{ik} - \delta^h_k \omega_{ij} \tag{7}$$

where a^h, Ω_{ijk}, ω_{ik}, $\tilde{\omega}_{ij}$ are certain tensors.

It is easy to establish that spaces A_n satisfying (6) satisfy con-ditions (7), but the converse apparently does not hold. Obviously, when a^h is vanishing, spaces A_n are projective Euclidean. There-fore, these spaces A_n satisfying conditions (6) and (7) naturally generalize projectively Euclidean spaces. Naturally, there are many more spaces that do not satisfy these conditions.

Let us prove the following theorem:

Theorem 2. *Let A_n be an n-dimensional projective symmetric or projective recurrent equiaffine space A_n which does not satisfy the condition (7). Then A_n does not admit a conformal Fedosov structure.*

Proof. Let there exist a conformal Fedosov structure J in a projective symmetric or projective recurrent equiaffine space A_n which satisfying equations (1). Next, we assume that in A_n does not satisfy the conditions (7).

The equiaffine space A_n is characterized by symmetric the Ricci tensor *Ric*, and the Weyl projective curvature tensor W in equiaffine space defined in the following formula

$$W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}\left(\delta^h_j R_{ik} - \delta^h_k R_{ij}\right),$$

where R^h_{ijk} and $R_{ik}\left(= R^\alpha_{i\alpha k} = R_{ki}\right)$ are components of the Riemann curvature tensor and the Ricci tensor. Note that the projective curvature tensor is traceless and satisfies the conditions $W^\alpha_{\alpha jk} = 0$ and $W^\alpha_{i\alpha k} = 0$.

Since for *projective symmetric spaces* $A_n : W^h_{ijk,l} = 0$, and for *projective recurrent spaces* $A_n : W^h_{ijk,l} = \varphi_l W^h_{iijk}$, $\varphi_l \neq 0$.

therefore from first differential prolongation (5) we obtain

$$a^jW^i_{mkl} - a^iW^j_{mkl} - \delta^j_m a^\alpha W^i_{\alpha kl} + \delta^i_m a^\alpha W^j_{\alpha kl} =$$
$$= a^i_{lm}\delta^j_k - a^j_{lm}\delta^i_k - a^i_{km}\delta^j_l + a^j_{km}\delta^i_l, \tag{8}$$

where a^i_{lm} is a certain new tensor.

Therefore, from equation (8), by contracting over indices *j* and *k*, we obtain:

$$a^\alpha W^i_{m\alpha l} - a^\alpha W^i_{\alpha ml} = (n-2)a^i_{lm} + a^\alpha_{\alpha m}\delta^i_l.$$

Using the Bianchi $R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0,$ and also $W^h_{ijk} + W^h_{jki} + W^h_{kij} = 0,$ from above, it is easy to derive:

$$(n-2)a^i_{lm} = a^\alpha W^i_{l\alpha m} - a^\alpha_{\alpha m}\delta^i_l. \tag{9}$$

On the other hand, by contracting (8) over indices *j* and *m:*

$$(n-2)a^\alpha W^i_{\alpha kl} = a^i_{kl} - a^i_{lk} + a^\alpha_{l\alpha}\delta^i_k - a^\alpha_{k\alpha}\delta^i_l.$$

Using equation (9), we eliminate the tensor a^i_{kl}:

$$(n-2)^2 a^\alpha W^i_{\alpha kl} = (a^\alpha W^i_{kl\alpha} - a^\alpha W^i_{lk\alpha}) + \delta^i_m \overset{1}{a}_l + \delta^i_l \overset{2}{a}_m, \tag{10}$$

where $\overset{\sigma}{a}_l$ are vectors.

Since aα $a^\alpha W^h_{lk\alpha} - a^\alpha W^h_{ki\alpha} = a^\alpha W^h_{\alpha kl},$ from (10) we get:

$$a^\alpha W^i_{\alpha kl} = \delta^i_k \overset{3}{a}_l + \delta^i_l \overset{4}{a}_k, \tag{11}$$

Symmetrizing (11) respective indices k and l gives: $\delta^i_k\left(\overset{3}{a}_l + \overset{4}{a}_l\right) + \delta^i_l\left(\overset{3}{a}_k + \overset{4}{a}_k\right) = 0.$

Clearly, $\overset{4}{a}_l = -\overset{3}{a}_l,$ and formula (11) has the following form:

$$a^\alpha W^i_{\alpha kl} = \delta^i_k \overset{3}{a}_l - \delta^i_l \overset{3}{a}_k. \tag{12}$$

After substitution (12) to (8) obtain

$$a^jW^i_{mkl} - a^iW^j_{mkl} = b^i_{lm}\delta^j_k - b^j_{lm}\delta^i_k - b^i_{km}\delta^j_l + b^j_{km}\delta^i_l, \tag{13}$$

where b^l_{lm} is a certain tensor.

(13) Recall that $a^i \neq 0$ and that dimension $n \geq 4$. Then it is straight forward to check that there exist non-collinear covectors α_j and β_j, and a vector b^i, such that $\alpha_\alpha a^\alpha = \beta_\alpha a^\alpha = 0$ and $\alpha_\alpha b^\alpha = \beta_\alpha b^\alpha = 1.$

Contracting equation (13) with $\alpha_j b^k$, we obtain

$$b^i_{lm} = b^\alpha b^i_{\alpha m}\alpha_l + \alpha_\alpha b^\alpha_{lm} b^i - \alpha_\alpha b^\beta b^\alpha_{\beta m}\delta^i_k - \alpha_\alpha b^\beta W^\alpha_{m\beta l} a^i \tag{14}$$

After contraction (13) with $\alpha_j b^k$ we get

$$b^i_{lm} = b^\alpha b^i_{\alpha m}\beta_l + \beta_\alpha b^\alpha_{lm} b^i - \beta_\alpha b^\beta b^\alpha_{\beta m}\delta^i_k - \beta_\alpha b^\beta W^\alpha_{m\beta l} a^i \tag{15}$$

Subtracting (15) from (14), and after slight modifications we derive

$$b^\alpha b^i_{\alpha m}(\alpha_l - \beta_l) + a^i a_{lm} + b^i b_{lm} + c_m \delta^i_k = 0$$

(16)

where a_{lm}, b_{lm}, c_m are a tensors. (16)

If a vector c_m is nonvanishing, then there exists a vector C^m such that $C^m c_m = 1$. By contracting (16) with C^m, we see that rank $(\delta^i_k) \leq 3$.

This contradicts n ≥ 4, so $c_m = 0$ and formula (16) are simplified, and by contracting with ε^l such that $\varepsilon^l(\alpha_l - \beta_l) = 1$, it follows that

$$b^\alpha b^i_{\alpha m} = a^i A_m + b^i B_m,$$

where A_m and B_m are covectors.

Then (14) simplifies to

$$b^i_{lm} = a^i \Omega_{lm} + b^i \omega_{lm} + c_m \delta^i_l,$$

where Ω_{lm}, ω_{lm} and c_m are tensors.

From equation (13) we eliminate b^i_{lm} and obtain

$$a^j Z^i_{mkl} - a^i Z^j_{mkl} = b^i(\omega_{lm}\delta^j_k - \omega_{km}\delta^j_l) - b^j(\omega_{lm}\delta^i_k - \omega_{km}\delta^i_l) + 2c_m(\delta^i_l\delta^j_k - \delta^i_k\delta^j_l), \tag{17}$$

where

$$Z^h_{ijk} = W^h_{ijk} - \delta^h_j \omega_{ik} + \delta^h_r \omega_{ij}. \tag{18}$$

Now, contracting (17) with α_j, we obtain (18)

$$-a^i \alpha_\alpha Z^\alpha_{mkl} = b^i(\omega_{lm}\alpha_k - \omega_{km}\alpha_i) - (\omega_{lm}\delta^i_k - \omega_{km}\delta^i_l) + 2c_m(\delta^i_l \alpha_k - \delta^i_k \alpha_l).$$

After minor modifications, we have

$$\delta^i_k(\omega_{lm} + 2c_m\alpha_l) - \delta^i_l(\omega_{km} + 2c_m\alpha_k) = a^i \alpha_j Z^j_{mkl} + b^i(\omega_{im}\alpha_k - \omega_{km}\alpha_l).$$

In the case that $\omega_{lm} + 2c_m\alpha_l \neq 0$, there exists a bivector $\xi^l\zeta^m$ such that $\xi^l\zeta^m(\omega_{lm} + 2c_m\alpha_l) = 1$. After contracting the above formula with $\xi^l\zeta^m$, we see that rank $(\delta^i_k) \leq 3$. This contradicts n ≥ 4, so condition

$$\omega_{lm} + 2c_m\alpha_l = 0$$

is necessarily satisfied.

Analogously, contracting (17) with β_j, we obtain $\omega_{im}+2c_m\beta_l=0$.

By comparing the last two formulas, we obtain $2c_m\alpha_l=2c_m\beta_l$.

Since α_l and β_l are non-collinear, it follows that $c_m=0$ and in turn $\omega_{im}=0$.

Consequently, from (17) we obtain

$$a^j Z^i_{mkl} - a^i Z^j_{mkl} = 0 \qquad (19)$$

After contraction (19) with γ_j for which $\gamma_j a^j = 1$ we get

$$Z^h_{ijk} = a^h \Omega_{ijk},$$

where Ω_{ijk} is a tensor, and based on formula (18), we obtain

$$W^h_{ijk} = a^h\Omega_{ijk} + \delta^h_j\omega_{ik} - \delta^h_k\omega_{ij}.$$

Therefore, conditions (7) are fulfilled, which is a contradiction to the assumptions of the Theorem 2 we formulated, and thus it is proven.

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ABOUT UNIQUENESS OF SOLUTIONS OF FREDHOLM LINEAR INTEGRAL EQUATIONS OF THE THIRD KIND ON THE SEGMENT

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In this paper, the uniqueness of the solution for a new class of Fredholm linear integral equations of the third kind on a segment is investigated. Based on the method of integral transformations and the method of non-negative quadratic forms, the uniqueness theorem of the solution for this class of integral equations of the third kind are proved. Examples are given.

Keywords: Uniqueness, solutions, segment, Fredholm linear integral equations, third kind.

Бул макалада Фредгольмдун үчүнчү түрдөгү сызыктуу интегралдык теңдемелеринин жаңы классынын чыгарылышынын жалгыздыгы изилденет. Интегралдык өзгөртүү ыкмасынын жана терс эмес квадраттык формалар методунун негизинде үчүнчү түрдөгү интегралдык теңдемелердин берилген классы үчүн чыгарылышынын жалгыздык теоремасы далилденген. Мисалдар келтирилген.

Урунттуу сөздөр: Жалгыздыгы, чыгарылыштары, сегмент, үчүнчү түрдөгү Фредгольмдун сызыктуу интегралдык теңдемелери.

В данной работе исследуется единственность решения нового класса линейных интегральных уравнений Фредгольма третьего рода на отрезке. На основе метода интегральных преобразований и метода неотрицательных квадратичных форм доказаны теорема единственности решения для данного класса интегральных уравнений третьего рода. Приведены примеры.

Ключевые слова: Единственность, решения, отрезок, линейные интегральные уравнения Фредгольма третьего рода.

1. Introduction

Various problems of the theory of integral equations of the first kind were studied in [1-17, 19-20]. But fundamental results for Fredholm integral equations of the first kind were obtained in [5, 6], where regularizing operators in the sense of M.M. Lavrent'ev were constructed. Results on non-classical Volterra integral equations of the first kind can be found in [1]. In [2, 8], problems of regularization, uniqueness and existence of solutions for Volterra integral and operator equations of the first kind are studied. In [7], for linear Volterra integral equations of the first and the third kind with smooth kernel, the existence of multiparameter family of solutions was proved. In [3, 4], on the basis of the theory of Volterra integral equations of the first kind, various inverse problems were studied. In [9, 10], uniqueness theorems were proved and requlalizing operators in the sense of Lavrent'ev were constructed for systems of nonlinear Volterra integral equations of the third kind and for systems of linear Fredholm integral equations of the third kind. In [12], problems of uniqueness and stability of solutions for linear integral equations of the first kind with two independent variables were investigated. In [11], based on a new approach, the existence and uniqueness of solutions of Fredholm integral equations of the third kinds were studied. In [12], uniqueness theorems were proved and an estimate of the stability of the solution for two-dimensional integral equations of the first kind was obtained. The method of integral transformations and method of non-negative quadratic forms is used here. In [13], based on a new approach, the existence and uniqueness of solutions of the system linear Fredholm integral equations of the third kinds were investigated. In [17], on the basis of the method of integral transformation, uniqueness theorems for the new class of linear Volterra integral equations of the third kind on the segment are proved. In [18], information on integral equations of the first kind was used to determine the perturbation coefficient of an elliptic differential equation. In [19], based on a new approach, the issues of existence, uniqueness and non-uniqueness of solutions for systems of linear and nonlinear Fredholm integral equations of the third kind on a segment were investigated. In [20], based on a modification of the new approach, the issues of existence, uniqueness and non-uniqueness of solutions for systems of linear Fredholm integral equations of the third kind on the real line were studied.

In the present paper, on the basis of the method of integral transformations and the method of non-negative quadratic forms, uniqueness theorems for the new class of linear Fredholm integral equations of the third kind on the segment are proved.

2. Fredholm linear integral equations of the third kind on the segment

We will consider the equation

$$m(t)u(t) + \int_a^b K(t,s)u(s)ds = f(t), \qquad t \in [a,b], \qquad (1)$$

where $m(t), \varphi(t),\ K(t,s)$ and $f(t)$ are known functions,

$m(t) \in C[a,b]$, $0 \le m(t)$ for all $t \in [a,b]$, $m(t)$ is equal to zero at least at one point of the segment $[a,b]$, $u(t)$ is unknown function,

$$K(t,s) = \begin{cases} A(t,s), & a \le s \le t \le b, \\ B(t,s), & a \le t \le s \le b. \end{cases}$$

Suppose that

$$A(t,s) + B(s,t) = \sum_{i=1}^{n} P_i(t) H_i(t,s) P_i(s), \ (t,s) \in G = \{(t,s): a \le s \le t \le b\}, \qquad (2)$$

where $P_i(t)$ and $H_i(t,s)$ are known functions, $i = 1,2,\dots,n$.

Assume that the following conditions are satisfied:

a) $P_i(t) \in C[a,b]$, $P_i(t) \neq 0$ for almost all $t \in [a,b]$, $\frac{\partial H_i(t,s)}{\partial t}$, $\frac{\partial^2 H_i(t,s)}{\partial t \partial s} \in C(G)$, $(H_i(t,a))'$, $(H_i(b,t))' \in C[a,b]$ $i = 1,2,\dots,n$;

b) $m(t) \ge 0$, $H_i(t,a) \ge 0, (H_i(t,a))' \le 0$, $\forall t \in [a,b]$, $\frac{\partial H_i(b,s)}{\partial s} \ge 0, \forall s \in [a,b]$,

$\frac{\partial^2 H_i(t,s)}{\partial t \partial s} \le 0$, $\forall (t,s) \in G$, $i = 1,2,\dots,n$;

c) At least one of the following four conditions holds:

1) $m(t) > 0$ for almost all $t \in [a,b]$;

2) Exists $i_0 \in \{1,2,\dots,n\}$ such that $(H_{i_0}(t,a))' < 0$ for almost all $t \in [a,b]$;

3) Exists $i_0 \in \{1,2,\dots,n\}$ such that $(H_{i_0}(b,t))' > 0$ for almost all $t \in [a,b]$;

4) Exists $i_0 \in \{1,2,\dots,n\}$ such that $\frac{\partial^2}{\partial t \partial s} H_{i_0}(t,s) < 0$ for almost all $(t,s) \epsilon G$.

The integral equation (1) is written as

$$m(t)u(t) + \int_a^t A(t,s)u(s)ds + \int_t^b B(t,s)u(s)ds = f(t), t \in [a,b]. \qquad (3)$$

Multiplying both sides of the equation (3) by $u(t)$, integrating over the domain $[a,b]$ and applying Dirichlets formulas, we obtain

$$\int_a^b m(s)u^2(s)ds + \int_a^b \int_a^t [A(t,s) + B(s,t)]u(s)u(t)dsdt = \int_a^b f(s)u(s)ds. \qquad (4)$$

If we take into account (2), then from (4) we get

$$\int_a^b m(s)u^2(s)ds + \sum_{i=1}^n \int_a^b \left[\int_a^t H_i(t,s)P_i(s)u(s)ds\right] P_i(t)u(t)dt =$$

$$= \int_a^b f(s)u(s)ds. \tag{5}$$

We introduce the notation

$$Z_i(t,s) = \int_s^t P_i(\tau)u(\tau)d\tau, \quad i = 1,2,\dots,n. \tag{6}$$

Then

$$P_i(s)u(s)ds = -d_s Z_i(t,s), \quad i = 1,2,\dots,n, \tag{7}$$

$$P_i(t)u(t)dt = d_t Z_i(t,s), \quad i = 1,2,\dots,n, \tag{8}$$

$$Z_i(t,s)P_i(t)u(t)dt = \frac{1}{2} d_t Z_i^2(t,s), \quad i = 1,2,\dots,n, \tag{9}$$

$$Z_i(t,s)P_i(s)u(s)ds = -\frac{1}{2} d_s Z_i^2(t,s), \quad i = 1,2,\dots,n. \tag{10}$$

Applying (6), (7), (8), (9), (10), integrating by parts and Dirichlet formula from (5) we have

$$\sum_{i=1}^n \int_a^b \int_a^t H_i(t,s)P_i(s)u(s)dsP_i(t)u(t)dt = -\frac{1}{2}\sum_{i=1}^n \int_a^b \left[\int_a^t H_i(t,s)d_s Z_i(t,s)\right] *$$

$$*P_i(t)u(t)dt = -\frac{1}{2}\sum_{i=1}^n \int_a^b \left[H_i(t,s)z_i(t,s)\Big|_{s=a}^{s=t} - \int_a^t \frac{\partial}{\partial s} H_i(t,s)z_i(t,s)ds \right] u(t) *$$

$$* P_i(t)dt = \frac{1}{2}\sum_{i=1}^n \int_a^b H_i(t,a)Z_i(t,a)P_i(t)u(t)dt$$

$$+\frac{1}{2}\sum_{i=1}^n \int_a^b \int_a^t \frac{\partial}{\partial s} H_i(t,s)Z_i(t,s)P_i(t)u(t) *$$

$$* dsdt = \frac{1}{4}\sum_{i=1}^n \int_a^b H_i(t,a)d_t Z_i^2(t,a) + \frac{1}{4}\sum_{i=1}^n \int_a^b \int_s^b \frac{\partial}{\partial s} H_i(t,s)d_t Z_i^2(t,s)ds =$$

$$= \frac{1}{4}\sum_{i=1}^{n}\Big[H_i(t,a)Z_i^2(b,a) - \int_a^b \frac{\partial}{\partial t}H_i(t,a)Z_i^2(t,a)dt + \int_a^b \frac{\partial}{\partial s}H_i(b,s)Z_i^2(b,s)ds - -\int_a^b\int_s^b \frac{\partial^2}{\partial t\partial s}H_i(t,s)Z_i^2(t,s)dtds\Big]$$

.

Hence, by virtue of (6) we obtain

$$\sum_{i=1}^{n}\int_a^b\int_a^t H_i(t,s)P_i(s)u(s)dsP_i(t)u(t)dt =$$

$$= \frac{1}{4}\sum_{i=1}^{n}\Big\{H_i(t,a)\left[\int_a^b P_i(s)u(s)ds\right]^2 - \int_a^b \frac{\partial}{\partial t}H_i(t,a)\left[\int_a^t P_i(s)u(s)ds\right]^2 dt +$$

$$+\int_a^b (H_i(b,s))'\left[\int_s^b P_i(s)u(s)ds\right]^2 ds - \int_a^b\int_s^b \frac{\partial^2}{\partial t\partial s}H_i(t,s)\left[\int_s^t P_i(s)u(s)ds\right]^2 dtds\Big\}.$$

(11)

Taking into account (11), from (5) we have

$$\int_a^t m(s)u^2(s)ds + \frac{1}{4}\sum_{i=1}^{n}\Bigg\{H_i(t,a)\left[\int_a^b P_i(s)u(s)ds\right]^2 -$$

$$-\int_a^b \frac{\partial}{\partial t}H_i(t,a)\left[\int_a^t P_i(s)u(s)ds\right]^2 dt + \int_a^b (H_i(b,s))'\left[\int_s^b P_i(s)u(s)ds\right]^2 ds -$$

$$-\int_a^b\int_s^b \frac{\partial^2}{\partial t\partial s}H_i(t,s)\left[\int_s^t P_i(s)u(s)ds\right]^2 dtds\Bigg\} = \int_a^b f(s)u(s)ds. \qquad (12)$$

If $f(t) = 0$ for all $t \in [a,b]$, then by virtue of the conditions a), b) and c) from (12) we get

$$u(t) = 0, \forall t\epsilon[a,b].$$

Thus the following theorem is proved.

Theorem. Let conditions a), b and c) be satisfied. Then the solution of the integral equation (1) is unique in the space C[a,b].

3. Examples

Example 3.1. Consider the integral equation (1) for $a = 0,\ b = 1,$ $m(t) = t,\ A(t,s) = \sqrt{t}(1-t)^3s^3\sqrt{s} + \sqrt[3]{t}(2+t)^{-1}s\sqrt[3]{s},$

$B(t,s) = \sqrt{t}(1--s)^3 t^3 \sqrt{s} + +\sqrt[3]{t}(2+s)^{-1} t\sqrt[3]{s}.$

Then from (2) we obtain

$m(t) = t,\ n = 2, P_1(t) = \sqrt{t}, H_1(t,s) = 2(1-t)^3 s^3, P_2(t) = \sqrt[3]{t},$

$\mathrm{H}_2(\mathrm{t},\mathrm{s}) = 2(2+t)^{-1} s.$

In this case, all the conditions of the theorem are fulfilled. Since

$$H_1(t,0) = 0,\ \left(H_1(t,0)\right)' = 0,\ \frac{\partial}{\partial s} H_1(t,s) = 6(1-t)^3 s^2,$$

$$\frac{\partial^2}{\partial t \partial s} H_1(t,s) = -18(1-t)^2 s^2, (t,s) \in G = \{(t,s): 0 \le s \le t \le 1\},$$

$$H_2(t,0) = 0, \quad (H_2(t,0))' = 0,\ \frac{\partial}{\partial s} H_2(t,s) = 2(2+t)^{-1},\ \frac{\partial^2}{\partial t \partial s} H_2(t,s) =$$

$$= -2(2+t)^{-2}, \quad (t,s) \in G = \{(t,s): 0 \le s \le t \le 1\}.$$

Example 3.2. Consider the integral equation (1) for $a = 0,\ b = 2,$
$m(t) = t(2-t),\ A(t,s) = \sin\sqrt[5]{t}\,(2-t)^5 s^2 \sin\sqrt[5]{s} + \cos\sqrt[8]{t}\,(3+t)^{-3} s^6 \cos\sqrt[8]{s}$
$, B(t,s) = 5\sin\sqrt[5]{t}\,(2-s)^5 t^2 \sin\sqrt[5]{s} + 2\cos\sqrt[8]{t}\,(3+s)^{-3} t^6 \cos\sqrt[8]{s}.$

Then from (2) we get

$m(t) = t(2-t),\ n = 2, P_1(t) = \sin\sqrt[5]{t}, H_1(t,s) = 6(2-t)^5 s^2, P_2(t) = \cos\sqrt[8]{t},$

$\mathrm{H}_2(\mathrm{t},\mathrm{s}) = 3(3+t)^{-3} s^6.$

In this case, all the conditions of the theorem are fulfilled. Since

$$H_1(t,0) = 0,\ \left(H_1(t,0)\right)' = 0,\ \frac{\partial}{\partial s} H_1(t,s) = 12(2-t)^5 s,$$

$$\frac{\partial^2}{\partial t \partial s} H_1(t,s) = -60(2-t)^4 s, \qquad (t,s) \in G = \{(t,s): 0 \le s \le t \le 2\},$$

$$H_2(t,0) = 0,$$

$$(H_2(t,0))' = 0,\ \frac{\partial}{\partial s} H_2(t,s) = 18(3+t)^{-3} s^5,\ \frac{\partial^2}{\partial t \partial s} H_2(t,s) =$$

$$= -54(3+t)^{-4} s^5, \quad (t,s) \in G = \{(t,s): 0 \le s \le t \le 2\}.$$

4. Conclusion.

The proposed methods can be used to study the uniqueness of solutions for various classes of linear integral and integro-differential equations, as well as for solving specific applied problems reduced to the integral equations of the third kind.

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THE LEVEL LINE METHOD FOR THE STUDY OF SINGULARLY PERTURBED EQUATIONS WITH ANALYTIC FUNCTIONS

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Singularly perturbed equations with analytic functions constitute a relatively underexplored area within the theory of singularly perturbed equations. To date, no unified methodology has been developed for the analysis of this class of equations. In this paper, we propose a method based on the level lines of a certain harmonic function, which plays a crucial role in the investigation of the asymptotic behavior of solutions to singularly perturbed equations. The core idea of the proposed method is demonstrated through a representative example.

Key words: Singular perturbation, analytic and harmonic functions, level lines, monotonicity, curves, asymptotic estimates.

Аналитикалык функциялары бар сингулярдуу бузулган теңдемелер сингулярдуу бузулган теңдемелердин теориясында аз изилденген аймакты билдирет. Бул класстагы теңдемелерди изилдөөнүн бирдиктүү методологиясы иштелип чыга элек. Бул эмгекте сингулярдуу бузулган теңдемелердин чечимдеринин асимптотикалык жүрүм-турумун изилдөөдө маанилүү роль ойногон белгилүү бир гармониялык функциянын деңгээл сызыктарына негизделген метод сунушталат. Сунушталган ыкманын маңызын көрсөтүү үчүн мисал колдонулат.

Урунттуу сөздөр: Сингулярдык пертурбация, аналитикалык жана гармоникалык функциялар, деңгээл сызыктары, монотондуулук, ийри сызыктар, асимптотикалык баалоо.

Сингулярно возмущенные уравнения с аналитическими функциями относятся к мало исследованной области в теории сингулярно возмущенных уравнений. Еще не разработана единая методология исследованная таких классов уравнений. В данной работе предложен метод основанный на линиях уровня некоторой гармонической

функции, которая играет существенную роль при исследовании асимптотического поведения решений сингулярно возмущенных уравнений. На одном примере описана суть предлагаемого метода.

Ключевые слова: Сингулярное возмущение, аналитические и гармонические функции, линии уровня, монотонность, кривые, асимптотические оценки.

Introduction. Let us consider the system of equations

$$\varepsilon x' = f(x,y), \tag{1}$$

$$y' = g(x,y), \tag{2}$$

where $0 < \varepsilon$ is a small real parameter; $t \in [t_0, T]$ is an interval on the real axis; $x([t_0,T] \to R^n), y([t_0,T] \to R^m)$.

Systems of the form (1), (2) are called autonomous singularly perturbed equations [1, 2]. Non-autonomous singularly perturbed equations were studied in [3, 4, 5]. In [6, 7], new phenomena and effects were discovered within the theory of singularly perturbed equations, and a theory and methodology for identifying such new phenomena and effects were developed. The investigation of systems of the form (1)–(2) has been carried out by many researchers. The most comprehensive results were obtained by L. S. Pontryagin and his students [1, 2]. He proposed to begin the study of the asymptotic behavior of solutions of system (1)–(2) depending on the nature of the stationary solutions of system (1). This system, in the terminology of L. S. Pontryagin, is called the system of fast motions. This terminology arises for the following reason. If one considers the phase velocity of system (1)–(2) in the space R^{n+m} then

$$v = \left(\frac{1}{\varepsilon} f(x,y), g(x,y)\right).$$

The first component of the phase velocity tends to infinity as ε→0, provided that $f(x,y) \neq 0$. If $f(x,y)$ vanishes or takes values comparable to ε, then the components of the phase velocity are of the same order. The function $f(x,y)$ vanishes at those points (x,y) that correspond to stationary solutions of system (1). As a rule, such solutions represent equilibrium states. L. S. Pontryagin studied system (1)–(2) under the assumption that system (1) possesses equilibrium states. The equilibrium states of system (1) form a surface Γ of dimension m.

The surface Γ contains all equilibrium states. The surface consisting of stable equilibrium states is denoted by Γ_-, while the surface consisting of unstable equilibrium states is denoted by Γ_+. The surface on which stability is lost is denoted by Γ_0. All equilibrium states are determined from the equation

$$f(x,y) = 0. \qquad (3)$$

If we set ε=0 in system (1)–(2), we obtain a degenerate system.

$$f(x,y) = 0, \qquad (4)$$

$$g(x,y) = y', \qquad (5)$$

If $x = \varphi(y)$ is a solution of system (4), then from (5), under certain additional assumptions, we can determine a solution

$$y = \varphi(t)$$

defined, for example, on the interval $t_0 \leq t \leq T$. Then the solution of the degenerate system (4)–(5) can be represented in the form

$$x = \varphi(\varphi(t)), y = \varphi(t), \qquad t_0 \leq t \leq T. \qquad (6)$$

Let $x(t,\varepsilon)$, $y(t,\varepsilon)$be a solution of system (1)–(2) satisfying the initial condition

$$x(t_0,\varepsilon) = x^0, y(t_0,\varepsilon) = y^0 \qquad (7)$$

which exists under certain general assumptions.

One may now pose the following problem: under what conditions does the solution of system (1)–(2) satisfying condition (7) converge to the solution (6) of the degenerate system?

L. S. Pontryagin solved this problem in the following way. Suppose that system (1) has an equilibrium point and that the determinant of the matrix

$$A(t) = \|\partial f(x,y)|\partial x\|$$

has one eigenvalue that vanishes at a certain value of the slow variable, while the remaining eigenvalues have negative real parts. This corresponds to the fact that on the surface Γ the equilibrium point is exponentially stable, and when the slow variable crosses the surface Γ_0, the stability of the equilibrium point is lost. For values $y \in \Gamma_+$, the equilibrium point is unstable.

In works [1], an asymptotic representation of solutions of system (1)–(2) was obtained, including the point at which the stability of the equilibrium is lost.

For the general case, it was proved that when an equilibrium point becomes unstable, the solution of system (1) rapidly departs from the emerging unstable equilibrium. In the 1970s, under the leadership of L. S. Pontryagin, a new phenomenon in the theory of autonomous singularly perturbed equations was discovered. The essence of this phenomenon is as follows: when an equilibrium point becomes unstable, the solution of the "fast subsystem" does not immediately leave the emerging unstable equilibrium, but follows it for a finite time interval and only afterward rapidly departs from it. A fast–slow system consisting of two first-order fast equations and one slow equation was constructed in [8]. This phenomenon was further generalized in [9]. In [10], a method for studying singularly perturbed equations with analytic functions under a change in the stability of equilibrium points was developed.

An analysis of the existing studies shows that, in all these works, the stability of equilibrium points is determined by one or several pairs of complex conjugate eigenvalues [11, 12]. In such cases, the determinant of the matrix $A(t)$ does not vanish at the point where the stability of the equilibrium is lost. Therefore, two cases should be considered: first, when the determinant of $A(t)$ vanishes at the points where the stability of the equilibrium is lost, and second, when the determinant of $A(t)$ does not vanish at these points.

As shown in $[13, 14, 15, 16, 17]$, in the first case a delay of the solution may occur. This happens only when the trajectory of the degenerate equation has a specific form (a so-called “duck” or *canard* trajectory).

As noted above, for the study of singularly perturbed equations possessing equilibrium points that lose stability at one or several points, a method based on the use of level lines of certain harmonic functions has been developed.

The application of this method will be demonstrated using a single model equation.

Let us consider the equation

$$\varepsilon x' = (t + i)x + b(t), \tag{8}$$

with the initial condition

$$x(t_0, \varepsilon) = x^0, \tag{9}$$

where $t_{0,} t \in D \subset C$ is a domain in the complex plane, and

$D = \{t \in C,\ |t| < r_0 \in R$ the set of real numbers, $r_0 \gg 1.\}$

Assume that the following condition holds:

(A) $b(t) \in Q(D)$ denotes the space of analytic functions in D, and $b(-i) \neq 0$.

In equation (8), setting ε=0, we obtain the degenerate equation

$(t + i)\xi + b(t) = 0,$

which has the solution

$$\xi = -\frac{b(t)}{(t + i)}. \tag{10}$$

The solution (10) of the degenerate equation has a first-order pole at the point $i = -i \in D$.

Problem. Investigate the asymptotic behavior of the solution of problem (8)–(9) in the domain D.

The peculiarity of the problem under consideration lies in the fact that the real part of the function $(t + i)$, for real values of ttt, changes sign on the interval $(-r_0, r_0)$, and the degenerate equation possesses a singularity.

In equation (8), introduce a new unknown function

$$x = z + \xi, \qquad (11)$$

where z is a new unknown function. Substituting (11) into (8), we obtain the equation

$$\varepsilon z' = (t+i)z + \varepsilon\varphi(t) \qquad (12)$$

with the initial condition

$$z(t_0, \varepsilon) = z^0, \qquad (13)$$

where

$$\varphi(t) = -\left(\frac{b(t)}{(t+i)}\right)'.$$

Problem (12)–(13) can be rewritten in the following integral form:

$$z = z^0 exp\frac{(t+i)^2 - (t_0+i)^2}{2\varepsilon} +$$

$$+\int_{t_0}^{t} \varphi(\tau) exp\frac{(t+i)^2 - (\tau+i)^2}{2\varepsilon} d\tau. \qquad (14)$$

As shown in earlier studies [8,9,10,11,12], the asymptotic behavior of expression (14) is essentially influenced by the function $Re(t+i)^2$. The following proposition (P) holds: if there exists a subdomain subset $D_0$and a set $\Omega\big(p(t_0,t)\big)$, where $p(t_0,t)$ is a smooth or piecewise smooth curve connecting the fixed point t_0 with an arbitrary point $t \in D$, such that along the curve $\big(p(t_0,t)\big)$ the function $Re(t+i)^2$ is nonincreasing, then expression (14) is bounded for $\forall t \in D_0$.

Indeed, according to (P), for all t, $\forall t,\tau \in D_0 (Re[(t+i)^2 - (t_0+i)^2] \leq 0) \wedge$
$\wedge Re[\ (t+i)^2 - (\tau+i)^2 \leq 0]$. Hence, $|z| \leq C_1$.

Thus, it follows from (P) that, in order to obtain an asymptotic estimate of (14), it is necessary to construct a domain D_0 and a set Ω.

One of the ways to construct the $\Omega(p(t_0,t))$ is to use the level lines of the function $(p(t_0,t))$. The set $\Omega(p(t_0,t))$ contains curves connecting the point t_0 with an arbitrary point $t \in D_0$, such that along the curve $(p(t_0,t))$ the function $Re(t+i)^2$ does not increase. If the level lines of $Re(t+i)^2$ are introduced, then the curve $(p(t_0,t))$ must intersect each level line only once or contain a segment of a level line. Thus, the curve $(p(t_0,t))$ It intersects the level lines, then may proceed along one of them, after which it again intersects the level lines. This process continues until the path reaches the point t.

The method under consideration resembles the descent of a traveler down a mountain slope toward the foothills. At first, the traveler moves downward (descending gradually), then follows a certain segment along a single level, after which he descends again, and so on. This process continues until the traveler reaches the foot of the mountain. Note that during the descent the traveler may choose different possible routes.

Similarly, in the choice of the path $(p(t_0,t))$, there also exist many possible variants. Below, we consider one such variant.

Let us consider the function $Re(t+i)^2 = t_1^2 - (t_2+1)^2$, $t = t_1 + it_2$ and t_1, t_2 are real variables, and examine its level lines.

Introduce the level set

$$(p_0) = \{t \in D\,, Re(t+i)^2 = 0\,\}.$$

The level lines (p_0) divide the domain D_0 into four regions, in each of which $Re(t+i)^2$ takes either negative or positive values (Fig. 1).

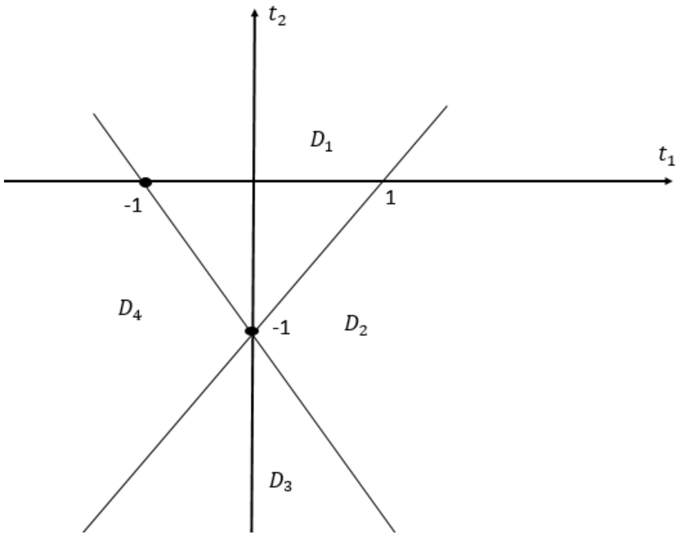

Fig. 1. Level lines (p_0) and the partition of the domain D_0.

These regions are denoted by $D_j(j = 1,2,3,4)$. Let D_1 denote the region containing the segment $[-1;1]$ of the real axis. Moving clockwise, the regions are labeled D_1, D_2, D_3, D_4 (Fig. 1).

It is easy to verify the relations

$$\forall t \in (D_1 \cup\ D_3)(Re(t+i)^2 \leq 0\ ),$$
$$\forall t \in (D_2 \cup\ D_4)(Re(t+i)^2 \geq 0\ ).$$

Let us take $t_0 < 1$ and consider the straight-line segment connecting the points $(t_0;\ 0), (0;\ -1)$. The equation of this segment has the form

$$t_1 = \frac{1}{t_0} t_1 - 1. \qquad (K_1)$$

Let us check the monotonicity of the function $Re(t+i)^2$ along the curve (K_1). We have

$$Re(t+i)^2 = t_1^2 - \frac{1}{t_0^2} t_1^2.$$

Hence,

$$\left(t_1^2 - \frac{1}{t_0^2} t_1^2\right)' = \left(1 - \frac{1}{t_0^2}\right) 2t_1.$$

Since $t_0 < 1$, it follows that $\left(1 - \frac{1}{t_0^2}\right) > 0$. Therefore, $Re(t+i)^2$ is strictly decreasing along (K_1). Now consider the segment connecting the points $(0;\ -1+\delta), (1-\delta;\ 0)$, where $0 < \delta$ is a sufficiently small number independent of ε.

Let us write the equation of this segment:

$$t_2 = t_1 - 1 + \delta. \qquad (K_2)$$

As in the previous case, we examine the monotonicity of the function $Re(t+i)^2$ along (K_2). We have

$$Re(t+i)^2 = t_1^2 - (t_1 + \delta)^2.$$

Then

$$(Re(t+i)^2)' = -2\delta < 0.$$

It follows that $Re(t+i)^2$ is strictly decreasing along (K_2).

The domain bounded by $(K_1), (K_2)$, and the segment $[t_0;\ 1-\delta]$ of the real axis is denoted by D_0 (Fig. 2).

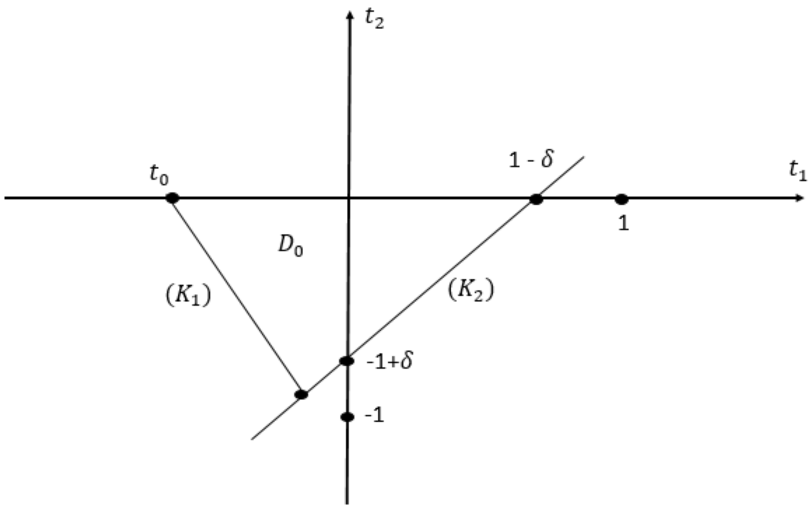

Fig. 2. The domain D_0.

It remains to construct the set $\Omega\big(p(t_0, t)\big)$. For any $\forall t \in D_0$, the path $\big(p(t_0, t)\big)$ consists of the union $(K_1) \cup (K_2)$, connecting the points $(t_0;\ 0)$ and

$\tilde{t} \in (K_1) \cup (K_2)$), and a straight-line segment (p_1) connecting the points $(\tilde{t} = t_1 + i\tilde{t}_2, t = t_1 + it_2)$.

By construction, along $(K_1) \cup (K_2)$ the function $Re(t+i)^2$ is strictly decreasing. We now check whether $Re(t+i)^2$ also decreases along (p_1)?

$(Re(\tau+i)^2)'_{\tau_2} = (t_1^2 - (\tau_2+1)^2)'_{\tau_2} = -2(\tau_2+1)$.

Since $-1 < \tau_2$ by assumption, it follows that $Re(\tau+i)^2$ is strictly decreasing along (p_1).

Thus, the set $\Omega\big(p(t_0,t)\big)$ has been constructed. According to proposition (**P**), for $\forall t \in D_0(|z| \leq c_1)$.

The obtained estimate is relatively rough and does not provide a more precise description of the asymptotic behavior of (14) in the domain D_0. To obtain more detailed information about the asymptotic behavior of (14) in D_0, we proceed as follows.

Consider the level line

$$t_1^2 - (t_2+1)^2 = (t_0 - \varepsilon ln\varepsilon)^2 - 1. \qquad (p_2)$$

This level line passes through the point $(t_0 - \varepsilon ln\varepsilon; 0)$. The curves (p_2) and (K_1) intersect at a point A_1. Denote by (K_{10}) the part of (K_1) connecting the points $(t_0;\ 0)$ and A_1, and by (p_{20}) the part of (p_2) connecting the points A_1 and $(t_0 - \varepsilon ln\varepsilon; 0)$. The domain bounded by (K_{10}), and (p_{20}) the segment $[t_0, t_0 - \varepsilon ln\varepsilon]$ of the real axis is denoted by D_{00}. Let A_2 be the intersection point of (K_1) and (K_2). Denote by (K_1) the part of (K_{11}), connecting the points A_1 and A_2, and by (K_{20}) the part of (K_2) connecting the points A_2 and $(1-\delta; 0)$.

The domain bounded by (p_{20}), the segment $[\,t_0 - \varepsilon ln\varepsilon, 1-\delta]$ of the real axis, and (K_{20}) is denoted by D_{01} (Fig. 3).

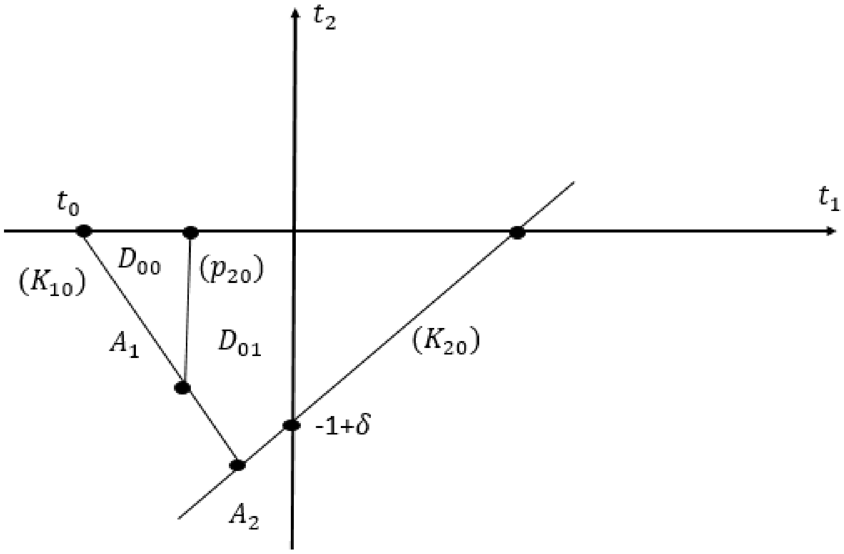

Fig. 3. The domains D_{00} and D_{01}.

Let us perform an asymptotic estimate of (14) in the domains D_{00}, D_{01}, D_{02}. We split expression (14) into two components

$$J_0(t_0,t,\varepsilon) = z^0 exp\frac{(t+i)^2-(t_0+i)^2}{2\varepsilon},$$

$$J_1(t_0,t,\varepsilon) = \int_{t_0}^{t} \varphi(\tau) exp\frac{(t+i)^2-(\tau+i)^2}{2\varepsilon} d\tau.$$

Let $t \in D_{00}$, then,

$$|J_0(t_0,t,\varepsilon)| = |z^0| exp\frac{t_1^2-(t_2+1)^2-(t_0^2-1)}{2\varepsilon}$$

Since

$$\forall t \in D_{00} (t_1^2-(t_2+1)^2-(t_0^2-1) \le 0),$$

it follows that

$$|J_0(t_0,t,\varepsilon)| = |z^0|.$$

If $\forall t \in D_{01},$ then

$$\forall t \in D_{01}\left(t_1^2-(t_2+1)^2-(t_0^2-1) \le (t_0-\varepsilon ln\varepsilon)^2-(t_0^2-1)\right)$$

Therefore

$$|J_0(t_0,t,\varepsilon)| \le c_1\varepsilon^n, n \in N.$$

Thus, $J_0(t_0,t,\varepsilon)$ is significant only in the domain D_{00}. Let us now consider $J_1(t_0,t,\varepsilon)$. Taking into account the construction of the set $\Omega(p(t_0,t))$, it is easy to verify that, since the function $\varphi(\tau)$ has no singularities in D_0, $|J_1(t_0,t,\varepsilon)| \le c_1\varepsilon, t \in D_0$.

Note that in different parts of D_0, the asymptotic behavior of the function (14) is different. According to the definitions adopted in [18], the domain D_{00} is a boundary-layer region, and

$\forall t \in D_{00}(z(t,\varepsilon) \le |z^0|)$,

the domain D_{01} is a regular region, and

$\forall t \in D_{01}(z(t,\varepsilon) \le c_1\varepsilon)$.

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GENERALIZATIONS OF THE ANALOGUE OF THE CLASSICAL GOURSAT AND DARBOUX-GOURSAT-BODOT PROBLEM IN THE CLASS OF GENERALIZED FUNCTIONS OF INFINITE ORDER

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The paper investigates the possibility of continuing solutions of systems of linear differential equations with constant coefficients, defined in the neighborhood of the faces of a parallelepiped, in the class of generalized functions of infinite order.The proven extension theorem is a significant generalization of the classical Goursat and Darboux-Goursat-Bodo problems, since it considers solutions in wider functional spaces that are closely related to spaces of analytic functionals and hyperfunction theory.

Keywords: Generalized functions of infinite order, continuation problem, Goursat problem, weakly hyperbolic operator, characteristic set, hyperfunctions.

Бул макалада чексиз тартиптеги жалпыланган функциялар классындагы параллелепипедтин беттеринин кошунасында аныкталган туруктуу коэффициенттери бар сызыктуу дифференциалдык теңдемелердин системаларына чечимдерди жайылтуу мүмкүнчүлүгү каралат. Далилденген кеңейтүү теоремасы классикалык Гурсат жана Дарбу-Гурсат-Бодо маселелерин олуттуу жалпылоо болуп саналат, анткени ал аналитикалык функциялардын жана гиперфункция теориясынын мейкиндиктери менен тыгыз байланышкан чоң функциялык мейкиндиктерде чечимдерди карайт.

Урунттуу сөздөр: Чексиз тартиптеги жалпыланган функциялар, улантуу маселеси, Гурсат маселеси, алсыз гипербола оператору, мүнөздүү көптүк, гиперфункциялар.

В работе исследуется возможность продолжения решений систем линейных дифференциальных уравнений с постоянными коэффициентами, определенных в окрестности граней параллелепипеда, в классе обобщенных функций бесконечного порядка. Доказанная теорема о продолжении является существенным обобщением классических задач Гурса и Дарбу-Гурса-Бодо, поскольку она рассматривает решения в более широких функциональных пространствах, тесно связанных с пространствами аналитических функционалов и теорией гиперфункций.

Ключевые слова: Обобщенные функции бесконечного порядка, задача продолжения, задача Гурса, слабо гиперболический оператор, характеристическое множество, гиперфункции.

Introduction

Classical problems of mathematical physics, such as the Cauchy problem or boundary value problems (including Darboux-Goursat-Bodot problems), are traditionally considered in spaces of smooth or finite-ordinal generalized functions (distributions).However, in modern research related to the theory of D-modules, hyperfunction theory, and analysis on complex manifolds, it becomes necessary to study solutions of differential equations in broader classes, namely in spaces of generalized functions of infinite order.

Let $P(D)$ - be a linear differential operator with constant coefficients in $\mathbb{R}^n$, where $D=(D_1,\ldots,D_n)$ and $D_j=-i\dfrac{\partial}{\partial\xi_j}$. We consider a homogeneous system:

$$P\left(D\right)u=0 \qquad (1)$$

wherec $P(D)=(P_{ij}\left(D\right))$ is the matrix $t\times s$ of differential operators, and $u=\left(u_1,\ldots,u_s\right)$ is the desired vector function.

Let be F compact in $\mathbb{R}^n$ and $\forall\beta$ satisfying the condition $\forall\beta$ and the whole $m>0$ by denote $S_{m,F}^{\beta,D}$ the space of infinitely differentiable functions ψ in $\mathbb{R}^n$ for which $\left|D^j\psi\left(x,y\right)\right|\le c\mathrm{J}_F\left(y\right)\exp\left(-D\left|z\right|^{\frac{1}{\beta}}\right)$ for any $j,\left|j\right|\le m$ .Consider the space : $S_F^{\beta,D}=\bigcap\limits_{m>0}S_{m,F}^{\beta,D}$.

Let's introduce a system of norms in it: $\left\|\psi\right\|_{m,F}^{\beta,D}=\max\limits_{|j|\le m}\sup\limits_{z}\dfrac{\left|D^j\psi\right|\exp\left(D\left|z\right|\right)^{\frac{1}{\beta}}}{\mathrm{J}_F\left(y\right)},m=0,1,\ldots$

Setting the task

Let be $\pi\subset\mathbb{R}^n$ a parallelepiped and π_i its $\left(n-1\right)$ -dimensional face lying in subspaces $\xi_i=0$.We assume that the operator is weakly hyperbolic with respect to the normals to the faces.

For an arbitrary linear weakly hyperbolic operator $P(D)$, we construct extensions of generalized solutions (1) defined in the neighborhood of the faces of the parallelepiped n to the neighborhood of the parallelepiped π in the class of generalized functions of infinite order.

The theorem. *Let be* $P(D)$ *a weakly hyperbolic operator , if the characteristic set* $N^{'}=char(P)$ *satisfies the condition* $N^{'}\subset\bigcup_{k=1}^{n}\{z_k=0\}$*, then there exists a number* $h<1$ *that depends only on* P *, and for any* $\forall B>0$ *neighborhood* L *of a compact* $\bigcup_{k=1}^{n}\pi_k$ *there is a neighborhood* $L^{'}$ *of a parallelepiped* π *such that any generalized* $u\in\left[\mathrm{U}_L^{\beta}\right]^s$ *solution of system (1) defined on* L *can be continued by a function* $\upsilon\in\left[\mathrm{U}_{L^{'}}^{\beta}\right]$ *that is a solution of system(1) on* $L^{'}$*, and the inequality holds:* $\|\upsilon\|_{L^{'}}^{\beta,B^{'}}\le c\|u\|_L^{\beta,B}$ *where are constants* $B^{'}$ *and* c *are independent of* u*.*

The proof is based on the fundamental result of Palamodov, see [1](chap. IV, ,theor.2) on the representation of generalized solutions in the form of linear functionals on the space of basic functions , the carrier of which lies on the characteristic space $N^{'}=char(P)$.

Proof. By definition, an arbitrary functional is a linear functional on a countably normalized space. According to Palamodov's theorem, see [1] (ch.IV, theory.2) the functionality has representations: , .and the vector measures are such that. Let us introduce generalized functions. Obviously we have where. From here. Denoting by we will have. Let us show that, then where. For any we have.

It is obvious that. By virtue of the corollary of Lemma 3.1.5 (see [3]) we have

Where. By virtue of Lemma 2.1.3 (see [3]) for any we have:

Hence, By virtue of statement a) of Lemma 3.1.6, see [3], we have.

It can be proved that there is a solution to (1) in the neighborhood, then we have

For anyone we will have:

Based on Lemma 3.1.4, see [3], we obtain th e following:

where. Generalized functions and coincide on a compact, i.e. we have.

The theorem has been proved.

Example. Consider the system:

In the domain, the Darboux-Goursat-Bodot problem requires finding a solution satisfying the edge conditions on the axes: where. The classic solution looks like this. Generalization. In the context of system (1), if we consider the operator in, the faces of and.

Our theorem states that if a generalized solution (in the OFBP class) is given in the neighborhood of the axes, then it can be continued as a generalized solution in the neighborhood of the entire quadrant.

The modern aspect: The classical solution shows that the solution is formed by the boundary. Generalization of this problem in the OFBP class means that even if the boundary conditions are extremely singular, the solution in the domain will be determined by them and will be continued correctly, while maintaining its structure, as described by the theory.

Conclusion

The proved theorem on the possibility of extending generalized solutions in spaces of infinite order is an essential step in the generalization of the classical theory of boundary value problems. The connection of these spaces with analytic functionals and hyperfunctions allows us to consider solutions of differential equations with constant coefficients in the broadest analytical classes, adapting the theory of linear operators to modern complex geometry and analysis.

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MSC 34E05

ENLARGING OF DYNAMIC SYSTEMS CLASSES WITH SPECIAL SOLUTIONS

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The purpose of the paper is unifying and enlarging the class of initial value problems for linear dynamic equations with special solutions. The peculiarity of such class is that presence of such phenomena is provided not by signs of coefficients but by absolute constants and domains of coefficients. The notion of asymptotical equivalence and the phenomenon "the dimension of the quotient space is less than one of the initial space" are also considered. These results can be applied for solutions of initial value problems for linear differential equations with bounded delay.

Key words: difference equation, delay-differential equation, special solution, initial value problem, asymptotic equivalence.

Маселенин максаты – атайын чыгарылышы болгон сызыктуу динамикалык теңдемелер үчүн баштапкы маселелеринин классын бириктирүү жана кеңейтүү. Мындай класстын өзгөчөлүгү - мындай кубулуштардын болушу коэффициенттердин белгилери менен эмес, коэффициенттердин абсолюттук константалары жана аймактары менен камсыздалат. Асимптотикалык эквиваленттик түшүнүгү жана "коэффициенттик мейкиндиктин өлчөмү баштапкы мейкиндиктин өлчөмүнөн азыраак" деген кубулуш да каралат. Бул натыйжалар чектелген кечигүү менен сызыктуу дифференциалдык теңдемелер үчүн баштапкы маселе-леринин чыгарылышы үчүн колдонулушу мүмкүн.

Урунттуу сөздөр: айырмалуу теңдеме, кечигүүчү аргументтүү дифференциалдык теңдеме, атайын чыгарылыш, баштапкы маселе, асимптотикалык эквиваленттик.

Целью статьи является объединение и расширение класса задач с начальными значениями для линейных динамических уравнений со специальными решениями. Особенность такого класса заключается в том, что наличие таких явлений определяется не знаками коэф-фициентов, а абсолютными константами и областями значений коэффициентов. Также рассматриваются понятие асимптотической эквивалентности и явление "размерность фактор-пространства меньше единицы из исходного пространства". Эти результаты могут быть применены для решения задач с начальными значениями для

линейных дифференциальных уравнений с ограниченной задержкой.

Ключевые слова: разностное уравнение, дифференциальное уравнение с запаздывающим аргументом, специальное решение, начальная задача, асимптотическая эквивалентность.

1. Introduction

The problem of asymptotic behavior (as time tends to infinity) of solutions of initial value problems is one of the main in the theory of dynamical systems.

First results are [1], [2], [3], [4], next results are [5], [6], [7].

Denote $N_0 := \{0,1, 2, 3,... \}$; $R_+ := [0, \infty)$; $R_{++} := (0, \infty)$, $h=const \in R_+$, $R_h := [-h, \infty)$.

Concerning the scalar equation with constant delay

$$z(t) = p(t)z(t-h),\ t\in R_+,\ p(t) \in C(R_+ \to p^*:= [p\_, p_+]),\ z \in C(R_h) \cap C^l(R_+) \qquad (1)$$

with the initial condition

$$z(t) = \varphi(t),\ -h\le t \le 0,\ \varphi(t) \in C([-h, 0]). \qquad (2)$$

If the function $z(t)$ has a physical dimension of length (L), then $z'(t)$ has a physical dimension of (L/T), T being time; the function $p(t)$ has a physical dimension of $(1/T)$; $p(t)h$ is a dimensionless value.

The mention results [1-4], yield the following

Theorem 1. *If*

$$|\, p^*|h<exp(-1) = 0.367... \qquad (3)$$

then there exists a positive "special" solution $Z(t)$ of the IVP (1)-(2) with "asymptotical approximation property": for each solution $z(t)$ there exists such $\gamma\in R$ that $lim\{|z(t) — \gamma Z(t)| : t \to \infty\} = 0$.

The constant exp(-1) is absolute and exact.

Other formulations of this phenomenon:

(E1) the (infinite-dimensional) space of solutions of the IVP (1)-(2) is asymptotically finite-dimensional;

(E2) any solution of the IVP (1)-(2) can be presented as a sum of a solution of a finite-dimensional space and a "fading" one.

Such properties are caused by complete continuity of the operator of shift along trajectories of solutions of (1) but it is not sufficient to prove such asymptotic.

2. Definitions of equivalence

There are many similar definitions of a dynamical system (flux) in the literature. We will use the following

Definition 1. *A dynamical system (with an initial condition) is a tuple of a number h, a totally ordered set $\Lambda \subset R$ with the least element but without the greatest one (domain of functions) (usually $\Lambda = R_h$ or $\Lambda = N_0$), a topological space Z [range of functions]; a set Φ of functions $[-h, 0] \to Z$ (initial conditions)] (if $h = 0$ then $\Phi = Z$); a function $W(t,\varphi : \Lambda \times Z$ such that its restriction on R_h equals φ [solutions of initial value problems]. If $\Lambda = R_h$ then $W(t, \varphi)$ is continuous with respect to t.*

We proposed

Definition 2. *The following equivalence is said to be asymptotical equivalence - AS (λ-exponential asymptotical equivalence, $\lambda \in R_{++}$).*

For a linear normed space Z

$(\varphi_1 \sim \varphi_2) \Leftrightarrow (\lim\{||W(t,\varphi_1) - W(t,\varphi_2)||_Z: t \to \infty\} = 0)$.

$(\varphi_1 \sim_\lambda \varphi_2) \Leftrightarrow (\sup\{ \exp (\lambda t)||W(t,\varphi_1) - W(t,\varphi_2)||_Z: t \in \Lambda\} < \infty)$.

For a metrical space Z

$(\varphi_1 \sim \varphi_2) \Leftrightarrow (\lim\{ \rho_Z(W(t,\varphi_1), W(t,\varphi_2)): t \to \infty\} = 0)$.

$(\varphi_1 \sim_\lambda \varphi_2) \Leftrightarrow (\sup\{ \exp(\lambda t) \rho_Z(W(t,\varphi_1), W(t,\varphi_2)): t \in \Lambda\} < \infty)$.

Definition 3. *The quotient space $\Phi^* := \Phi/ \sim$ of the space Φ by the AS is said to be an asymptotical quotient space - AQS; respectively, the quotient space $\Phi^*_\lambda := \Phi/ \sim_\lambda$ of the space by the λ-exponential AS is said to be λ-exponential AQS.*

The phenomenon "the dimension of the quotient space is less than one of the *ini-tial space" is said to be "asymptotical reduction of dimension of space of solutions".*

The well-known method of shift operators can be presented due to this definition as follows.

Consider the operator-difference equation for the sequence $U_n(t)\in C^{(1)}[-h,0]$, $n\in\mathbf{N}_0$:

$$U_0(t) = S_0(\varphi(\cdot))(t) := \varphi(0) + \int_{-h}^{t} p(s+h)\varphi(s)\,ds;$$

$$U_{n+1}(t) = S_n(U_n(\cdot))(t) := U_n(0) + \int_{-h}^{t} p(s+nh+h)U_n(s)\,ds, n\in\mathbf{N}_0,$$

with the final transformation

$z(t) = U_n(t+nh+h)$ $(nh\le t < nh+h)$, $n\in N_0$.

We will use another form. Add the equalities

$\varphi(t) = \varphi(0) + \int_0^t \varphi'(s)\,ds$ and $z(t) = z(0) + \int_0^t z'(s)\,ds$ to the IVP (1)-(2) and denote $x_n := z(nh)$, $y_n(t) := z'(t+nh)$, $-h\le t\le 0$, $n\in N_0$.

Then we obtain the system

$$y_{n+1}(t) = p(s+nh+h)\left(x_n + \int_0^t y_n(s)ds\right),$$

$$x_{n+1} = x_n + \int_{-h}^{t} y_{n+1}(s)ds, n\in\mathbf{N}_0, \tag{4}$$

with the inverse transformation

$$u(t) = x_n + \int_0^{t-nh} y_n(s)ds, nh\le t < nh+h, n\in\mathbf{N}_0. \tag{5}$$

3. Special solutions of operator-difference equations and method of splitting

We will expand denotations for operators onto sets of operators.

Let Ω be a normed linear space with norm $||\cdot||^*$. Consider the operator-difference equation

$$\text{Given } z_0 \in \Omega;\ z_{n+1} = F_n z_n,\ n\in N_0 \tag{6}$$

where $F_n : \Omega\to\Omega$ are linear operators, $F_n \in\Psi$, Ψ is a set of operators.

Denote the solution of (5) as $\zeta_n(z_0)$.

<u>Definition 4.</u> If there exist such connected closed set $\Omega_z \subset \Omega\backslash\{0\}$ and number $q\_ \in R_{++}$ that $\Psi\Omega_z \subset q\_\Omega_z$ then the family Ψ is said to be a *family with domination.*

<u>Lemma 1.</u> *If the family Ψ is a family with domination then there exist such $Z_0 \in \Omega$ that* $(\forall n)(\zeta_n(Z_0) \in q\_^n\Omega_z)$.

Extending the terminology [1-4] we called such solutions *special*.

<u>Definition 5.</u> If $(\forall z_0 \in \Omega)(\exists Z_0 \in \Omega_z)(\lim\{||\zeta_n(z_0) - \zeta_n(Z_0)||^*: n\to\infty\}= 0)$ then special solutions are said to be *asymptotically approximating*.

For a linear operator in a normed space denote

$||H||\_:=\sup\{\eta\in R_+: (||x||=1)\Rightarrow(||H x|| \geq \eta)\}$.

To obtain meaningful conditions we propose the following construction of *split-ting the space.*

Introduce a linear projection operator $P: \Omega\to\Omega$, $Q=I-P$, where I *is* the identity operator, Ω_1 is the kernel of operator P, Ω_2 is the image of operator P. A subscript denotes narrowing of a linear operator on the subspace Ω_1 or Ω_2 and two subscripts do corresponding operator norms. Denote variables $x_n=Pz_n\in\Omega_1$; $y_n=Qz_n\in\Omega_2$ and operators $a_n=P\ F_{n1}:\Omega_1\to\Omega_1$; $b_n=P\ F_{n2}:\Omega_2\to\Omega_1$; $c_n=Q\ F_{n1}:\Omega_1\to\Omega_2$; $d_n=QF_{n2}:\Omega_2\to\Omega_2$.

Thus the system (6) is presented as a system of two linear non-autonomous ope-rator-difference equations

$$\text{Given } x_0 \in\Omega_1,\ y_0 \in\Omega_2;\ x_{n+1} = a_nx_n + b_ny_n,\ y_{n+1} = c_nx_n + d_ny_n,\ n\in N_0 \qquad (7)$$

with the inverse transformation $z_n = x_n + y_n$, $n\in N_0$.

Denote the unit ball in Ω_1 as Ω_0.

Denote corresponding sets of operators as A, B, C, D.

Theorem 1. *If there exist such number $q\_ > 0$ and such bounded set Ω_y, $0\in \Omega_y \subset \Omega_2$, meeting the condition $(\forall\alpha \in [0,1])(\alpha\,\Omega_y \subset\Omega_y)$ that*

1) $||A||^*_{11\_} - ||B\Omega_y||^*_1 \geq q_-$; 2) $C\Omega_0 + D\Omega_y \subset q\_ \Omega_y$,

then there exists such solution $\{X_n, Y_n: n \in N_0\}$ *of (7) that*

$(\forall n \in N)(\ ||X_n||^*_1 \geq q_-^n ||X_0||^*_1;\ Y_n \subset ||X_n||^*_1 \Omega_y)$.

Proof. Let $||X_0||^*_1 = 1$, $Y_0 = 0$, then $Y_0 \subset ||X_0||^* \Omega_y$. By induction,

$||X_{n+1}||^*_1 = ||a_n X_n + b_n Y_n||^*_1 \geq ||AX_n||^*_1 - ||BY_n||^*_1 \geq$

$\geq ||A||^*_{11\_} ||X_n||^*_1 - ||B(||X_n||^*_1 \Omega_y)||^*_1 =$

$= ||X_n||^*_1 (\ ||A||^*_{11\_} - ||B\Omega_y||^*_1) \geq q\_ ||X_n||^*_1 \geq q_- \cdot q_-^n ||X_0||^*_1 = q_-^{n+1} ||X_0||^*_1$;

$Y_{n+1} = c_n X_n + d_n Y_n \in CX_n + DY_n \subset C||X_n||^*_1 (\ ||X_n||^*_1)^{-1} X_n + D||X_n||^*_1 \Omega_y =$

$= ||X_n||^*_1 (C(\ ||X_n||^*_1)^{-1} X_n + D\Omega_y) \subset ||X_n||^*_1 (C\Omega_0 + D\Omega_y) \subset ||X_n||^*_1 q\_ \Omega_y \subset$

$\subset ||X_{n+1}||^*_1 \Omega_y$.

Specify this theorem for the case $Q_1 = R$. Then x_n and a_n are scalars, $a_n \in A := [a\_, a_+] > 0$.

Theorem 2. *Let* $Q_1 = R$. *If there exist such number* $q_- > 0$ *and such bounded set* Ω_y, $0 \in \Omega_y \subset \Omega_2$, *meeting the condition* $(\forall \alpha \in [0,1])(\alpha\Omega_y \subset \Omega_y)$ *that*

1) $a_- - ||B\Omega_y||^*_1 \geq q_-$; 2) $C[-1,1] + D\Omega_y \subset q_- \Omega_y$,

then there exists such solution $\{X_n, Y_n: n \in N_0\}$ *of the system (3.2) that*

$(\forall n \in N)(X_n \geq q_-^n;\ Y_n \subset X_n \Omega_y)$.

Theorem 3. *Let* $\Omega_1 = R$; $b := ||B||^*_{21} < \infty$, $c := ||C||^*_{12} < \infty$, $d := ||D||^*_{22} < \infty$.

Denote $\beta := bc$. *If 1)* $a_- - d > 0$ *and* $(a_- - d)^2 > 4\beta$ *then then there exists such "special" solution* $\{X_n, Y_n: n \in N_0\}$ *of the system (3.2) that* $(\forall n \in N)(X_n \geq q_-^n;$ $||Y_n||^*_2 \leq wX_n)$ *where* $w = \left((a\_ - d) - \sqrt{(a\_ - d)^2 - 4\beta}\right)/(2b)$,

$q\_ = \left(a\_ + d + \sqrt{(a\_ - d)^2 - 4\beta}\right)/2$.

2) If additionally $a_+ d + \beta < q_-^2$ *then for every solution* $\{x_n, y_n: n \in N_0\}$ *of the system (3.2) there exists a limit* $\gamma(x_n, y_n) := \lim\{x_n/X_n;\ n \to \infty\}$.

3) If additionally $(a_+ d + \beta)\, q_-^{-2}\, (a_+ + bw) < 1$ *then special solutions are asymptotically approximating:* $\lim\{x_n - \gamma(x_n, y_n) X_n:\ n \to \infty\} = 0$.

Proof. *Let* $\Omega_y=\{y\in\Omega: ||y||^*_2\leq w\}$. Then the condition 2) in Theorem 3.2 can be written as $c+dw\leq q_-w$. The number w fulfills the equation $bw^2 - (a_- - d)w + c = 0$, or $c + dw = q_-w$. 1) is proven.

Denote $\Delta_n := a_nd_n - b_nc_n$. Consider and transform the difference

$$x_{n+1}/X_{n+1} - x_n/X_n = (x_{n+1}X_n - x_nX_{n+1})/(X_{n+1}X_n) =$$
$$= b_n(X_ny_n - x_nY_n)/(X_{n+1}X_n) = ,$$
$$= b_n\Delta_n(X_{n-1}y_{n-1} - x_{n-1}Y_{n-1})/(X_{n+1}X_n) =$$
$$= b_n\prod_{k=0}^{n-1}\Delta_k(X_0y_0 - x_0Y_0)/(X_{n+1}X_n).$$

Denote $\omega := (a_+d + \beta)\, q_-^{-2}$. We have

$$|x_{n+1}/X_{n+1} - x_n/X_n| \leq b\prod_{k=0}^{n-1}|\Delta_k|\cdot||X_0y_0 - x_0Y_0||_2^*/(q_-{}^{n+1}q_-{}^n) \leq$$
$$\leq b||X_0y_0 - x_0Y_0||_2^*q_-{}^{-2n-1}\prod_{k=0}^{n-1}(a_+d + \beta) =$$
$$= b||X_0y_0 - x_0Y_0||_2^*(a_+d + \beta)^nq_-{}^{-2n-1} = b||X_0y_0 - x_0Y_0||_2^*\omega^nq_-{}^{-1}.$$

Hence, the sequence $\{x_n/X_n: n\in N\}$ converges and has a limit. 2) is proven. It is well-known that for this limit $(\exists const>0)(\forall n\in N)(|\gamma(x,y) - x_n/X_n|\leq const\cdot\omega^n)$.

Further,

$$X_{n+1}\leq a_+X_n + b||Y_n||^*_2\leq a_+X_n + bwX_n = (a_+ + bw)X_n\leq(a_+ + bw)^{n+1};$$
$$|x_n - \gamma(x,y)X_n| = |\gamma(x,y) - x_n/X_n|X_n < const\cdot\omega^n(a_+ + bw)^n.$$

3) is proven. Theorem is proven.

4. Conclusion

The paper demonstrates that there exists a class of linear dynamical equations in infinite-dimensional spaces where the phenomenon of asymptotical finite-dimensionality of space of solutions takes place; there are absolute domains provid-ing this phenomenon within each type of equations.

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ONE CLASS OF SYSTEMS OF VOLTERRA LINEAR INTEGRAL EQUATIONS OF THE FIRST KIND ON A SEMI-AXIS

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In the present article the theorem about uniqueness of the linear integral equations of the first kind, with method of nonnegative quadratic forms and functional analysis methods.

Key words: linear, inteqral equations, first kind, uniqueness.

Бул макалада терс эмес квадраттык формалар усулунун, функционалдык анализдин усулдарынын жардамы менен биринчи түрдөгү сызыктуу интегралдык тендемелердин чечимдеринин жалгыздыгы далилденди.

Урунттуу сөздөр: сызыктуу, интегралдык тендемелер, биринчи түрдөгү, жалгыздык.

В данной работе, с помощью методом неотрицательных квадратичных форм, методам функционального анализа доказывается единственность решений линейных интегральных уравнений первого рода.

Ключевые слова: линейные, интегральные уравнения, первого рода, единственность.

1. Introduction

Inverse and ill-posed problems are currently attracting great interest. The theory and numerical methods for solving inverse and ill-posed problems were studied in [1-29].The notion of correctness in the works of A.N. Tikhonov [1], M.M. Lavrent'ev [1] and V.K. Ivanov [3], different from the classical one, provided a means for studying ill-posed problems and stimulated interest in integral equations with large applied value.

Ill-posed problems also occur in many other branches of mathematics, elementary examples being the Fredholm integral equation of the first kind, analytic continuation of a function, determination of the derivative of a function that is only approximately specified, and a singular linear system of algebraic equations. A further important class concerns inverse problems where it is typically required to determine the coefficients of an equation from a known ledge of certain functional of the solution.

The fundamental results for Fredholm integral equations of the first kind were obtained by M.M. Lavrentiev in [4], [5], where the regularizing operators in the sense of M.M. Lavrentiev. In [6], [7] results were obtained in the field of nonclassical Volterra equations of the first kind and numerical methods for solving nonclassical linear Volterra equations of the first kind, which describe the processes of aging and replacement of elements in developing systems. In [8], for linear Volterra integral equations of the first and the third kind with smooth kernel, the existence of multiparameter family of solutions was proved. In [9], [10], on the basis of the theory of Volterra integral equations of the first kind, various inverse problems were studied.

Modern methods for solving ill-posed problems are gaining more and more popularity. Results of the inverse problem of microtomography in [11] and numerical methods for solving ill-posed problems in [12] were obtained, inverse problems of biology and medicine [13].

In [14], [15] problems of regularization, uniqueness and of solutions for Volterra integral and operator equations of the first kind are studied.

In [16], a new approach was used to study the existence and uniqueness of solutions to scalar integral Fredholm equations of the third kind with multipoint singularities and their systems. In [17], [18], uniqueness theorems were proved and regularization operators in the sense of Lavrent'ev were constructed for systems of linear Volterra and Fredholm integral equations of the third kind.

In [19], [20] problems of uniqueness and stability of solutions for linear Fredholm integral equations of the first kind were investigated. In [21]-[24] based on the basis of the method of integral transformation, uniqueness theorems for the new class of linear integral equations of the first kind uniqueness theorems for the new class the uniqueness of solutions of linear integral equations and the system of the first kind with two independent variables in the in different areas.

The concept of a derivative with respect to an increasing function was introduced by A. Asanov in 2001 in [25] and plays a special role in the study. This notion is a generalization of the usual notion of a derivative of a function and is an inverse operator for one class of the Stieltjes integral. Based on the concept of such a derivative, using the method of integral transformations and the method of non-negative quadratic forms, linear integral Stieltjes equations of the first kind were studied in [26], [27]. In [28], [29] uniqueness theorems for solutions of the Fredholm and Volterra linear integral equations of the first kind on the semi-axis are proved.

In this paper, using functional analysis and the method of nonnegative quadratic forms, we prove a uniqueness theorem for one class of systems of linear Volterra integral equations of the first kind on the semi-axis.

Consider of the systems of linear Volterra integral equations

$$Ku \equiv \int_{-\infty}^{t} K(t,s)\ u\ (s)ds = f(t),\ t \in (-\infty, a], \qquad (1)$$

$$\int_{-\infty}^{a}\int_{-\infty}^{t} |A(t,s)|^2 dsdt < \infty,$$

$A(t,s) = a_{ij}(t,s)$

$$f(t) = \big(f_i(t)\big) = (f_1(t), \dots, f_n(t))^T, u(t) = \big(u_i(t)\big) = (u_1(t), \dots, u_n(t))^T.$$

Here $A(t,s)$ is a given matrix function, $f(t)$ is a known vector function, and $u(t)$ is an unknown vector function.

Let us introduce the following notations:

1. For vectors $u = (u_1, u_2, \dots, u_n),\ \vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_n) \in R^n$

define the dot product by the equality

$\langle u, \vartheta\rangle = \sum_{i=1}^{n} u_i\, \vartheta_i,$ norm - equality $\|u\| = \langle u, u\rangle^{1/2}$.

1. For $A = (a_{ij}) - n \times n -$ square matrix define the norm

$$\|A\| = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \left|a_{ij}\right|^2\right)^{\frac{1}{2}}.$$

2. Through $L_{2,n}(-\infty, t)$ denote the space of all n- dimensional vector functions $u(t) = (u_1(t), \ldots, u_n(t))$ meeting a condition $u_i(t) \in L_2(-\infty, t]$ for all $i = 1,2, \ldots, g = n$.

3. For $u(t) = (u_1(t), \ldots, u_n(t)) \in L_{2,n}(-\infty, t]$ define the norm

$$\|u(t)\|_{L_{2,n}} = \left(\int_a^b \|u(t)\|^2 dt\right)^{\frac{1}{2}}.$$

We will assume that

$\|K(t,s)\| \in L_2(-\infty, t] \times (-\infty, t]),\quad \|f(t)\| \in L_2(-\infty, t]$.

Both parts of (1) are scalar multiplied by the vector function $u(t)$ and integrating by area $(-\infty, a]$, let's receive

$$\int_{-\infty}^{a} \int_{-\infty}^{t} \langle K(t,s)u(s), u(t)\rangle\, ds dt = \int_{-\infty}^{a} \langle f(t), u(t)\rangle\, dt, \quad (2)$$

We introduce a new matrix function $M(t,s) = (M_{ij}(t,s))$ as follows

$$M(t,s) = \begin{cases} K(t,s), & -\infty < s \le t \le a, \\ K(s,t), & -\infty < t \le s \le a, \end{cases} \qquad (3)$$

$K(t,s) = (K_{ij}(t,s), \qquad K(s,t) = (K_{ij}(s,t))$.

It is clear that $M(t,s) = M(s,t)$, $(t,s) \in (-\infty, a] \times (-\infty, a]$ and fair

$$\int_{-\infty}^{a} \int_{-\infty}^{a} \|M(t,s)\|^2 ds dt < +\infty.$$

Then it is known that

$$M(t,s)=\sum_{v=1}^{m}\lambda_v\begin{pmatrix}\varphi_1^{(v)}(t)\\ \dots\\ \varphi_n^{(v)}(t)\end{pmatrix}(\overline{\varphi}_1^{(v)}(s)\dots\overline{\varphi}_n^{(v)}(s)),\, m\le\infty, \qquad (8)$$

where

$\lambda_v -$ the eigenvalues of the matrix kernel $M(t,s)$, arranged in descending order of their modules, and the vector function

$\varphi^{(v)}(t)=(\varphi_1{}^{(v)})(t),\dots,\varphi_n{}^{(v)}(t))-$ proper orthonormal vector functions corresponding to eigenvalues λ_v.

In the future, we assume that all eigenvalues λ_v the matrix kernel $M(t,s)$ are positive. Due to the complete continuity and self-adjoint of the operator M generated by the matrix kernel $M(t,s)$, the orthonormal sequence of eigenvector functions is complete in $L_2(-\infty,a]$.

For $u(t)=(u_i(t)\in L_2(-\infty,a]$ define the norm

$$\|u(t)\|_2=\left(\sum_{i=1}^{n}\int_{-\infty}^{a}|u_i(t)|^2dt\right)^{\frac{1}{2}}=\left(\int_{-\infty}^{a}\|u(t)\|^2dt\right)^{\frac{1}{2}}=\left(\sum_{v=1}^{\infty}\left|u^{(v)}\right|^2\right)^{\frac{1}{2}},$$

where

$$u^{(v)}=\int_{-\infty}^{a}\langle u(t),\varphi^{(v)}(t)\rangle dt=\int_{-\infty}^{a}\left(\sum_{i=1}^{n}u_i(t)\,\varphi_i^{(v)}(t)\right)dt, \qquad (v=1,2,\dots).$$

Theorem. Suppose $M(t,s)$ is a complete kernel and $0<\lambda_1\le\lambda_2\le\dots$. Then the solution of the equation (1) is unique in $L_{2,n}(-\infty,a]$.

Proof. Let the system of integral equations (1) have a nonzero solution $u(t)\in L_{2,n}(-\infty,a]$ for $f(t)\equiv 0$. Then

$$\int_{-\infty}^{t}K(t,s)u(s)ds=f(t)\equiv 0, \quad for\ all\ t\in(-\infty,a].$$

Hence by virtue of (6) we have

$$\int_{-\infty}^{a}\int_{-\infty}^{t}\langle K(t,s)u(s),u(t)\rangle\,dsdt=0 \qquad (9)$$

Taking into account (7) and (8) of (9), we obtain

$$\int_{-\infty}^{a}\int_{-\infty}^{t}\sum_{v=1}^{\infty}\lambda_v\left\langle\begin{pmatrix}\varphi_1^{(v)}(t)\\ \cdots\cdots\\ \varphi_n^{(v)}(t)\end{pmatrix}(\varphi_1^{(v)}(s)\ldots\varphi_n^{(v)}(s))\begin{pmatrix}u_1(s)\\ \cdots\\ u_n(s)\end{pmatrix},u(t)\right\rangle dsdt=0,$$

$$\sum_{v=1}^{\infty}2\int_{-\infty}^{a}\left[\int_{-\infty}^{t}\langle u(s),\varphi^{(v)}(s)\rangle ds\right]\langle u(t),\varphi^{(v)}(t)\rangle dt=0.$$

From here

$$\sum_{v=1}^{\infty}\lambda_v\left|\int_{-\infty}^{a}\langle u(t),\varphi^{(v)}(t)\rangle dt\right|^2=0,$$

obtain

$$\int_{-\infty}^{a}\langle u(t),\varphi^{(v)}(t)\rangle dt=0.$$

Therefore, $u(t)=0$, for all $t\in(-\infty,a]$. The theorem is proven.

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ON HUTCHINSON POPULATION MODEL FOR DIFFERENTIAL EQUATIONS WITH MAXIMA AND JUMPS

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The article is devoted to study G. E. Hutchinson population model for functional-differential equations. However, the functional-differential equation with maxima we consider as a mathematical model of population dynamics of a species. The equation of population model we consider with initial value condition. The method of steps is used. In general, the impulsive nonlinear functional-differential equation with maxima is considered under the boundary conditions. The method of successive approximations is used with combination of the contracting mapping method.

Keywords: G. E. Hutchinson population model, boundary problem, impulsive functional-differential equation, solvability.

Бул макала Г.Е. Хатчинсондун популяция моделин максималдуу функционалдык дифференциалдык теңдемелер үчүн колдонууга арналган. Бирок, максималдуу сызыктуу эмес функционалдуу дифференциалдык теңдеме түрдүн популяциясынын динамикасынын математикалык модели катары каралат. Популяция моделинин теңдемеси баштапкы шарттар менен каралат. Кадамдоо ыкмасы колдонулат. Жалпы учурда, чектик шарттарда максималдуу импульстук сызыктуу эмес функциялык дифференциалдык теңдеме каралат. Удаалаш жакындатуу ыкмасы менен айкалышта элесин кысуу ыкмасы колдонулат.

Урунттуу сөздөр: Г.Э. Хатчинсондун популяциялык модели, чектик маселе, импульсивдүү функционалдык дифференциалдык теңдеме, чечилиши.

Статья посвящена изучению модели популяции Г. Э. Хатчинсона для функционально-дифференциальных уравнений с максимумами. Однако, в качестве математической модели динамики популяции вида рассматривается нелинейное функционально-дифференциальное уравнение с максимумами. Уравнение модели популяции рассматривается с начальными условиями. Используется метод шагов. В общем случае, рассматривается импульсное нелинейное функционально-дифференциальное уравнение с максимумами при граничных условиях. Используется метод последовательных приближений в сочетании с методом сжимающих отображений.

Ключевые слова: Популяционная модель Г. Э. Хатчинсона, краевая задача, импульсивное функционально-дифференциальное уравнение, разрешимость.

Introduction. Statement of the problem

In many studies, differential equations are a tool for mathematical modeling of processes and studying their properties (see, for example, [1]-[4] and see also [5]-[13]). Mathematical modelling in ecology has a long history leading back to the famous works of Malthus [9], Quetelet [10] and Verhulst [11] on principals of demographic regulation. In the first part of the 20th century, the classical works of Lotka [8] and Volterra [12] on dynamics of interacting species radically transformed the entire idea of modelling in ecology. Their works (for a short history of the implementation of mathematics in ecology, see [13]) eventually resulted in the emergence of theoretical ecology as a new science. Note that the questions on the main mechanisms of populations regulation addressed by Lotka and Volterra are still in the focus of theoretical ecology! In partially, to describe the population dynamics of a species, it was proposed a mathematical model (see, [1], [3])

$$\dot{x}(t)+rx(t)+rx^2(t))=0, \quad t\geq 0, \tag{1}$$

where $x(t)=N(t)+K$, $N(t)$ is the population size of the species, K is the capacity of the habitat, r is the reproduction rate.

Equation (1) with initial data $x(0)=x_0$ has the following properties:

1). $N(t)>0$ for $x_0(t)>0$;

2). $\lim\limits_{t\to\infty} N(t)=K$;

3). The function $N(t)$ is increasing for $0<x_0<K$, and is decreasing for $x_0>K$;

4). The equilibrium position $N(t)\equiv K$ is globally asymptotically stable.

Equation (1) describes well the dynamics of population growth of protozoan microorganisms. In this regard, in 1948, G. E. Hutchinson proposed a model described by a differential equation with delay [5], [6]

$$\dot{x}(t)+rx(t-h)+rx(t-h)x(t)=0, \quad t\geq 0, \tag{2}$$

where $x(t)=N(t)+K$, h characterizes the average reproductive age of the species.

Hutchinson used equation (2) to describe changes in the size of a herd of cows on a pasture. However, equation (2) has been used by other scientists to describe other processes (see, for examples, [2]). Let us consider the equation (2) with the initial condition $x(t)=\varphi(t)$, $t\in[-h,0]$, $x(0)=\varphi_0$. For small $r>0$ the equation (2) has a stability domain. According to this model, asymptotic stability of the equilibrium position occurs when $N(t)\equiv K$, $t\geq 0$, $0<rh<\pi/2$. It should be noted that periodic fluctuations appear in the mathematical model for $rh>\pi/2$. Consequently, in Hutchinson's model, the reproductive rate r and the average reproductive age h of the species play a fundamental role.

In the present paper we propose the following model to describe the population dynamics of the species

$$\dot{x}(t)+r\max_{\tau\in[t-h,t]}x(\tau)+r\,x(t)\max_{\tau\in[t-h,t]}x(\tau)=0, \quad t\geq 0, \tag{3}$$

where $x(t)=N(t)+K$, τ is a variable value that provides the maximum number of species on the segment $[t-h,t]$, h characterizes the size of the reproductive age of the species, r is reproduction rate.

Theoretically, differential equations are considered, in partially, in [14]-[18].

The increasing solutions of the differential equations (3) we are looking in the form

$$x(t)=e^{\lambda t}, \quad 0<\lambda=const.$$

Then from the equation (3) we obtain

$$\lambda+r+r\,e^{\lambda t}=0, \quad t\geq 0.$$

This equation we rewrite as

$$\lambda+r+r\,(N(t)+K)=0, \quad t\geq 0.$$

Hence, we obtain that

$$N(t) = -\frac{\lambda + r}{r} - K, \quad t \geq 0,$$

where $N(t)$ is the population size of the species, K is the capacity of the habitat, $r \neq 0$ is the reproduction rate.

It is interesting that

$$-\frac{\lambda + r}{r} - K > 0 \quad \text{or} \quad rK + r < -\lambda.$$

Hence, we get the condition $\lambda < -r(K+1)$ for the increasing solution of the differential equation (3). The solution of the equation (3) depends from the $r \neq 0$. It is clear, that $\lim_{r \to 0} x(t) = 0.$ Since $r > 0,\ K > 0,$ then the equation (3) has no monotone increasing solution.

The decreasing solutions of the differential equations (3) we are looking in the form

$$x(t) = e^{-\lambda t}, \quad 0 < \lambda = const.$$

Then from the equation (3) we obtain

$$-\lambda + re^{\lambda h} + re^{\lambda h}(N(t) + K) = 0, \quad t \geq 0.$$

Hence, we obtain that

$$N(t) = \frac{\lambda - re^{\lambda h}}{re^{\lambda h}} - K, \quad t \geq 0.$$

We consider the following estimate

$$\frac{\lambda - re^{\lambda h}}{re^{\lambda h}} - K > 0 \quad \text{or} \quad re^{\lambda h}K + re^{\lambda h} < \lambda.$$

Hence, we get the condition $0 < r < \lambda / e^{-\lambda h}(K+1)$ for the decreasing solution of the differential equation (3). The solution of the equation (3) depends from the $r \neq 0$. It is clear, that $\lim_{r \to 0} x(t) = 0.$ Consequently, for all $\lambda > re^{\lambda h}(K+1)$ the solutions of differential equation (3) are decreasing. In this case, the population size $N(t)$ of the species is decreasing.

Thus, monotonicity of population size $N(t)$ change is true only in the case of population decline. An increasing population size $N(t)$ is associated with non-monotonicity (oscillatory). From the condition $\lambda > re^{\lambda h}(K+1)$ we obtain the condition for reproductive age of the species $h < \left(\frac{\lambda}{r(K+1)}\right)^{\frac{1}{\lambda}}$. Hence, we deduce that population size $N(t)$ is decreasing, when reproductive age h of the species is smaller than $\left(\frac{\lambda}{r(K+1)}\right)^{\frac{1}{\lambda}}$.

In general, to solve the equation (3) we use the method of steps with initial value conditions

$$x(t) = \varphi(t), \quad t \in [t-h, t], \quad x(0) = \varphi(0) = \varphi_0.$$

In general, we suppose that the solution at every point of the beginning every interval is continuous:

$$x^{+}((n-1)h) = x^{-}((n-1)h).$$

Zero step. On the initial interval $[-h,0]$ the solution has the form

$$x(t) = \varphi(t) = \varphi_0(t). \tag{4}$$

First step. On the interval $[0,h]$ we have $\max\limits_{\tau\in[t-h,t]} x(\tau) = \max\limits_{\tau\in[t-h,t]} \varphi_0(\tau)$. We consider the equation (3) with condition $x(0) = \varphi_0(0)$. Then we have

$$\dot{x}(t) = -r \max_{\tau\in[t-h,t]} \varphi_0(\tau)(1+x(t)), \quad t \in [0,h].$$

By integrating last equation, we obtain

$$x(t) = \varphi_1(t) = \varphi_0(0) - r\int_0^t H_1(s)(1+x(s))ds,$$

where $H_1(s) = \max\limits_{\tau\in[s-h,s]} \varphi_0(\tau)$.

So, on the interval $[0,h]$ we have solution of the equation

$$x(t) = \varphi_1(t), \quad t \in [0,h]. \tag{5}$$

Second step. On the interval $[h,2h]$ we have $\max\limits_{\tau\in[t-h,t]} x(\tau)=\max\limits_{\tau\in[t-h,t]}\varphi_1(\tau)$. We consider the condition $x(h)=\varphi_1(h)$. Then, by integrating the equation

$$\dot{x}(t)=-r\max_{\tau\in[t-h,t]}\varphi_1(\tau)(1+x(t)),\quad t\in[h,2h],$$

we obtain

$$x(t)=\varphi_2(t)=\varphi_1(h)-r\int_h^t H_2(s)(1+x(s))ds,$$

where $H_2(s)=\max\limits_{\tau\in[s-h,s]}\varphi_1(\tau)$.

So, on the interval $[h,2h]$ we have solution of the equation (3)

$$x(t)=\varphi_2(t),\quad t\in[h,2h]. \tag{6}$$

Third step. On the interval $[2h,3h]$ we have $\max\limits_{\tau\in[t-h,t]} x(\tau)=\max\limits_{\tau\in[t-h,t]}\varphi_2(\tau)$. Then, by integrating the equation

$$\dot{x}(t)=-r\max_{\tau\in[t-h,t]}\varphi_2(\tau)(1+x(t)),\quad t\in[2h,3h]$$

with the condition $x(2h)=\varphi_2(2h)$, we obtain

$$x(t)=\varphi_3(t)=\varphi_2(2h)-r\int_{2h}^t H_3(s)(1+x(s))ds,$$

where $H_3(s)=\max\limits_{\tau\in[s-h,s]}\varphi_2(\tau)$.

So, on the interval $[2h,3h]$ we have solution of the equation (3)

$$x(t)=\varphi_3(t),\quad t\in[2h,3h]. \tag{7}$$

$k-$**step.** Continuing the process (4)-(7) and integrating the following equation

$$\dot{x}(t)=-r\max_{\tau\in[t-h,t]}\varphi_{k-1}(\tau)(1+x(t)),\quad t\in[(k-1)h,kh]$$

with condition $x((k-1)h)=\varphi_{k-1}((k-1)h)$, we obtain

$$x(t)=\varphi_k(t)=\varphi_{k-1}((k-1)h)-r\int\limits_{(k-1)h}^{t}H_k(s)(1+x(s))ds,$$

where $H_k(s)=\max\limits_{\tau\in[s-h,s]}\varphi_{k-1}(\tau)$.

So, on the interval $[(k-1)h,kh]$ we have solution of the equation (3)

$$x(t)=\varphi_k(t),\quad t\in[(k-1)h,kh]. \tag{8}$$

Thus, by the formulas (4)-(8) we can find the solution of the equation (3) with corresponding initial conditions.

If in the model (3) we take into account the heterogeneous habitat, migration factors associated with the amount of available food, and the influence of predators, then the mathematical model takes on a more complex form

$$\dot{x}(t)+r\max_{\tau\in[t-h,t]}x(\tau)+rx(t)\max_{\tau\in[t-h,t]}x(\tau)=f(t,x(t)),\ \ t\in[0,T],\ t\neq t_i,\ \ i=1,2,...,p, \tag{9}$$

where f is an external disturbing influence on the population dynamics of a species.

We will consider impulsive equation (9) with the following conditions

$$x(t)=\varphi(t),\ \ t\in[-h,0],\quad x(0)+x(T)=\varphi_0. \tag{10}$$

We integrate the initial value problem (9), (10) with impulsive conditions

$$x\left(t_i^+\right)-x\left(t_i^-\right)=I_i\left(x\left(t_i\right)\right),\qquad i=1,2,...,p,$$

and obtain

$$x(t)=x(0)+\int\limits_0^t\left[-r\max_{\tau\in[s-h,s]}x(\tau)-rx(s)\max_{\tau\in[s-h,s]}x(\tau)+f(s,x(s))\right]ds+\sum_{0<t_i<t}I_i\left(x\left(t_i\right)\right). \tag{11}$$

Taking into account condition (2), from equation (11) we obtain

$$x(0)=\frac{\varphi_0}{2}-\frac{1}{2}\int\limits_0^T\left[-r\max_{\tau\in[s-h,s]}x(\tau)-rx(s)\max_{\tau\in[s-h,s]}x(\tau)+f(s,x(s))\right]ds+\frac{1}{2}\sum_{0<t_i<T}I_i\left(x\left(t_i\right)\right). \tag{12}$$

Substituting (12) into equation (11), we obtain the nonlinear Fredholm integral equation

$$x(t) = J(t; x(t)) \equiv \frac{\varphi_0}{2} +$$

$$+\frac{K_0}{2}\int_0^T \left[-r \max_{\tau\in[s-h,s]} x(\tau) - rx(s) \max_{\tau\in[s-h,s]} x(\tau) + f(s, x(s)) \right] ds + \frac{K_0}{2} \sum_{0<t_i<T} I_i\left(x(t_i)\right), \quad (13)$$

where

$$K_0 = \begin{cases} -\frac{1}{2}, & t \le s \le T, \\ \frac{1}{2}, & 0 \le s < t. \end{cases}$$

We use a Banach space $C\left([0,T], R\right)$, that contains a function $u(t)$, defined and continuous on a segment $[0,T]$, with the norm

$$\| u(t) \|_{C[0,T]} = \max_{0\le t\le T} | u(t) |.$$

By the aid of $PC\left([0,T], R\right)$ we denoted

$$PC\left([0,T], R\right) = \left\{ x : [0,T] \to R;\ x(t) \in C\left((t_i, t_{i+1}], R\right),\ i = 1, \ldots, p \right\},$$

$x\left(t_i^+\right)$ and $x\left(t_i^-\right)$, $i = 0, 1, \ldots, p$, are exist and finite; $x\left(t_i^-\right) = x\left(t_i\right)$. It is easy to see that the linear space $PC\left([0,T], R\right)$ is Banach space with norm

$$\| x(t) \|_{PC} = \max \left\{ \|x\|_{C((t_i, t_{i+1}])},\ i = 1, 2, \ldots, p \right\}.$$

Let us also consider the Banach space

$$BD\left([0,T], R\right) = \left\{ u : [0,T] \to R;\ u(t) \in C\left([0,T], R\right) \right\},$$

in which the function exists and is bounded by the norm

$$\| u(t) \|_{BD[0,T]} = \| u(t) \|_{PC[0,T]} + h \| \dot{u}(t) \|_{PC[0,T]},$$

where $0 < h = const$.

Lemma [14], [17]. For the difference of two functions with maxima there holds the following estimate

$$\left\| \max_{\tau\in[t-h,t]} x(\tau) - \max_{\tau\in[t-h,t]} y(\tau) \right\|_{C[0,T]} \le \| x(t) - y(t) \|_{C[0,T]} + h \| \dot{x}(t) - \dot{y}(t) \|_{C[0,T]}.$$

Theorem. Let be fulfilled the following conditions:

1). $M_f = \| f(t,x) \| < \infty,\ 0 < M_f = const;$

2). $M_I = \max_i \left\| I_i\left(x(t_i)\right) \right\| < \infty,\ 0 < M_I = const;$

3). $| f(t,x_1) - f(t,x_2) | \le L_f | x_1 - x_2 |,\ 0 < L_f = const;$

4). $| I_i(t,x_1) - I_i(t,x_2) | \le L_{I_i} | x_1 - x_2 |,\ 0 < L_{I_i} = const;$

5). $\rho_i = \max\{\rho_{i1}; h\rho_{i2}\} < 1,\ i = 1,2,\dots,n-1.$

Then for the equation (13) has a unique solution in the class $BD([0,T],R)$, which can be found by following iteration process:

$$x_0(t) = \frac{\varphi_0}{2}, \quad x_n(t) = J(t; x_{n-1}(t)),\ n = 1,2,3,\dots \tag{14}$$

For the first approximation from (13) and (14) we have

$$\| x_1(t) \| \le \frac{|\varphi_0|}{2} + \sum_{i=1}^{p} \left\| I_i\left(x\left(t_i\right)\right) \right\| +$$

$$+\frac{1}{4}\int_0^T \left[|r| \left\| \max_{\tau\in[s-h,s]} x_0(\tau) \right\| + |r| \| x_0(s) \| \left\| \max_{\tau\in[s-h,s]} x_0(\tau) \right\| + \| f(s, x_0(s)) \| \right] ds \le$$

$$\le \frac{|\varphi_0|}{2} + \frac{p}{4} M_I +$$

$$+\frac{T}{4}\left[|r| \max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + |r| \frac{|\varphi_0|}{2} \max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + M_f \right] =$$

$$= \frac{|\varphi_0|}{2} + \rho_{01} < \infty, \tag{15}$$

where

$$\rho_{01} = |r| \frac{T}{4} \max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + |r| \frac{T}{4} \frac{|\varphi_0|}{2} \max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + \frac{T}{4} M_f + \frac{p}{4} M_I .$$

Taking into account estimate (15) from (14) we have estimates

$$\| x_1(t) - x_0(t) \| \le \sum_{i=1}^{p} \left\| I_i\left(x\left(t_i\right)\right) \right\| +$$

$$+\frac{1}{4}\int_0^T \left[|r| \left\| \max_{\tau\in[s-h,s]} x_0(\tau) \right\| + |r| \| x_0(s) \| \left\| \max_{\tau\in[s-h,s]} x_0(\tau) \right\| + \| f(s, x_0(s)) \| \right] ds \le \rho_{01}. \tag{16}$$

Analogously, from (9) we have

$$\| \dot{x}_1(t) - \dot{x}_0(t) \| \le$$

$$\le |r| \left\| \max_{\tau\in[s-h,s]} x_0(\tau) \right\| + |r| \| x_0(s) \| \left\| \max_{\tau\in[s-h,s]} x_0(\tau) \right\| + \| f(s, x_0(s)) \| \le$$

$$\le |r| \max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + |r| \frac{|\varphi_0|}{2} \max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + M_f = \rho_{02}. \tag{17}$$

From (16) and (17) we obtain that there holds the estimate

$$\| x_1(t) - x_0(t) \|_{BD[0,T]} \le \rho_{01} + h\rho_{02}.$$

For the second difference we have

$$\| x_2(t) - x_1(t) \| \le \frac{1}{4} \sum_{i=1}^{p} L_{I_i} | x_1(t_i) - x_0(t_i) | + \frac{T|r|}{4} \left\| \max_{\tau\in[t-h,t]} x_1(\tau) - \max_{\tau\in[t-h,t]} x_0(\tau) \right\| +$$

$$+ \frac{T|r|}{4} \| x_1(t) \| \left\| \max_{\tau\in[t-h,t]} x_1(\tau) - \max_{\tau\in[t-h,t]} x_0(\tau) \right\| +$$

$$+ \frac{T|r|}{4} \| x_1(t) - x_0(t) \| \left\| \max_{\tau\in[t-h,t]} x_0(\tau) \right\| + \frac{T}{4} L_F \| x_1(t) - x_0(t) \|.$$

We apply the lemma to the last inequality:

$$\| x_2(t) - x_1(t) \| \le \frac{T|r|}{4} \left(\| x_1(t) - x_0(t) \| + h \| \dot{x}_1(t) - \dot{x}_0(t) \| \right) +$$

$$+ \frac{T|r|}{4} \left(\frac{|\varphi_0|}{2} + \rho_{01} \right) \left(\| x_1(t) - x_0(t) \| + h \| \dot{x}_1(t) - \dot{x}_0(t) \| \right) +$$

$$+ \frac{T|r|}{4} \| x_1(t) - x_0(t) \| \max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} +$$

$$+ \left[\frac{T}{4} L_f + \frac{1}{4} \sum_{i=1}^{p} L_{I_i} \right] \| x_1(t) - x_0(t) \|.$$

Hence, we obtain

$$\| x_2(t) - x_1(t) \| \le \chi_{11} \| x_1(t) - x_0(t) \| + \chi_{12} h \| \dot{x}_1(t) - \dot{x}_0(t) \|, \tag{18}$$

where

$$\chi_{11}=|r|\frac{T}{4}\left(1+\left(\frac{|\varphi_0|}{2}+\rho_{01}\right)+\max\left\{\|\varphi(t)\|,\frac{|\varphi_0|}{2}\right\}\right)+\frac{T}{4}L_f+\frac{1}{4}\sum_{i=1}^{p}L_{I_i},$$

$$\chi_{12}=\frac{T|r|}{4}\left(1+\frac{|\varphi_0|}{2}+\rho_{01}\right).$$

Since $\chi_{11}\ge\chi_{12}$, the estimate (18) we rewrite as

$$\|x_2(t)-x_1(t)\|\le\rho_{11}\left(\|x_1(t)-x_0(t)\|+h\|\dot{x}_1(t)-\dot{x}_0(t)\|\right)\le$$

$$\le\rho_{11}\left(\rho_{01}+h\rho_{02}\right), \tag{19}$$

where $\rho_{11}=\chi_{11}$.

Analogously we have

$$\|\dot{x}_2(t)-\dot{x}_1(t)\|\le|r|\left(\|x_1(t)-x_0(t)\|+h\|\dot{x}_1(t)-\dot{x}_0(t)\|\right)+$$

$$+|r|\left(\frac{|\varphi_0|}{2}+\rho_{01}\right)\left(\|x_1(t)-x_0(t)\|+h\|\dot{x}_1(t)-\dot{x}_0(t)\|\right)+$$

$$+|r|\|x_1(t)-x_0(t)\|\max\left\{\|\varphi(t)\|,\frac{|\varphi_0|}{2}\right\}+\frac{1}{4}L_f\|x_1(t)-x_0(t)\|\le$$

$$\le\chi_{21}\|x_1(t)-x_0(t)\|+\chi_{22}h\|\dot{x}_1(t)-\dot{x}_0(t)\|,$$

$$\chi_{21}=|r|\left(1+\left(\frac{|\varphi_0|}{2}+\rho_{01}\right)+\max\left\{\|\varphi(t)\|,\frac{|\varphi_0|}{2}\right\}\right)+\frac{1}{4}L_f+\frac{1}{4}\sum_{i=1}^{p}L_{I_i},$$

$$\chi_{22}=|r|\left(1+\frac{|\varphi_0|}{2}+\rho_{01}\right).$$

From last estimate we get

$$\|\dot{x}_2(t)-\dot{x}_1(t)\|_{BD[0,T]}\le\rho_{12}\left(\|x_1(t)-x_0(t)\|+h\|\dot{x}_1(t)-\dot{x}_0(t)\|\right)\le$$

$$\le\rho_{12}\left(\rho_{01}+h\rho_{02}\right), \tag{20}$$

where $\rho_{12}=\chi_{21}$.

From (19) and (20) we obtain that there hold the estimates

$$\| x_2(t) - x_1(t) \|_{BD[0,T]} \le \rho_1 \| x_1(t) - x_0(t) \|_{BD[0,T]},$$

where $\rho_1 = \max\{\rho_{11}; h\rho_{12}\} < 1;$

$$\| x_2(t) - x_1(t) \|_{BD[0,T]} \le \rho_{11}\left(\rho_{01} + h\rho_{02}\right) + h\rho_{12}\left(\rho_{01} + h\rho_{02}\right) =$$

$$= \left(\rho_{11} + h\rho_{12}\right)\left(\rho_{01} + h\rho_{02}\right). \tag{21}$$

We need in the following estimate

$$\| x_2(t) \| \le \frac{|\varphi_0|}{2} +$$

$$+ \frac{1}{4}\int_0^T \left[|r| \left\| \max_{\tau\in[s-h,s]} x_1(\tau) \right\| + |r| \| x_1(s) \| \left\| \max_{\tau\in[s-h,s]} x_1(\tau) \right\| + \| f(s, x_1(s)) \| \right] ds \le$$

$$\le \frac{|\varphi_0|}{2} + |r| \frac{T}{4} \left(\max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + \rho_{01} \right) +$$

$$+ \frac{T}{4} |r| \left(\frac{|\varphi_0|}{2} + \rho_{01} \right) \left(\max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + \rho_{01} \right) + \frac{T}{4} M_f + \frac{p}{4} M_I =$$

$$\le \frac{|\varphi_0|}{2} + |r| \frac{T}{4} \left(\max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + \rho_{01} \right) \left(1 + \frac{|\varphi_0|}{2} + \rho_{01} \right) + \frac{T}{4} M_f + \frac{p}{4} M_I < \infty. \tag{22}$$

Taking into account (22), analogously to (20) and (21) we consider next difference

$$\| x_3(t) - x_2(t) \| \le \frac{T|r|}{4} \left\| \max_{\tau\in[t-h,t]} x_2(\tau) - \max_{\tau\in[t-h,t]} x_1(\tau) \right\| +$$

$$+ \frac{T|r|}{4} \| x_2(t) \| \left\| \max_{\tau\in[t-h,t]} x_2(\tau) - \max_{\tau\in[t-h,t]} x_1(\tau) \right\| +$$

$$+ \frac{T|r|}{4} \| x_2(t) - x_1(t) \| \left\| \max_{\tau\in[t-h,t]} x_1(\tau) \right\| + \left[\frac{T}{4} L_f + \frac{1}{4} \sum_{i=1}^{p} L_{I_i} \right] \| x_2(t) - x_1(t) \| \le$$

$$\le \rho_{21}\left(\| x_2(t) - x_1(t) \| + h \| \dot{x}_2(t) - \dot{x}_1(t) \| \right), \tag{23}$$

where we denoted

$$\rho_{21} = \frac{T}{4}\left[|r|\left(\| x_2(t) \| + \left\| \max_{\tau\in[t-h,t]} x_1(\tau) \right\| \right) + L_f \right] + \frac{1}{4}\sum_{i=1}^{p} L_{I_i}.$$

Similarly, from (20) and (21) we obtain

$$\| \dot{x}_3(t) - \dot{x}_2(t) \| \le \rho_{22}\left(\| x_2(t) - x_1(t) \| + h \| \dot{x}_2(t) - \dot{x}_1(t) \| \right), \tag{24}$$

where

$$\rho_{22} = |r| \frac{T}{4}\left(\| x_2(t) \| + \left\| \max_{\tau\in[t-h,t]} x_1(\tau) \right\| \right) + \frac{T}{4} L_f + \frac{1}{4}\sum_{i=1}^{p} L_{I_i}.$$

From the estimates (23) and (24) we obtain that there hold the estimates

$$\| x_3(t) - x_2(t) \|_{BD[0,T]} \le \rho_2 \| x_2(t) - x_1(t) \|_{BD[0,T]},$$

where $\rho_2 = \max\{\rho_{21}; h\rho_{22}\} < 1;$

$$\| x_3(t) - x_2(t) \|_{BD[0,T]} \le \rho_{21}(\rho_{11} + h\rho_{12})(\rho_{01} + h\rho_{02}) + h\rho_{22} =$$

$$= (\rho_{21} + h\rho_{22})(\rho_{11} + h\rho_{12})(\rho_{01} + h\rho_{02}). \tag{25}$$

Continuing the process (25), we obtain an estimate for arbitrary n:

$$\| x_n(t) - x_{n-1}(t) \|_{BD[0,T]} \le \rho_{n-1} \| x_{n-1}(t) - x_{n-2}(t) \|_{BD[0,T]},$$

where $\rho_{n-1} = \max\{\rho_{(n-1)1}; h\rho_{(n-1)2}\} < 1;$

$$\| x_n(t) - x_{n-1}(t) \|_{BD[0,T]} \le$$

$$\le (\rho_{(n-1)1} + h\rho_{(n-1)2}) \cdots (\rho_{21} + h\rho_{22})(\rho_{11} + h\rho_{12})(\rho_{01} + h\rho_{02}), \tag{26}$$

where

$$\rho_{(n-1)1} = \frac{T}{4}\left[|r|\left(\| x_{n-1}(t) \| + \left\| \max_{\tau\in[t-h,t]} x_{n-2}(\tau) \right\| \right) + L_f \right] + \frac{1}{4}\sum_{i=1}^{p} L_{I_i},$$

$$\rho_{(n-1)2} = |r|\left(\| x_{n-1}(t) \| + \left\| \max_{\tau\in[t-h,t]} x_{n-2}(\tau) \right\| \right) + L_f.$$

From (26) we obtain the following estimate:

$$\| x_n(t) - x_{n-1}(t) \|_{BD[0,T]} \leq \rho_0 \rho_1 \rho_2 \cdot \ldots \cdot \rho_{n-2} \rho_{n-1}, \tag{26}$$

where $\rho_i = \max\{\rho_{i1}; h\rho_{i2}\} < 1,\ i = 1, 2, \ldots, n-1.$

The quantities r, h, L_f, L_{I_i} we choose such that from (26) implies

$$\lim_{n\to\infty} \| x_n(t) - x_{n-1}(t) \|_{BD[0,T]} = 0.$$

Hence, we deduce that the right-hand side of the equation (13) is contracting operator and the problem (1), (2) has a unique solution in the space $BD([0,T], R)$.

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HÖLDER'S GENERALIZED INEQUALITY ON THE SPACE OF UPPER SEMI-CONTINUOUS FUNCTIONS

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For a compact Hausdorff space X, we consider the space are defined as set-functions on the σ -algebra of the Borel subsets of X, and also the space $USC(X)$ of all normalized max-plus linear functionals on the linear space of all upper semi-continuous functions on X, equipped with idempotent operations. The main result establishes that a max-plus version of the Hölder inequality holds on the space of upper semi-continuous functions.

Key words: idempotent measure, max-plus linear functional, Borel sets, upper semi-continuous functions.

Компакттуу Гаусдорф X мейкиндиги үчүн Борелдин көптүктөрүнүн σ -алгебрасында функциялардын жыйындысы катары аныкталган X ички мейкиндикти, ошондой эле идемпотенттик операциялар менен жабдылган X боюнча бардык жогорку жарым үзгүлтүксүз функциялардын сызыктуу мейкиндигиндеги бардык нормалдаштырылган макс-плюс сызыктуу функционалдардын $USC(X)$ мейкиндигин карайбыз. Негизги жыйынтык Гёльдердин теңсиздигинин макс-плюс версиясы жогорку жарым үзгүлтүксүз функциялар мейкиндигинде орундаларын аныктайт.

Урунттуу сөздөр: идемпотенттик чара, макс-плюс сызыктуу функционалдуу, Борел комплекттери, жогорку жарым үзгүлтүксүз функциялар.

Для компактного хаусдорфова пространства X мы рассматриваем пространство, определенное как множество функций на σ -алгебре Борелевских подмножеств X, а также пространство $USC(X)$ всех нормированных макс-плюс линейных функционалов на линейном пространстве всех верхне-полунепрерывных функций на X, снабженном идемпотентными операциями. Основной результат устанавливает, что макс-плюс версия неравенства Гёльдера выполняется на пространстве верхне-полунепрерывных функций.

Ключевые слова: идемпотентная мера, макс-плюс линейный функционал, борелевские множества, верхнеполунепрерывные функции.

Introduction

Idempotent mathematics is a branch of mathematical sciences, rapidly developing and gaining popularity over the last four decades. An important stage of development of the subject was presented in the book “Idempotency” [7] edited by J. Gunawardena. This book arose out of the well-known international workshop that was held in Bristol, England, in October 1994. Idempotent mathematics is based on replacing the usual arithmetic operations with a new set of basic operations, i. e., on replacing numerical fields by idempotent semirings and semifields. Typical example is the so-called max-plus algebra $\mathbb{R}_{max}$ [7], [12].

In [16] M. Zarichnyi considered categorical properties of the space of idempotent probability measures on compact Hausdorff spaces. Later the space of idempotent probability measures was investigated on the class of compact metric spaces [17]. The study of spaces of idempotent probability measures leads to similar problems for topological spaces which are wider than the class of compact Hausdorff (or metrizable) spaces, in particular, for the case of Tychonoff spaces. In this case it is natural to apply the methods proposed in works [1], [2] or [14].

The results obtained in [1], [2], [15] show that in order to establish “good” properties of the space of idempotent probability measures, methods are required which are different from classical methods (i. e., from methods suitable for probability measures which have been applied in [14] and others). Note that idempotent probability measures in general are not linear and for a given compact Hausdorff space X they are defined as max-plus functionals on $C(X)$ [15], while the usual probability measures on X are a positive, linear, normalized functional on $C(X)$ [4], [5].

In this article, we introduce the space $L^{\oplus p}$ for a Tychonoff space X. Then a max-plus version of the well-known Hölder’s generalized inequality and Minkowskii equality on the space of upper semi-continuous functions is proved.

Main part

Let X be a compact Hausdorff space and $\mathfrak{B}(X)$ be the family of Borel subsets of X. We denote $\overline{\mathbb{R}}_+ = [0, +\infty]$. The symbol $\mathfrak{A}$ denotes the directed set. We give the following definition as [11].

Definition 2.1. A set function $\mu: \mathfrak{B}(X) \to \overline{\mathbb{R}}_+$ is said to be an idempotent measure on X if the following conditions hold

1. $\mu(\emptyset) = 0$;
2. $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for any $A, B \in \mathfrak{B}(X)$;
3. $\mu(\bigcup_{\alpha\in\mathfrak{A}} A_\alpha) = \sup\limits_{\alpha\in\mathfrak{A}}\{\mu(A_\alpha)\}$ for every increasing net $\{A_\alpha : \alpha \in \mathfrak{A}\} \subset \mathfrak{B}(X)$ such that $\bigcup_{\alpha\in\mathfrak{A}} A_\alpha \in \mathfrak{B}(X)$.

The set of all idempotent measures on X we denote by $IM(X)$. If $\mu(X) = 1$, then the idempotent measure μ is called an idempotent probability measure on X. We denote

$$I(X) = I_{\mathfrak{B}}(X) = \{\mu \in IM(X) : \mu(X) = 1\}.$$

Let (X, μ) be an idempotent measure space such that $\mu(X) < \infty$. We adopt the convention that $\infty \cdot 0 = 0$. For a set $A \subset X$, the characteristic function χ_A is defined as $\chi_A(x) = 1$ at $x \in A$ and $\chi_A(x) = 0$ at $x \in X \setminus A$.

Definition 2.2. For a measurable function $f: X \to \overline{\mathbb{R}}_+$ we define the idempotent integral of f with respect to μ by

$$\int_X^{\oplus} f d\mu = \sup_{t\in\overline{\mathbb{R}}_+}\{t \cdot \mu\{x \in X : f(x) \geq t\}\}.$$

For $A \subset X$, we let $\int_A^{\oplus} f d\mu = \int_X^{\oplus} f \chi_A d\mu$.

Lemma 2.3 (Chebyshev's inequality). Let f be a $\overline{\mathbb{R}}_+$ - valued measurable function on X and $0 < a < +\infty$. Then we have the following inequality

$$\mu\{x \in X : f(x) \geq a\} \leq \frac{1}{a}\int_X^{\oplus} f(x) d\mu.$$

Proof. Let $A = \{x \in X : f(x) \geq a\}$. Then by monotony idempotent integral one has

$$\int_A^{\oplus} f(x) d\mu \geq a \cdot \mu(A).$$

According to $A \subset X$ and the last inequality, we have

$$a \cdot \mu(A) \le \int_A^{\oplus} f(x)d\mu \le \int_X^{\oplus} f(x)d\mu.$$

Thus, Chebyshev's inequality is valid.

Lemma 2.4 (Borel-Cantelli). [11] Let $\{A_\alpha : \alpha \in \mathfrak{A}\}$ be a net of subsets of X. If $\mu(A_\alpha) \to 0$, then we have the following equality

$$\limsup_{\alpha \in \mathfrak{A}}\{\mu(A_\alpha)\} = \mu\left(\bigcap_{\beta \in \mathfrak{A}} \bigcup_{\alpha \ge \beta} A_\alpha\right).$$

Definition 2.6. Let X be a Tychonoff space. A function $\varphi: X \to \mathbb{R}_{\max}$ is said to be upper semi-continuous if for every $a \in \mathbb{R}$, the set $\{x \in X: \varphi(x) < a\}$ is open [6].

We denote by $USC(X)$ the linear space of all upper semi-continuous functions defined on X and endowed with the sup-norm $\|\varphi\| = \sup\{|\varphi(x)|: x \in X\}$ and let $USC_b(X)$ be the space of real-valued upper semi continuous functions on X bounded above [8].

For each $c \in \mathbb{R}_{\max}$ we denote by c_X the constant function in $USC(X)$ defined by the formula $c_X(x) = c$ for each $x \in X$. Define on the set $USC(X)$ operations $\oplus$ and $\odot$ by $\varphi \oplus \psi = \max\{\varphi, \psi\}$ and $\varphi \odot \psi = \varphi + \psi$, where $\varphi, \psi \in USC(X)$.

Definition 2.7. We call that a functional $\nu: USC(X) \to \mathbb{R}_{\max}$ is max-plus-linear, if it has the following properties:

1) $\nu(\varphi \oplus \psi) = \nu(\varphi) \oplus \nu(\psi)$ for any $\varphi, \psi \in USC(X)$;
2) $\nu(c \odot \varphi) = c \odot \nu(\varphi)$ for every $c \in \mathbb{R}_{\max}$ and $\varphi \in USC(X)$.

Now, by the following equality [3], [9], [18] we define an inner product on the linear space $USC_b(X)$:

$$(\varphi, \psi) = \ln\left(\frac{1}{\mu(X)} \int_X^{\oplus} e^{\varphi(x) \odot \psi(x)} d\mu\right), \varphi, \psi \in USC_b(X).$$

For $\varphi \in USC_b(X)$, define a $\oplus$-norm of φ by $\|\varphi\|_{\oplus 2} = \sqrt{(|\varphi|, |\varphi|)}$. For $p > 2$ we put

$$\|\varphi\|_{\oplus p} = \ln\left(\frac{1}{\mu(X)} \int_X^{\oplus} e^{p \cdot |\varphi|(x)} d\mu\right)^{\frac{1}{p}}.$$

Let $L^{\oplus p}$ denote the set of all μ-measurable functions φ such that $|\varphi|^p$ is μ-integrable, i.e.

$$L^{\oplus p} = \{\varphi \in USC_b(X): \|\varphi\|_{\oplus p} < +\infty\}.$$

Theorem 2.8 (Hölder's generalized inequality). 1) Let $p \in [1, +\infty]$ and $q \in [1, +\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then we have the following inequality

$$|(\varphi,\psi)|_{\oplus r} \le \|\varphi\|_{\oplus p} \odot \|\varphi\|_{\oplus q}, \text{ for every } \varphi,\psi \in USC_b(X);$$

2) If $0 < p < q$, then $\|\varphi\|_{\oplus p} \le \|\varphi\|_{\oplus q}$, i.e. $L^{\oplus q} \subset L^{\oplus p}$.

Proof. 1) From [10], [13], we have the following inequality:

$$ab \le \frac{a^{\frac{p}{r}}}{\frac{p}{r}} + \frac{b^{\frac{q}{r}}}{\frac{q}{r}}, \text{ where } p,q,r \in [1,+\infty], p \ge r, q \ge r \text{ and } \frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Now, let $I_1 = \int_X^{\oplus} \varphi^p(x)d\mu \ne 0$ and $I_2 = \int_X^{\oplus} \psi^q(x)d\mu \ne 0$. Then

$$\frac{\varphi^r}{I_1^{\frac{r}{p}}} \cdot \frac{\psi^r}{I_2^{\frac{r}{q}}} \le \frac{r\varphi^p}{pI_1} + \frac{r\psi^q}{qI_2} \Rightarrow$$

$$\int_X^{\oplus} \frac{\varphi^r(x)}{I_1^{\frac{r}{p}}} \cdot \frac{\psi^r(x)}{I_2^{\frac{r}{q}}} d\mu \le \int_X^{\oplus} \left(\frac{r\varphi^p(x)}{pI_1} + \frac{r\psi^q(x)}{qI_2}\right) d\mu \le$$

$$\int_X^{\oplus} \frac{r\varphi^p(x)}{pI_1} d\mu + \int_X^{\oplus} \frac{r\psi^q(x)}{qI_1} d\mu =$$

$$= \frac{r}{pI_1} \int_X^{\oplus} \varphi^p(x)\, d\mu + \frac{r}{qI_2} \int_X^{\oplus} \psi^q(x)\, d\mu = \frac{r}{p} + \frac{r}{q} = 1.$$

Consequently, we obtain

$$\int_X^{\oplus} \varphi^r(x) \cdot \psi^r(x) d\mu \le I_1^{\frac{r}{p}} \cdot I_2^{\frac{r}{q}} \Rightarrow$$

$$\left(\frac{1}{\mu(X)} \int_X^{\oplus} \varphi^r(x) \cdot \psi^r(x) d\mu\right)^{\frac{1}{r}} \le \left(\frac{1}{\mu(X)} I_1\right)^{\frac{1}{p}} \cdot \left(\frac{1}{\mu(X)} I_2\right)^{\frac{1}{q}} \Rightarrow$$

$$\ln\left(\frac{1}{\mu(X)} \int_X^{\oplus} \varphi^r(x) \cdot \psi^r(x) d\mu\right)^{\frac{1}{r}} \le \ln\left(\frac{1}{\mu(X)} I_1\right)^{\frac{1}{p}} + \ln\left(\frac{1}{\mu(X)} I_2\right)^{\frac{1}{q}}.$$

If $I_1 = 0$ or $I_2 = 0$, then $0 \le \int_X^{\oplus} \varphi^r(x) \cdot \psi^r(x) d\mu \le I_1^{\frac{r}{p}} \cdot I_2^{\frac{r}{q}} = 0$.

Hence, $\int_X^{\oplus} \varphi^r(x) \cdot \psi^r(x) d\mu = 0$. Thus, $|(\varphi,\psi)|_{\oplus r} \le \|\varphi\|_{\oplus p} \odot \|\varphi\|_{\oplus q}$.

2) If $0 < p < q$, then $k = \frac{q}{p} > 1$. By the first part of Theorem 1, we have

$$\int_X^{\oplus} (1 \cdot |\varphi|^p(x))\, d\mu \le \left(\int_X^{\oplus} 1^m\, d\mu\right)^{\frac{1}{m}} \cdot \left(\int_X^{\oplus} |\varphi|^{p \cdot k}(x)\, d\mu\right)^{\frac{1}{k}} = (\mu(X))^{\frac{1}{m}} \cdot \left(\int_X^{\oplus} |\varphi|^{p \cdot k}(x)\, d\mu\right)^{\frac{1}{k}} \Rightarrow$$

$$\frac{1}{\mu(X)}\int_X^{\oplus} |\varphi|^p(x)\,d\mu \le (\mu(X))^{\frac{1}{m}-1}\cdot\left(\int_X^{\oplus} |\varphi|^q(x)\,d\mu\right)^{\frac{p}{q}} = (\mu(X))^{-\frac{p}{q}}\cdot\left(\int_X^{\oplus} |\varphi|^q(x)\,d\mu\right)^{\frac{p}{q}} \Rightarrow$$

$$\left(\frac{1}{\mu(X)}\int_X^{\oplus} |\varphi|^p(x)\,d\mu\right)^{\frac{1}{p}} \le \left(\frac{1}{\mu(X)}\int_X^{\oplus} |\varphi|^q(x)\,d\mu\right)^{\frac{1}{q}} \Rightarrow$$

$$\ln\left(\frac{1}{\mu(X)}\int_X^{\oplus}|\varphi|^p(x)\,d\mu\right)^{\frac{1}{p}} \le \ln\left(\frac{1}{\mu(X)}\int_X^{\oplus}|\varphi|^q(x)\,d\mu\right)^{\frac{1}{q}}.$$

Thus, $\|\varphi\|_{\oplus p} \le \|\varphi\|_{\oplus q}$. Hence, $L^{\oplus q} \subset L^{\oplus p}$.

Theorem 2.9 (Minkowski equality). Let $p \ge 1$ and $\varphi, \psi \in L^{\oplus p}(X, \mu)$. Then

$$\|\varphi \oplus \psi\|_{\oplus p} = \|\varphi\|_{\oplus p} \oplus \|\psi\|_{\oplus p}.$$

Proof. By definition, we obtain

$$\|\varphi \oplus \psi\|_{\oplus p} = \ln\left(\frac{1}{\mu(X)}\int_X^{\oplus} e^{p\cdot\max\{|\varphi|(x),|\psi|(x)\}}d\mu\right)^{\frac{1}{p}}.$$

Since $e^{p\cdot\max\{|\varphi|(x),|\psi|(x)\}} = \max\{e^{p\cdot|\varphi|(x)}, e^{p\cdot|\psi|(x)}\}$ and the Shilkret integral is max-additive, we have the following equality

$$\int_X^{\oplus} e^{p\cdot\max\{|\varphi|(x),|\psi|(x)\}}d\mu = \max\left\{\int_X^{\oplus} e^{p\cdot|\varphi|(x)}d\mu, \int_X^{\oplus} e^{p\cdot|\psi|(x)}d\mu\right\}.$$

Taking logarithms yields the desired equality.

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Π-completeness of the space of idempotent probability measures with finite support

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For a Tychonoff space X, we consider the subspace $I_f(X)$ of idempotent probability measures with finite support equipped with the topology of pointwise convergence. Using an open cover of X we construct an open cover of $I_f(X)$ in a natural way, and show that disjointness of the cover is preserved by the functor I_f. We prove that $I_f(\beta X)$ is a perfect compactification of $I_f(X)$. Finally, applying the Π-completeness criterion, we establish that $I_f(X)$ is Π-complete if and only if X is Π-complete.

Key words: Π-complete space; perfect compactification; idempotent probability measure; finite support; disjoint cover.

Тихоновдун X мейкиндиги үчүн чектүү таянычтуу, чектүү конвергенция топологиясы менен жабдылган идемпотенттик ыктымалдык чараларынын $I_f(X)$ мейкиндигин карайбыз. X ачык капкагын колдонуу менен биз табигый түрдө ачык $I_f(X)$ капкагын курабыз жана капкактын ажырагыстыгы I_f функтору тарабынан сакталып турганын көрсөтөбүз. $I_f(\beta X)-$ $I_f(X)$тин кемчиликсиз компактификациясы экенин далилдейбиз. Акырында, Π -толуктуулук критерийин колдонуу менен, $I_f(X)$ Π -толук, эгерде X качан жана качан гана Π -толук болсо гана аныктайбыз.

Урунттуу сөздөр: Π -толук мейкиндик, кемчиликсиз компактификация, идемпотенттик ыктымалдык өлчөмү, чектүү колдоо, ажыратылган жабуу.

Для пространства Тихонова X, мы рассматриваем подпространство $I_f(X)$ идемпотентных вероятностных мер с конечным носителем, снабженное топологией поточечной сходимости. Используя открытое покрытие X, мы естественным образом строим открытое покрытие $I_f(X)$ и показываем, что непересекаемость покрытия сохраняется функтором I_f. Мы доказываем, что $I_f(\beta X)$ является совершенной компактификацией $I_f(X)$. Наконец, применяя критерий Π-полноты, мы устанавливаем, что

$I_f(X)$ является Π -полным тогда и только тогда, когда X является Π -полным.

Ключевые слова: Π-полное пространство, совершенная компактификация, идемпотентная вероятностная мера, конечный носитель, непересекающееся покрытие.

Introduction

The notion of Π-completeness, introduced and developed in [8], [9], plays an important role in the theory of compact-type spaces. Functorial behavior of compact-type properties under hyperspace construction was investigated, particularly, in [10], [4]. On the other hand, idempotent (max-plus) analysis produces the functor I of idempotent probability measures on compact spaces [12]; several extensions of I to Tychonoff spaces were constructed in [7]. Topological properties of spaces of idempotent measures and related set systems were studied in [11]. In [3] the authors show that the functor of idempotent probability measures with compact supports preserves the uniform openness of maps, the weight and the completeness index of uniform spaces.

In [1] it was shown that every idempotent probability measure is uniquely determined by its density function. This provides a convenient description of supports and allows one to build natural neighborhood systems in spaces of idempotent measures.

In this paper we focus on the finite-support subspace $I_f(X)$, which can be viewed as an idempotent analogue of probability measures with finite support. Unlike $I_\beta(X)$, the space $I_f(X)$ has a stronger combinatorial structure controlled by finite supports. Our main result states that Π-completeness is preserved and reflected by the assignment $X \mapsto I_f(X)$:

$$I_f(X) \text{ is } \Pi\text{-complete} \Leftrightarrow X \text{ is } \Pi\text{-complete}.$$

The proof uses: (i) lifting a disjoint cover from X to $I_f(X)$; (ii) a closure lemma controlling supports inside βX; and (iii) a Π-completeness criterion formulated in terms of perfect compactifications.

The passage from $I_\beta(X)$ to the finite-support subspace $I_f(X)$ is not purely formal. Indeed, $I_f(X)$ is sensitive to the combinatorics of finite supports and to the choice of a distinguished support point, which affects the behavior of natural set systems in $I_f(X)$ and their closures in $I_f(\beta X)$. Therefore, even if Π-completeness is known to be preserved by I_β, it is not automatic for I_f.

Our approach is based on constructing, from a cover ω of X, a canonical family of subsets in $I_f(X)$ described by support restrictions. We prove that disjointness and the condition $Kp\omega = 1$ are preserved by this construction. Then we establish a closure lemma controlling supports in βX, which allows us to prick out a measure $\mu \in I_f(\beta X) \setminus I_f(X)$ using a lifted cover. This is combined with perfectness of the compactification $I_f(\beta X)$.

The paper is organized as follows. In Section 2 we recall Π-completeness and basic facts on idempotent probability measures. In Section 3 we develop set systems in $I_f(X)$ associated with open families in X and study their basic properties. In Section 4 we prove that $I_f(\beta X)$ is a perfect compactification of $I_f(X)$ and establish the main theorem on Π-completeness of $I_f(X)$.

Preliminaries.

In this paper, by a space we mean a topological T_1-space, and by a map a continuous map. We use standard notions from general topology [6].

A collection ω of subsets of a set X is *star-finite* if each element of ω intersects only finitely many elements of ω. A star-finite open cover is *finite-component* if each component contains only finitely many elements.

For a collection $\omega = \{O_\alpha : \alpha \in A\}$ in a space Y we put $[\omega]_Y = \{[O_\alpha]_Y : \alpha \in A\}$. For a subspace $W \subset Y$ and a point $y \in Y \setminus W$, an open cover λ of W *pricks out* y in Y if $y \notin \cup[\lambda]_Y$ [9].

Let X be Tychonoff and βX its Stone-Čech compactification.

Definition 2.1.[9] A Tychonoff space X is *Π-complete* if for every $x \in \beta X \setminus X$ there exists a finite-component open cover ω of X which pricks out x in βX.

Let bX be a perfect compactification of X. The following criterion is crucial.

Theorem 2.2. [4] A Tychonoff space X is Π-complete if and only if for every $x \in bX \setminus X$ there exists an open cover ω of X with $Kp\omega = 1$ pricking out x in bX.

Definition 2.3. For a cover ω of a space X we write $Kp\ \omega = 1$ if ω has multiplicity 1, i.e., each point $x \in X$ belongs to at most one member of ω. Equivalently, the family ω is pairwise disjoint: if $U, V \in \omega$ and $U \neq V$, then $U \cap V = \emptyset$.

Since βX is perfect, we also use:

Corollary 2.4. A Tychonoff space X is Π-complete if and only if for every $x \in \beta X \setminus X$ there exists an open cover ω of X with $Kp\ \omega = 1$ pricking out x in βX.

We will also use the fact that a closed subspace of a Π-complete space is Π-complete[8].

Perfect compactifications. Let υX be a compact extension of a Tychonoff space X. For an open set $H \subset X$ denote by $O_{\upsilon X}(H)$ the maximal open subset of υX such that $O_{\upsilon X}(H) \cap X = H$.

A compactification υX of X is called *perfect* if for any two disjoint open sets $U_1, U_2 \subset X$ one has

$$O_{\upsilon X}(U_1 \cup U_2) = O_{\upsilon X}(U_1) \cup O_{\upsilon X}(U_2).$$

It is well known (see [2]) that the Stone-Čech compactification βX is a perfect compactification of X.

Idempotent probability measures and supports. Let K be a compact Hausdorff space. An idempotent probability measure on K is a functional $\mu: C(K) \to \mathbb{R}$ satisfying [12]:

1) $\mu(c_K) = c$ for constants c;

2) $\mu(c \odot \varphi) = c \odot \mu(\varphi)$, where $c \odot \varphi = c + \varphi$;

3) $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$, where $\varphi \oplus \psi = \max\{\varphi, \psi\}$.

The space of all idempotent probability measures is denoted by $I(K)$ and carries the topology τ_p of pointwise convergence.

A neighborhood base of $\mu \in I(K)$ in the topology τ_p consists of sets of the form

$\langle \mu; \varphi_1, \dots, \varphi_m; \varepsilon\rangle = \{\nu \in I(K): |\nu(\varphi_i) - \mu(\varphi_i)| < \varepsilon, i = 1, \dots, m\}$,

where $\varphi_i \in C(K)$ and $\varepsilon > 0$.

By [1], [11], each $\mu \in I(K)$ admits a representation

$$\mu = \bigoplus_{x \in K} \lambda(x) \odot \delta_x,$$

where $\lambda: K \to [-\infty, 0]$ is upper semicontinuous and $\max \lambda = 0$.

The support of μ is

$$\operatorname{supp} \mu = \{x \in K: \lambda(x) > -\infty\}.$$

Finite-support subspace $I_f(X)$. Let $I_\omega(K)$ be the set of all $\mu \in I(K)$ with finite support. For a compact Hausdorff space K define $I_f(K)$ as the set of all $\mu \in I_\omega(K)$ such that:

$$I_f(K) = \left\{\mu = \bigoplus_{i=1}^{n} \chi_\mu(x_i) \odot \delta_{x_i} \in I_\omega(K) \middle| \exists\, i_0: \chi_\mu(x_{i_0}) = 0,\right.$$
$$\left.\chi_\mu(x_i) \le -\frac{n}{n+1}, \qquad (i \ne i_0)\right\}$$

or a Tychonoff space X we view X as dense in βX and put

$$I_f(X) = \{\mu \in I_f(\beta X): \operatorname{supp} \mu \subset X\}.$$

Thus $I_f(X)$ is a subspace of $I(\beta X)$.

For an open set $U \subset X$ we use:

$$\langle U \rangle = \{\mu \in I_f(X): \operatorname{supp} \mu \subset U\}.$$

Set systems in $I_f(X)$ associated with families in X.

Distinguished-point decomposition. Recall that each $\mu \in I_f(\beta X)$ admits a finite max-plus representation

$$\mu = \oplus_{i=1}^{n} \chi_\mu(x_i) \odot \delta_{x_i}, \max_{1 \le i \le n} \chi_\mu(x_i) = 0,$$

and in $I_f(\beta X)$ there exists a (unique) *distinguished atom* x_{i_0} with $\chi_\mu(x_{i_0}) = 0$, while all other coefficients satisfy $\chi_\mu(x_i) \leq -\frac{n}{n+1}$.

For $\mu \in I_f(X)$ we denote this distinguished point by $d(\mu) \in X$.

For a point $x \in X$ define

$$I_f^x(X) = \{\mu \in I_f(X) : d(\mu) = x\}.$$

Then

$$I_f(X) = \bigcup_{x \in X} I_f^x(X), \quad I_f^x(X) \cap I_f^y(X) = \emptyset \ (x \neq y)$$

For a set $M \subset X$ we put

$$\langle M \rangle^* = \{\mu \in I_f(X) : d(\mu) \in M\}.$$

Thus $\langle M \rangle^*$ consists of measures whose distinguished support point belongs to M.

Proposition 3.1. For any sets $M, N \subset X$,

$$\langle M \rangle^* \cap \langle N \rangle^* \neq \emptyset \Leftrightarrow M \cap N \neq \emptyset.$$

Proof. If $x \in M \cap N$, then $I_f^x(X) \subset \langle M \rangle^* \cap \langle N \rangle^*$. Conversely, if $\mu \in \langle M \rangle^* \cap \langle N \rangle^*$, then $d(\mu) \in M$ and $d(\mu) \in N$, hence $d(\mu) \in M \cap N$ and $M \cap N \neq \emptyset$.

Proposition 3.2. If $M \subset X$ is open (respectively, closed), then $\langle M \rangle^*$ is open (respectively, closed) in $I_f(X)$.

Proof. We show openness; closedness is similar. Let $\mu \in \langle M \rangle^*$ and $d(\mu) = x \in M$. Since M is open, choose an open neighborhood $V \subset M$ of x in X. Consider any continuous function $\varphi \in C(\beta X)$ such that $\varphi(x) = 0$ and $\varphi \leq -1$ on $\beta X \setminus V$. In the max-plus representation, the distinguished atom of μ is characterized by achieving the maximum coefficient 0, and the normalization keeps all other coefficients uniformly below 0. Hence there exists $\varepsilon > 0$ such that for every $\nu \in I_f(X)$ with $|\nu(\varphi) - \mu(\varphi)| < \varepsilon$, the distinguished point $d(\nu)$ must belong to $V \subset M$. Therefore a τ_p-neighborhood of μ is contained in $\langle M \rangle^*$, proving openness.

Support-controlled basic sets. For an open set $U \subset X$ we use the standard support-controlled subset

$$\langle U \rangle = \{ \mu \in I_f(X) : \operatorname{supp} \mu \subset U \}.$$

Lemma 3.3. If $U \subset X$ is open, then $\langle U \rangle$ is open in $I_f(X)$.

Proof. Let $\mu \in \langle U \rangle$. Then $\operatorname{supp} \mu$ is finite and contained in U. For each $x \in \operatorname{supp} \mu$ choose an open neighborhood $V_x \subset U$ in X. Pick continuous functions $\varphi_x \in C(\beta X)$ such that $\varphi_x(x) = 0$ and $\varphi_x \leq -1$ on $\beta X \setminus V_x$. A basic τ_p-neighborhood of μ controlling the values on finitely many functions φ_x forces every nearby measure $\nu \in I_f(X)$ to have support points inside $V_x \subset U$. Thus $\nu \in \langle U \rangle$, and $\langle U \rangle$ is open.

Lemma 3.4. For open sets $U, V \subset X$,

$$\langle U \rangle \cap \langle V \rangle \neq \emptyset \Leftrightarrow U \cap V \neq \emptyset.$$

Proof. If $x \in U \cap V$, then $\delta_x \in I_f(X)$ and $\operatorname{supp}(\delta_x) = \{x\} \subset U \cap V$, so $\delta_x \in \langle U \rangle \cap \langle V \rangle$. Conversely, if $\mu \in \langle U \rangle \cap \langle V \rangle$, then $\operatorname{supp} \mu \subset U$ and $\operatorname{supp} \mu \subset V$, hence $\operatorname{supp} \mu \subset U \cap V$ and $U \cap V \neq \emptyset$.

Lifting covers from X to $I_f(X)$. Let ω be an open cover of X. In order to cover a measure whose finite support may lie in several elements of ω, we use unions of finitely many sets from ω.

Define $\quad \Omega(\omega) = \{ \langle U_1 \cup \cdots \cup U_n \rangle : U_i \in \omega, n \in \mathbb{N} \}.$

Lemma 3.5. If ω is an open cover of X, then $\Omega(\omega)$ is an open cover of $I_f(X)$.

Proof. Each set $\langle U_1 \cup \cdots \cup U_n \rangle$ is open in $I_f(X)$ by Lemma 3.3., because $U_1 \cup \cdots \cup U_n$ is open in X.

Let $\mu \in I_f(X)$. Then $\operatorname{supp} \mu = \{x_1, \dots, x_k\}$ is finite. Choose $U_i \in \omega$ with $x_i \in U_i$ for each $i = 1, \dots, k$. Then $\operatorname{supp} \mu \subset U_1 \cup \cdots \cup U_k$, hence

$$\mu \in \langle U_1 \cup \cdots \cup U_k \rangle \in \Omega(\omega).$$

So $\Omega(\omega)$ covers $I_f(X)$.

Proposition 3.6. If ω is an open cover of X with $Kp\, \omega = 1$, then $Kp\, \Omega(\omega) = 1$.

Proof. Assume that a point $\mu \in I_f(X)$ belongs to two members of $\Omega(\omega)$:

$$\mu \in \langle U_1 \cup \cdots \cup U_n \rangle \cap \langle V_1 \cup \cdots \cup V_m \rangle, U_i, V_j \in \omega.$$

Then $\operatorname{supp} \mu \subset (U_1 \cup \cdots \cup U_n) \cap (V_1 \cup \cdots \cup V_m)$. In particular, $\operatorname{supp} \mu \neq \emptyset$ implies the intersection is nonempty, so there exist i, j with $U_i \cap V_j \neq \emptyset$. Since $Kp\, \omega = 1$, two elements of ω cannot intersect unless they are equal; hence $U_i = V_j$.

Now observe that if ω has $Kp\, \omega = 1$, then distinct elements of ω are pairwise disjoint. Therefore the unions $U_1 \cup \cdots \cup U_n$ and $V_1 \cup \cdots \cup V_m$ are equal exactly when they consist of the same elements of ω. Since $\operatorname{supp} \mu$ is contained in both unions and each $U \in \omega$ is disjoint from every other, the family of those $U \in \omega$ that meet $\operatorname{supp} \mu$ is uniquely determined. Consequently, the two members of $\Omega(\omega)$ containing μ must coincide. Thus no point of $I_f(X)$ lies in two distinct elements of $\Omega(\omega)$, i.e. $Kp\, \Omega(\omega) = 1$.

Remark 3.7. The family $\Omega(\omega)$ is the smallest natural "support-cover" of $I_f(X)$ generated by ω: it covers each measure by collecting finitely many members of ω that cover the finite support. Proposition 3.6. shows that the pricking-out condition $Kp = 1$ is preserved by this construction.

Perfect compactification and Π-completeness of $I_f(X)$.

In this section we complete the proof of the main result. We first establish a closure lemma controlling supports inside βX, then show that $I_f(\beta X)$ is a perfect compactification of $I_f(X)$, and finally apply the Π-completeness criterion.

A closure lemma for support-controlled sets.

Lemma 4.1. Let X be a Tychonoff space and $U \subset X$ be open. Then

$$[\langle U \rangle]_{I_f(\beta X)} \subset \{\mu \in I_f(\beta X) \colon \operatorname{supp} \mu \subset [U]_{\beta X}\}.$$

Equivalently, if $\mu \in I_f(\beta X)$ and $\mu \in [\langle U \rangle]_{I_f(\beta X)}$, then $\operatorname{supp} \mu \subset [U]_{\beta X}$.

Proof. Assume $\mu \in [\langle U \rangle]_{I_f(\beta X)}$ but $\operatorname{supp} \not\subset [U]_{\beta X}$. Choose $x_0 \in \operatorname{supp} \mu \backslash [U]_{\beta X}$.

Since $[U]_{\beta X}$ is closed in βX, there exists an open neighborhood W of x_0 in βX such that $W \cap [U]_{\beta X} = \emptyset$.

Because $x_0 \in \operatorname{supp} \mu$, there exists a function $\varphi \in C(\beta X)$ with $\varphi(x_0) = 0$ and $\varphi \leq -1$ on $\beta X \setminus W$. In the max-plus evaluation of $\mu(\varphi)$, the atom x_0 contributes the maximal value 0. By continuity of the evaluation maps $\nu \mapsto \nu(\varphi)$, there exists a basic τ_p-neighborhood $\mathcal{O}$ of μ in $I_f(\beta X)$ such that for every $\nu \in \mathcal{O}$ we have $\nu(\varphi) > -\frac{1}{2}$. Hence $\operatorname{supp} \nu \cap W \neq \emptyset$, so $\operatorname{supp} \nu \not\subset [U]_{\beta X}$.

But if $\nu \in \langle U \rangle$, then $\operatorname{supp} \nu \subset U \subset [U]_{\beta X}$, which is impossible for $\nu \in \mathcal{O}$. Thus $\mathcal{O} \cap \langle U \rangle = \emptyset$, contradicting $\mu \in [\langle U \rangle]_{I_f(\beta X)}$.

$I_f(\beta X)$ as a perfect compactification.

Theorem. For a Tychonoff space X, the space $I_f(\beta X)$ is a perfect compactification of $I_f(X)$.

Proof. It suffices to verify the perfectness condition

$$O(H_1 \cup H_2) = O(H_1) \cup O(H_2)$$

for basic open sets H_i in $I_f(X)$ corresponding to disjoint open sets in X.

Let $U_1, U_2 \subset X$ be disjoint open sets and consider

$$H_i = \langle U_i \rangle = \{\mu \in I_f(X) : \operatorname{supp} \mu \subset U_i\}, \; i = 1,2.$$

Then $H_1 \cap H_2 = \emptyset$.

Assume $\mu' \in I_f(\beta X) \setminus \big(O(H_1) \cup O(H_2)\big)$. Using the standard identity for perfect compactifications (see [2]),

$$I_f(\beta X) \setminus O(H_i) = \left[I_f(X) \setminus H_i\right]_{I_f(\beta X)}, \qquad i = 1,2,$$

we get

$$\mu' \in \left[I_f(X) \setminus H_1\right]_{I_f(\beta X)} \cap \left[I_f(X) \setminus H_2\right]_{I_f(\beta X)} = \left[I_f(X) \setminus (H_1 \cup H_2)\right]_{I_f(\beta X)}.$$

Hence $\mu' \notin O(H_1 \cup H_2)$.

This proves

$$O(H_1 \cup H_2) \subset O(H_1) \cup O(H_2),$$

and the reverse inclusion is trivial. Therefore $I_f(\beta X)$ is perfect compactification of $I_f(X)$.

Π-completeness of $I_f(X)$.

Theorem. For a Tychonoff space X, the space $I_f(X)$ is Π-complete if and only if X is Π-complete.

Proof.($\Rightarrow$) The set $\{\delta_x : x \in X\}$ is closed in $I_f(X)$ and homeomorphic to X. Since a closed subspace of a Π-complete space is Π-complete, it follows that X is Π-complete.

($\Leftarrow$)Assume X is Π-complete. Let $\mu \in I_f(\beta X) \setminus I_f(X)$. Then $\operatorname{supp} \mu \not\subset X$, so choose

$$x_0 \in \operatorname{supp} \mu \cap (\beta X \setminus X).$$

By Corollary 2.4., there exists an open cover ω of X with $Kp\ \omega = 1$ such that

$$x_0 \notin \cup[\omega]_{\beta X}.$$

Consider the lifted family

$$\Omega(\omega) = \{\langle U_1 \cup \cdots \cup U_n \rangle : U_i \in \omega\}.$$

By Lemma 3.5., $\Omega(\omega\$$ is an open cover of $I_f(X)$, and by Proposition 3.6., $Kp\ \Omega(\omega) = 1$.

Suppose $\mu \in \cup[\Omega(\omega)]_{I_f(\beta X)}$. Then $\mu \in [\langle U \rangle]_{I_f(\beta X)}$ for some union U of elements of ω. By Lemma 4.1.,

$$\operatorname{supp} \mu \subset [U]_{\beta X} \subset \cup[\omega]_{\beta X}.$$

In particular $x_0 \in \cup[\omega]_{\beta X}$, contradicting the choice of ω.

Thus $\Omega(\omega)$ pricks out μ in $I_f(\beta X)$. Since $I_f(\beta X)$ is perfect by Theorem 4.2., Theorem 2.2. yields that $I_f(X)$ is Π-complete.

Conclusion. In this paper we studied the finite-support subspace $I_f(X)$ of idempotent probability measures on a Tychonoff space X. Using the structure of finite supports and distinguished-point decomposition, we constructed natural set systems in $I_f(X)$ associated with open families in X.

It was shown that disjoint open covers of X can be lifted to disjoint support-controlled covers of $I_f(X)$, and that the pricking-out condition $Kp\,\omega = 1$ is preserved by this construction. We also proved a closure lemma describing the behavior of supports inside the Stone - Čech compactification.

As a consequence, we established that $I_f(\beta X)$ is a perfect compactification of $I_f(X)$. Applying the Π-completeness criterion for perfect compactifications, we obtained the main result: the space $I_f(X)$ is Π-complete if and only if the original space X is Π-complete.

Thus the functor of idempotent probability measures with finite support preserves and reflects Π-completeness of Tychonoff spaces.

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ON RESULTS OF OPERATIONS WITH KINEMATICAL SPACES

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Usually, operations with spaces yield spaces of same category (topological, uniform, metric). Examples show that for kinematic spaces this is not always the case. The article finds conditions when a set in a two-dimensional kinematic space is also a kinematic space, and when the gluing of two kinematic spaces is also a kinematic space.

Key words: topological space, kinematic space, uniform space, metric space, gluing.

Адатта, мейкиндиктер менен амалдар ошол эле категориядагы мейкиндиктерди (топологиялык, бир калыптуу, метрикалык) берет. Мисалдар муну көрсөтүп турат
кинематических жайлардын бул нерсе дайыма аткарылат. - Беренесинде найдены шарттары, учурда көптөгөн айыл двумерном кинематическом мейкиндигинде эле болуп кинематическим мейкиндиги, жана качан склейка эки кинематических жайлардын эле болуп кинематическим мейкиндиги.

Урунттуу сөздөр: топологиялык мейкиндик, кинематикалык мейкиндик, бир калыптуу мейкиндик, метрикалык мейкиндик, чаптоо.

Как правило, операции над пространствами приводят к пространствам одной и той же категории (топологическим, равномерным, метрическим). Примеры показывают, что для кинематических пространств это не всегда выполняется. В статье найдены условия, когда множество в двумерном кинематическом пространстве также является кинематическим пространством, и когда склейка двух кинематических пространств также является кинематическим пространством.

Ключевые слова: топологическое пространство, кинематическое пространство, равномерное пространство, метрическое пространство, склейка.

1. Introduction

We consider kinematic spaces. Definition of a M (the metric ρ_K is the minimal time of passing between points) with controlled motion of points was introduced in [1] (see also [2]), see Section 2 below.

Section 3 contains conditions on a metrical kinematizable an algorithm to implement such motion on a topological torus with obstacles [4]. Section 4 considers motion in Cartesian products of kinematical spaces.

2. Definitions for motion with bounded velocity

As a codifying the ancient idea of controlled motion with bounded velocity, the notion of kinematical space was introduced [1].

D e f i n i t i o n 1 (non-formal). A computer program is said to be a **presentation** of a computer kinematic space if:

P1) there is an (infinite) metrical space X of points and a set X_1 of display-presentable points being sufficiently dense in X;

P2) the user can pass from any point x_1 in X_1 to any other point x_2 by a sequence of adjacent points in X_1 by their will;

P3) the minimal time to reach x_2 from x_1 is (approximately) equal of the minimal time to reach x_2 from x_1.

The space X is said to be a **kinematic space**; the space X_1 is said to be a **computer kinematic space**; this minimal time is said to be the **kinematic distance** ρ_X between x_1 and x_2; a sequence of adjacent points is said to be a **route**. Passing to a limit as X_1 tends to X yields

D e f i n i t i o n 2 (formal). There is a set K of **routes**; each route M, in turn, consists of the positive real number T_M (**time** of route) and the function $m_M : [0, T_M] \to X$ (**trajectory** of route);

(K1) For $x_1 \neq x_2 \in X$ there exists such $M \in K$ that $m_M(0) = x_1$ and $m_M(T_M) = x_2$, and the set of values of such T_M is bounded with a positive number below;

(K2) If $M = \{T_M, m_M(t)\} \in K$ then the pair $\{T_M, m_M(T_M - t)\}$ is also a route of K (the reverse motion with same speed is possible); (cf. P3).

(K3) If $M=\{T_M, m_M(t)\} \in K$ and $T^* \in (0, T_M)$ then the pair: T^* and function $m^*(t)=m_M(t)$ $(0 \le t \le T^*)$ is also a route of K (one can stop at any desired moment);

(K4) concatenation of routes for three distinct points x_1, x_2, x_3.

$\rho_X(x_1, x_1)=0;$ for $x_1 \neq x_2$ $\rho_X(x_1, x_2):=\inf\{T_M: m_M(0) = x_1, m_M(T_M) = x_2\}$.

D e f i n i t i o n 3. If for each x_1, x_2 there exists such route M, that $m_M(0) = x_1$, $m_M(T_M) = x_2$, $T_M = \rho_X(x_1, x_2)$ then the kinematic space X is said to be **flat.**

D e f i n i t i o n 4. If there exists a kinematic consistent with the given metric then the metric space is said to be **kinematizable**.

D e f i n i t i o n 5. Routes from x_1 to x_2 and from x_2 to x_1 whose trajectories have no other common points form a **cyclic route** with time being the sum of two times.

D e f i n i t i o n 6. Let a set K^* of routes in the set X be given. If (K6) for all $x_1 \neq x_2 \in X$ there exists such $M \in K^*$ that $m_M(0) = x_1$ and $m_M(T_M) = x_2$ and (K7) sums of times of concatenations from x_1 to x_2 are bounded with a positive number below then the set K^* is a **base of kinematic**.

T h e o r e m 1. The set of routes being concatenations of elements of K^* is kinematic. It is said to be **generated** by K^*.

R e m a r k. (K1) instead of (K7) is not sufficient to be a basic of kinematic.

E x a m p l e 1. Let $X=R$ and the only $M(u,v)$ has the time $T_M(u,v)=|u-v|^2$. Then for any $n \in N$ the concatenation from 0 to 1: $M(0,1/n)+M(1/n,2/n)+\ldots+M((n-1)/n,n/n)$ has the sum $n\cdot(1/n)^2 = 1/n$ which is not bounded below.

T h e o r e m 2. The Cartesian product of several kinematical spaces $\{X_1,K_1\}$, $\{X_2,K_2\},\ldots, \{X_p,K_p\}$ with the following base of kinematics: $\{T \text{ of } M_j;\ x_1 \in X_1, \ldots, x_{j-1} \in X_{j-1}, M_j \in K_j, x_{j+1} \in X_{j+1}, \ldots, x_p \in X_p\}$, $j=1..p$ is a kinematic space.

P r o o f. If $(u_1,u_2,\ldots, u_p) \neq (v_1,v_2,\ldots, v_p)$ then $(\exists j)(u_j \neq v_j)$ and any concatenation contains a trajectory between u_j and v_j in X_j.

3. Subspace

E x a m p l e 2 of non-kinematizable subspace D of $\boldsymbol{R}^2$. The union of segments $\{(0,0)\text{-}(z_k:=)(\cos(\pi/k), \sin(\pi/k)):k\in N\}$ and the segment $\{0, (z_0:=)(1,0)\}$. In metric of $\boldsymbol{R}^2$ $z_k\to z_0$, but the length of a way from $z_k\to z_0$ within D is not less than 2.

T h e o r e m 3. A cyclic route in $\boldsymbol{R}^2$ is the boundary of a kinematizable space.

P r o o f. Let points z_1 and z_2 be within the cyclic route with trajectory T_0. There is a route z_1 - x_1. Let z_3 be the first point where this route intersects the trajectory T_0. Correspondingly, let z_4 be the first point where a route z_2 - x_1 intersects the trajectory T_0. Then a route z_1 - z_3 – (along T_0)- z_4 – x_2 is obtained by concatenation.

4. Gluing

An arbitrary gluing of two kinematic spaces may not be a kinematic space, as follows:

E x a m p l e 2. Consider two points z_0 , z_1, connected by a countable number of segments $U_n= [z_0 , z_1] \sim [0,1]$, $n = 2,3,\ldots$ of unit length that do not intersect each other at other points. Then the kinematic distance $\rho_K(z_0 , z_1)$ is equal to 1. Denote this kinematic space as W.

Take a copy W' of the same space and identify the points $z_0 \sim z_0'$, $z_1 \sim z_1'$ as well as the point $u_k:=1/k$ of the segment U_k with the point $w_k':= (1–1/k)$ of the segment U_k' for all $k = 2,3,\ldots$

Then in the new glued space we get:

$$\rho_K (z_0 , z_1)\leq \rho_K (z_0 , u_k)+ \rho_K (w_k' , z_1')=1/k+(1-(1-1/k))=2/k \to 0 \quad (k\to \infty).$$

Hence, the new glued space is not kinematic.

D e f i n i t i o n 7. Given kinematic spaces W, W' with sets of routes K, K', respectively, flat closed isokinematic sets $V\subset W$, $V'\subset W'$ and reciprocal functions $\varphi :V\to V'$, $\varphi ':V'\to V$, which carry out a one-to-one mapping of these sets.

Identify the corresponding points from the sets V and V', then we obtain a set that can be denoted as $W\cup W'$.

Denote $p: W \to W \cup W'$, $p': W' \to W \cup W'$ as canonical (identity) mappings of subsets into a set.

Define the basic routes for gluing the kinematical spaces W, W' along the sets V' and V as follows:

- the routes of K, considered in $W \cup W'$, that is, obtained in the following way: for any $\{T, m(t)\} \in K$ we take the pair $\{T, p(m(t))\}$;

- the routes from K', obtained in a similar way.

As the routes in $W \cup W'$ we take all possible pairs obtained from basic routes by applying a finite number of times of corresponding operations. It is obvious that for the resulting set properties K2), K3) and K4) will be satisfied.

Call the resulting space as a **kinematic gluing** of the kinematic spaces W, W'.

The correctness of this definition:

T h e o r e m 4. The gluing of two kinematic spaces along flat isokinematic closed sets is a kinematic space.

P r o o f. It is enough to prove the fulfillment of property K1). Here, in turn, it is enough to consider the routes passing along both components of the gluing. We will prove this property by contradiction.

If the beginning and end of the route are in one component then sections of the route located in another component can be replaced, without increasing its length, by sections passing through the gluing set, that is, this route is not shorter than the route passing through one of component, where property K1) is satisfied by condition.

If the beginning and end of the route are in different components, then, similar to the previous case, we can assume that the route consists of only two parts, each of which is located entirely in one of the components.

Assume that for given two points $z_0 \in W \backslash V$ and $z_1' \in W' \backslash V'$ there is a sequence of routes $\{M_k | k \in \mathbf{N}\}$ (with times T_k) such that $T_k \to 0 \, (k \to \infty)$.

Hence, there is a sequence of points $u_k \in V \equiv V'$ such that

$\rho_K(z_0, u_k) + \rho_K(u_k, z_1') \to 0;$.

i.e $\rho_K(z_0, u_k) \to 0$, $\rho_K(u_k, z_1') \to 0\,(k \to \infty)$.

Due to the closedness of $V \equiv V'$, it follows that $z_0 \in V$, $z_1' \in V'$, which contradicts the assumption.

Thus, Theorem has been proven.

E x a m p l e 4. Consider two flat kinematic spaces – unit squares *ABCD* and *A'B'C'D'*. Glue them by the segment A"(inclusive)-B"(exclusive). Then the resulting kinematic space is not flat: $\rho_K(C, C')=2$ but there is no route of length 2 between these points.

5. Conclusion

We hope that new facts of peculiarities of kinematical spaces and operations with them will distinguish specific of these spaces.

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MORPHISMS IN CATEGORY OF CONTROLLED PROCESSES IN COMPUTATIONAL MATHEMATICS

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Supra, the notion of category of equations was introduced by the second author with assistance of the notion “predicate” on the base of the principle of preservation of solution while transformations; elements of the category of equations and its subcategories were constructed on the base of well-known categories. As a specification of the second law of thermodynamic for isolated systems, supra the first author introduced new definitions, proposed general hypotheses and derived estimations from below on increasing of entropy on depending on time in permanently unstable (affectable) systems. On this base, the authors proposed the category of controlled processes

in computational mathematics. Definition and examples of morphisms in this category are proposed in this paper.

Keywords: category, morphism, entropy, control, differential equation, affectable system, motion.

Мурда экинчи автор, өзгөртүүлөрдө чыгарылышты сактоо принцибинин негизинде “предикат” түшүнүгүнүн жардамы менен теңдемелердин категориясынын түшүнүгүн киргизген; теңдемелердин категориясынын жана анын категориячаларынын элементтери белгилүү категориялардын негизинде курулган. Четтетилген системада термодинамиканын экинчи законун тактоо катары, биринчи автор жаңы аныктамаларды киргизген, жалпы гипотезаларды сунуштап, таасир этилүүчү системада материалдык чекитти сүрүлүүсүз жана сүрүлүүнүн негизинде кандайдыр бир аралыкка жылдырууда убакыттан көз каранды болгон энтропиянын өсүүсүнүн төмөнкү баасын алган. Бул макалада бул аныктамалар айкалыштырылган.

Урунттуу сөздөр: категория, морфизм, энтропия, башкаруу, дифференциалдык теңдеме, таасир этилүүчү система, кыймылдоо.

Ранее вторым автором было введено понятие категории уравнений с помощью понятия “предикат” на основе принципа сохранения решения при преобразованиях; построены элементы категории уравнений и ее подкатегорий на основе известных категорий.

Как уточнение второго закона термодинамики для изолированных систем, первый автор ввел новые определения, предложил общие гипотезы и получил оценки снизу для возрастания энтропии при передвижении материальной точки без трения и с трением на определенное расстояние в зависимости от времени в перманентно неустойчивых системах. В данной статье эти определения объединены.

Ключевые слова: категория, морфизм, энтропия, управление, дифференциальное уравнение, перманентно неустойчивая система, движение.

1. Introduction

Investigations in the category of topological spaces in Kyrgyzstan were initiated [1]. The notion of the category of equations was introduced [2] with assistance of the notion “predicate” on the base of the principle of preservation of solution while transformations; elements of the category of equations and its subcategories were constructed on the base of well-known categories.

As an improving of the second law of thermodynamic for isolated systems, on the base of [3], [4], [5], we put a general hypothesis on estimations from below on increment of entropy during motions in [6].

By using new definitions in [7] If low energetic outer influences can cause sufficiently various reactions and changings of the inner state of the system then it is said to be an “almost isolated”, “permanently unstable”, “affectable” system (A-system); such outer influences are said to be commands (these reactions and changings are implemented by means of inner free energy of the object) we found some estimations from below on increment of entropy while motion of a material point both without friction and with friction over definite distance on depending on time in such systems. Section 2 contains survey of known categories including the category of energy-entropy-optimization processes.

There are examples of some morphisms in this category in Section 3.

Remark. Morphisms are denoted as *Mor* or *Hom* in literature.

2. Survey of known categories

2.1) The basic category *Set* of sets. *Ob(Set)* are sets; *Mor(Set)* are functions.

2.2) The category *Func* of functions (synonyms in various branches of mathematics: maps, operators, transformations). It is mentioned in publications but we did not find its formal description. *Ob(Func) = Mor(Set)*, *Mor(Func)* are transformations of functions.

2.3) The category *Top* of topological spaces. *Ob(Top)* are topological spaces, *Mor(Top)* are continuous functions.

We proposed

2.4) Category of equations *Equa*. *Ob(Equa)* contains tuples $\{$Non-empty sets X, Y, predicate $P(x)$ on X, transformation $B: X \to Y\}$.

If $(\exists x \in X)(P(x) \wedge (y = B(x)))$ then $y \in Y$ is said to be a solution of the equation $\{X, Y, P, B\}$. Particularly, if B is the identity operator I, then we obtain the equation *"P(x)"*only. *Mor(Equa)* are such transformations of tuples $\{X, Y, P, B\}$ that solutions (or their absence) preserve.

We also defined some subcategories of *Equa:*

2.5) The category *Equa-Func* of equations for functions.

2.6) The category *Equa-Par* of equations with parameters.

2.7) The category *Equa-Func-Par* of equations for functions with parameters (including control).

2.8) The category *Equa-Func-Ent* of energy-entropy-optimization controlled processes.

Denote the physical dimension as *dim; dim(time)=* τ; *dim(length)=*λ; *dim(mass)=* μ; *dim* (*absolute temperature* Θ) $= \vartheta$. Then *dim(energy)=* $\mu\lambda^2/\tau^2$.

In processes considered below increment of entropy is defined as $\Delta H = \sum \Delta Q / \Theta$ where ΔQ is energy that was converted to heat irreversibly; $dim(\Delta H) = \mu\lambda^2/\tau^2/\vartheta$.

The second law of thermodynamic for isolated systems states increment of entropy $\Delta H > 0$ but does not give quantitative estimations. One estimation was proven in [4]: the minimal energy (increment of entropy) to treat one bit of information is the Shannon-von Neumann-Landauer boundary:

$\Delta H_{bit} = k_B \ln 2$ where k_B is the Boltzmann constant.

A hint to such estimation in general case was in [5]: "In any mechanical system the energy that must be expended to work against friction is equal to the product of the frictional force and the distance through which the system travels. Hence the faster a swimmer travels between two points, the more energy he or she will expend, although the distance traveled is the same whether the swimmer is fast or slow."

Basing on the notion of economical (cruising) speed, taking into account ide-as of Ecological Rallies for cars we specified the second law of thermodynamic for A-systems. We proved some estimations in mathematical models describing such concrete systems [8]-[9]-[10].

Let there is a (closed) system. Let it is in any stationary state *A* now and there it can pass to any other stationary state *B*.

Hypothesis [6]. There exists such time T_0 (the adiabatic time of the system), depending only on the initial state of the system, that ΔH is not less than any positive value for any transition from the state *A* to the state *B* during $T<T_0$. Moreover, there also exists such positive constant C_0 that $\Delta H \geq C_0 / T^2$; $dim(C_0) = \mu\lambda^2/\vartheta$.

By the principle of determinism, there is only scenario of the future for any iso-lated system, that is, there cannot be different possibilities of transitions. Hence, the system is to be A-system (with unbounded volume of inner free energy): different possible actions, transferring it from the state A to the state B by means of irreversibly converting free energy to heat which are controlled by any outer impetus of sufficie-ntly small energy.

Ob(Equa-Func-Ent) are A-systems with the task (*): to transfer the state A to the state B in time T with minimal increment of entropy ΔH.

Mor(Equa-Func-Ent) are transformations of A-systems preserving the minimal increment of entropy.

3. Examples of morphisms

Example 3.1. Let B be over A in gravitation field with constant q, $dim\ q = \lambda/\tau^2$. There is the load M of mass m based on many compressed short almost ideal springs at the point A, the massive brake with temperature Θ along the segment AB and the catcher at the point B; (*).

It is proven in [11] that the adiabatic time is $T_0 := \sqrt{(2h/q)}$ and

sup $\Delta H = mh^2/T^2 (1 - T^2/T_0{}^2)^2/2/\Theta$.

Example 3.2. Let A be over B at the height h in gravitation field and B be on an absolutely elastic plane. There is the load of mass m at the point A, the massive brake with the absolute temperature Θ along the segment AB and the catcher at the point A. We can apply unbounded force to the load. It is necessary to return the load to the point A by repelling at the point B during the time T.

It is proven in [11] that the adiabatic time is $T_0 := 2\sqrt{(2h/q)}$ and

sup $\Delta H = 2mh^2/T^2 (1 - T^2/T_0{}^2)^2/\Theta$.

Theorem 1. Changing $(m, h, q) \to (m/\eta^2, \eta h, \eta q)$, $\eta > 0$ yields a morphism.

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APPLICATION OF ARITHMETIC PROGRESSIONS IN THE CONSTRUCTION OF HIGH-ORDER FRACTAL MAGIC MATRICES

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High-order magic matrices that preserve row-, column-, and diagonal-sum invariants under scaling are of interest in discrete mathematics, combinatorics, and the theory of structured matrices. In this paper we propose a constructive framework for producing such matrices with an explicit self-similar (fractal) block structure. The method combines a decomposition principle with arithmetic progressions that serve as compact identifiers of block offsets. We state sufficient conditions under which the global magic property is inherited from a base magic square, and we describe a complete step-by-step construction of a 9×9 magic square derived from the classical 3×3 Lo Shu pattern. The construction yields a matrix whose rows, columns, and main diagonals sum to the same constant, while each embedded 3×3 block is itself a magic square, thereby exhibiting scale invariance. Brief remarks on possible uses in discrete modeling, permutation-based coding, and cryptography are included without shifting emphasis away from the theoretical results.

Keywords: magic squares; magic matrices; fractal structures; arithmetic progression; decomposition method; self-similarity

Дискреттик математика, комбинаторика жана структураланган матрицалар теориясын масштабдоодо катар-, мамычаларды жана диагоналдык суммаларды инварианттарды сактаган жогорку тартиптеги сыйкырдуу матрицалар кызыктырат. Бул макалада биз ачык өзүн-өзү окшош (фракталдык) блок структурасы менен мындай матрицаларды өндүрүү үчүн конструктивдүү негизди сунуштайбыз. Метод ажыратуу принцибин арифметикалык прогрессия менен айкалыштырат, алар блоктордун офсеттеринин компакт идентификаторлору катары кызмат кылат. Биз глоалдык сыйкырдуу касиет базалык сыйкырдуу квадраттан тукум кууп өткөн жетиштүү шарттарды айтабыз жана классикалык 3×3 Ло Шу үлгүсүнөн алынган 9×9 сыйкырдуу квадраттын толук этап-этабы менен курулушун сүрөттөп беребиз. Куруу матрицаны берет, анын саптары, мамычалары жана негизги диагоналдары бирдей константага барабар болот, ал эми ар бир киргизилген 3×3 блоктун өзү сыйкырдуу квадрат болуп, масштабдын өзгөрүүсүн көрсөтөт. Дискреттик моделдештирүү, алмаштырууга негизделген коддоо жана криптографияда мүмкүн болгон колдонуулар боюнча кыскача эскертүүлөр басымды теориялык натыйжалардан алыстатпастан камтылган.

Урунттуу сөздөр: сыйкырдуу квадраттар; сыйкырдуу матрицалар; фракталдык структуралар; арифметикалык прогрессия; ажыратуу ыкмасы; өзүнө окшоштук

Магические матрицы высокого порядка, сохраняющие инвариантность сумм строк, столбцов и диагоналей относительно масштабирования, представляют интерес в дискретной математике, комбинаторике и теории структурированных матриц. В данной статье мы предлагаем конструктивную основу для получения таких матриц с явной самоподобной (фрактальной) блочной структурой. Метод сочетает принцип разложения с арифметическими прогрессиями, которые служат компактными идентификаторами смещений блоков. Мы формулируем достаточные условия, при которых глобальное магическое свойство наследуется от базового магического квадрата, и описываем полное пошаговое построение магического квадрата 9×9, полученного из классического шаблона Ло Шу 3×3. Построение дает матрицу, сумма строк, столбцов и главных диагоналей которой равна одной и той же константе, в то время как каждый вложенный блок 3×3 сам по себе является магическим квадратом, демонстрируя тем самым масштабную инвариантность. Включены краткие замечания о возможных применениях в дискретном моделировании, кодировании на основе перестановок и криптографии без смещения акцента от теоретических результатов.

Ключевые слова: магические квадраты; магические матрицы; фрактальные структуры; арифметическая прогрессия; метод разложения; самоподобие

1. Introduction

Magic squares are classical objects in discrete mathematics and number theory and remain a convenient testbed for studying symmetry, invariants, and structured combinatorial constructions. Alongside their historical development, modern research has expanded the subject to families of related objects—magic rectangles, pandiagonal and multimagic squares, and structured magic matrices—where one seeks constructive principles that scale with matrix order. In a different direction, fractal geometry formalizes self-similarity and recursive generation of complex structures across scales. The notion of self-similarity is particularly natural for block-structured discrete objects: a large object may be assembled from smaller copies in a way that preserves defining constraints. Motivated by this viewpoint, we study magic matrices with a hierarchical block decomposition in which each block is itself a magic square and the arrangement of blocks reproduces the base pattern. The central technical question is: under what conditions does such a block assembly inherit the global magic property? This paper answers this question for a broad and practically useful class of constructions driven by arithmetic progressions.

Our contribution is threefold. First, we formalize a decomposition scheme in which a high-order magic matrix of order $N=n^2$ is built from n^2 blocks of order n. Second, we introduce arithmetic-progression identifiers that encode block offsets and allow a compact description of large constructions. Third, we provide an explicit worked example for $N=9$ based on the Lo Shu square, including schematics and the complete numerical 9×9 matrix. The presentation is intentionally constructive: every step can be reproduced directly and is suitable for implementation.

2. Definitions and Preliminaries

A magic square of order n is an $n\times n$ matrix $M=(m_{ij})$ with integer entries such that the row sums, column sums, and the sums along the two main diagonals are equal to a constant μ (the magic constant). For a normal magic square containing the integers $\{1,2,\dots,n^2\}$, the magic constant equals $\mu = n(n^2+1)/2$. We will use standard operations that preserve magic structure. In particular, adding a constant c to every entry of a magic square increases every row sum by nc and therefore produces another magic square with magic constant $\mu+nc$. In addition, applying any symmetry of the square (rotations/reflections) preserves the magic property.

To describe self-similar constructions, we view a large matrix as a block matrix. Let $N=n^2$. A matrix of order N is partitioned into an $n\times n$ grid of blocks, each block having size $n\times n$. We denote the block at block-row a and block-column b by $M^{(a,b)}$. A construction is called self-similar if each block $M^{(a,b)}$ is a copy of a base magic square B up to an additive shift (and possibly a symmetry), and the pattern of shifts itself follows the layout of a magic square.

3. Decomposition Method Based on Arithmetic Progressions

Let B be a fixed base magic square of order n with magic constant μ_B. We construct a matrix M of order $N=n^2$ as follows. First, we choose n^2 disjoint sets of integers, each of size n^2. For the canonical normal construction at $N=9$, we work with the set $\{1,\dots,81\}$ and partition it into nine consecutive blocks of size 9: Block 1: 1–9; Block 2: 10–18; …; Block 9: 73–81. Each block forms an arithmetic progression with common difference 1. More generally, one may use arithmetic progressions with common difference d and translate them to fit the desired numeric range.

Second, we turn each block into a magic square by placing its entries in the same relative positions as in B. Concretely, if B uses the labels $\{1,\dots,n^2\}$, then within a block we place the smallest element at the position of 1 in B, the second smallest at the position of 2 in B, and so on. This produces a block-matrix whose every $n\times n$ block is a magic square structurally identical to B.

Third, we arrange these n^2 blocks into an $n\times n$ array according to the layout of B (or another fixed magic square of order n acting as a block-addressing template). At this stage, the arithmetic progression associated with each block acts as an identifier of the block offset. The key observation is that the sums of block magic constants along any block-row, block-column, and block-diagonal are equal, because the identifiers are placed according to a magic pattern. This ensures that the assembled N×N matrix has equal row/column/diagonal sums.

4. Step-by-Step Construction of the 9×9 Matrix

We now present the complete construction for n=3 (N=9). Step 1: choose the base 3×3 Lo Shu magic square (Figure 1). This square serves as the positional template used both inside each block and for the arrangement of blocks. Step 2: partition the integers 1–81 into nine consecutive arithmetic blocks of length 9: [1–9], [10–18], …, [73–81]. Each block will populate one 3×3 submatrix. Step 3: for each block, place its nine entries in the Lo Shu positions. This guarantees that every 3×3 block is a magic square with magic constant 15 plus a block-dependent offset.
Step 4: arrange the nine blocks in a 3×3 block grid using the same Lo Shu layout as the block-addressing pattern. Figure 2 visualizes the resulting block structure (labels B1–B9). The final fully populated matrix is shown in Figure 3. Remark on readability: Figures 1–3 are intended to support different reading modes. Figure 1 specifies the base pattern, Figure 2 shows how blocks are positioned, and Figure 3 provides the explicit numerical matrix for verification and reuse.

Figure 1

Base 3×3 magic square (Lo Shu) used as the positional template.

8	1	6
3	5	7
4	9	2

Figure 1. The base 3×3 Lo Shu magic square.

Figure 2

Block representation of the 9×9 construction. Each label Bk denotes one 3×3 magic block.

B1	B1	B1	B2	B2	B2	B3	B3	B3

B1	B1	B1	B2	B2	B2	B3	B3	B3
B1	B1	B1	B2	B2	B2	B3	B3	B3
B4	B4	B4	B5	B5	B5	B6	B6	B6
B4	B4	B4	B5	B5	B5	B6	B6	B6
B4	B4	B4	B5	B5	B5	B6	B6	B6
B7	B7	B7	B8	B8	B8	B9	B9	B9
B7	B7	B7	B8	B8	B8	B9	B9	B9
B7	B7	B7	B8	B8	B8	B9	B9	B9

Figure 2. Block structure of the 9×9 fractal magic matrix (each 3×3 block is a copy of the base square up to an additive shift).

Figure 3

The complete 9×9 matrix obtained by the construction; it can be checked directly that every row, column, and main diagonal sums to 369.

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

Figure 3. Fully constructed 9×9 fractal magic square (global magic constant 369; each 3×3 block is magic with constant 15 plus an offset).

5. Properties of the Constructed Matrix

Global magic constant. For the normal 9×9 case, the magic constant is $\mu = 9(9^2+1)/2 = 9\cdot 82/2 = 369$. Direct inspection of Figure 3 verifies that each row and column sums to 369 and the two main diagonals also sum to 369. Local magic property. Each 3×3 block is constructed by placing a contiguous arithmetic block into the Lo Shu positions. Since the Lo Shu pattern is magic, each block is magic. If a block is formed by adding a constant c to the entries {1,…,9} positioned as Lo Shu, then its magic constant equals 15 + 3c. Scale invariance and self-similarity. The matrix exhibits self-similarity in a strict combinatorial sense: the same positional pattern appears both at the level of blocks and within blocks. The invariants defining a magic square are therefore preserved across two scales (3 and 9). The construction can be iterated further by treating the 9×9 matrix as a base pattern for a 27×27 matrix, provided the identifiers are chosen consistently.

Generalization. The same reasoning extends to any base order n where an appropriate base magic square exists. The decomposition scheme provides sufficient conditions for preserving magic sums under block assembly; the arithmetic progressions serve as a convenient family of identifiers for constructing blocks with controlled offsets.

6. Possible Applications

The present contribution is primarily theoretical. Nevertheless, high-order self-similar magic matrices provide structured test instances for combinatorial algorithms (e.g., symmetry detection, enumeration under constraints) and for discrete modeling where hierarchical constraints are desired. In coding theory and cryptography, the combination of regularity (deterministic construction) and combinatorial variability (choice of block identifiers, admissible symmetries of blocks, and block permutations that preserve the global constraints) suggests possible uses in permutation-based transforms and in the generation of key-dependent structured matrices. These directions require a separate security and statistical analysis and are mentioned here only as potential future work.

7. Conclusion

We proposed a constructive method for building high-order fractal magic matrices using arithmetic progressions within a decomposition framework. The method provides a transparent mechanism for preserving the magic property globally while ensuring that each embedded block remains magic, thus producing a clear self-similar structure. A complete worked example for the 9×9 case derived from the 3×3 Lo Shu square was presented together with schematic figures and the explicit final matrix. The approach is scalable and can be extended to higher orders and additional recursion levels; future work includes classification of admissible identifier families, study of symmetries and equivalence classes, and exploration of algorithmic and applied aspects.

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MSC 49M37

ABOUT ONE PROBLEM OF OBTAINING A PLANNED CROP YIELD WITH MINIMUM TOTAL FARM COSTS

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This article formulates an economic and mathematical model for determining the optimal area for each crop to achieve the planned yield with minimal total costs. The model's feasibility is demonstrated using a numerical example.

Key words: Peasant farm, sown area, optimal size, profit, mathematical model, price, unit of weight, products, costs, yield, agricultural crop.

Бул макалада ар бир өсүмдүк үчүн пландаштырылган түшүмдү минималдуу жалпы чыгымдар менен алуу үчүн оптималдуу аянтты аныктоонун экономикалык жана математикалык модели түзүлгөн. Моделдин ишке ашыруу мүмкүнчүлүгү сандык мисалды колдонуу менен көрсөтүлөт.

Урунттуу сөздөр: Дыйкан чарбасы, айдоо аянты, оптималдуу өлчөм, пайда, математикалык модель, баа, салмак бирдиги, продукциялар, чыгымдар, түшүмдүүлүк, айыл чарба өсүмдүктөрү.

В статье сформулирована экономико-математическая модель задачи определения оптимального размера посевной площади под каждый вид культуры с целью получения запланированного урожая при минимальных суммарных затратах. Работоспособность данной модели продемонстрирована на числовом примере.

Ключевые слова: Крестьянское хозяйство, посевные площади, оптимальный размер, прибыль, математическая модель, цена, единица веса, продукция, затраты, урожайность, сельхоз культура.

Introduction. Agriculture in Kyrgyzstan is crucial for providing employment for the working population, creating resources for the sustainable development of other sectors of the national economy, meeting the country's food needs, and promoting overall economic development [7, 9, 10, 12].

In agriculture, land is used as the primary, irreplaceable means of production. Compared to other means of production, land does not wear out, and when properly managed, it improves its quality.

Arable land in the Kyrgyz Republic has varying fertility potential depending on the climate.

The reform carried out in the agricultural sector has enabled rural residents to own land plots with varying soil structure and fertility, and in different locations

[5, 6]. Having received land plots, many owners began engaging in agricultural production, forming numerous peasant (farm) households, cooperatives, business partnerships, and joint-stock companies [1, 2, 3, 4].

Since the production activities of these farms are based on the use of land plots and the cultivation of crops on them, the question arises of how to determine the optimal sown area for a given crop in order to obtain the planned volume of production for each crop at minimal cost.

The problem of finding the optimal sown area and the crops grown on it is especially relevant now that society has transitioned to market relations.

Currently, there are a number of studies [8-11, 13] where optimization problems are widely used in economic and mathematical research.

When modeling the optimization of a farm's sown area structure, plot types and crop types are used as variables, which determine the plot size for a specific crop [1, 2].

This practice is key in modern agriculture, especially for those farmers or peasant farms who do not have sufficient land of their own to conduct business.

Below we will attempt to formulate an economic and mathematical model for determining the optimal size of the sowing area for each crop, where the possible planned harvest volume is indicated.

Problem statement. A farm has n plots of different arable land, each with an area equal to S_j, $j=1..n$, on which m types of crops are sown. When allocating crops to arable land, it is important to consider that the same crop may be sown in different plots of the arable land. Some plots are particularly suitable for certain crops, while others are completely unsuitable. For example, a crop that prefers moist soil cannot be sown on sandy soils, meaning the yield $a_{ij} = 0$.

It is assumed that the yield of the i-th crop on the j-th plot of the farm's cultivated area is equal to units, and the cost of obtaining the m-th output from a unit of the j-th cultivated area is $j = 1..\ n$.

The farm plans to obtain a yield of the i-th crop of at least P_i, $i = 1..\ m$.

It is necessary to determine the optimal size of the cultivated area for each crop and obtain the planned yield so that the total costs of growing the crop are minimized. Let's formulate a mathematical model of the problem.

Let's introduce the following notation:

i is the index of the type of agricultural crop sown by the farm, $i=1..m$;

j is the type of sown area, $j=1..n$;

a_{ij} is the yield of the *i*-th agricultural crop per unit of sown area of the *j*-th plot of the farm, *i=1..m, j=1..n*;

c_{ij} is expenses incurred for growing the *i*-th type of agricultural crop on a unit of the *j*-th sown area of the farm, $c_{ij}=(\frac{1}{2}\varepsilon^i_j+\delta^i_j)a_{ij}c_i$, *i=1..m, j=1..n;*

ε^i_j is share of management expenses per unit of volume of the *i*-th type of agricultural crop on the *j*-th category of sown area, *i=1..m, j=1..n*;

δ^i_j is share of wages per unit of volume of the *i*-th type of agricultural crop on the *j*-th category of sown area, *i=1..m, j=1..n*;

c^i_j is total payment for the service for growing the i-th type of agricultural crop on the j-th category of sown area, i=1..m, j=1..n;

P_i is planned maximum volume of each type of agricultural crop, *i=1..m*;

S_j is dimensions of the *j*-th plot of the sown area, *j=1.. n*;

x_{ij} is unknown values determining the size of the plot of the sown area for each agricultural crop on the farm, *i=1.. m, j=1..n*

According to the accepted notation, the mathematical model of the problem has the form.

Find the minimum $L(x)=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}$ (1)

under the conditions

$$\sum_{j=1}^{n}a_{ij}x_{ij}\geq P_i,\quad i=1..m, \tag{2}$$

$$\sum_{i=1}^{m}x_{ij}\leq S_j,\quad j=1..n, \tag{3}$$

$$x_{ij}\geq 0,\quad i=1..m,\quad j=1..n \tag{4}$$

where x= $\left|x_{ij}\right|_{m,n}$.

Inequality (2) indicates that the amount of produce obtained from the i-th agricultural crop, harvested from all plots, should not be less than the planned amount.

Equality (3) indicates that the sown area on all plots of the farm should be fully sown.

Equality (4) assumes that there should be no unsown plots.

The proposed model considers the optimization of sown area distribution among several crops to achieve the planned yield. Within this model, the farm's costs are determined.

To solve problems (1)-(4), we use the problem-solving methods presented in [14].

The economic-mathematical model of problem (1)-(4) for determining the optimal size of the sown area for each agricultural crop and obtaining the planned harvest volume is presented in Table 1.

Table 1

№	x_{11}	x_{12}	…	x_{1n}	x_{21}	x_{22}	…	x_{2n}	…	x_{m1}	x_{m2}	…	x_{mn}	
1	a_{11}	a_{12}	…	a_{1n}			…					…		P_1
2					a_{21}	a_{22}	…	a_{2n}						P_2
…			…						…			…		…
m										a_{m1}	a_{m2}	…	a_{mn}	P_m
1	1				1					1				S_1
2		1				1					1			S_2
…			…				…		…			…		…
n				1				1					1	S_n
	c_{11}	c_{12}	…	c_{1n}	c_{21}	c_{22}	…	c_{2n}	…	c_{m1}	c_{m2}	…	c_{mn}	*min*

Problem solving algorithm. Calculations begin with introducing the following notation into the formula: $c_{ij} = (\frac{1}{2}\varepsilon_j^i + \delta_j^i) a_{ij} c_i$, $i = 1.. m,\ j = 1.. n;$

Next, using the parameter values, we calculate the unit costs of each *i*-th crop type on the *j*-th category of crop area. Then, based on these data and P_i, S_j a numerical model of the problem is formulated according to (1)–(4) and the problem is solved using the method given in [14]. The solution method is complete.

From the solution of problem (1)-(4), we determine the size of the sown area for each crop type to obtain the planned yield, $i = 1.. m,\ j = 1..n.$

To test the validity of the presented economic-mathematical model, we will consider an example using data from a sole proprietor in the village of "Mramornoye" in the Alamudun district of the Chui region.

EXAMPLE. A sole proprietor has three types of land with different soil structures and fertility. The sown area of the first plot is 1 hectare, i.e., S_1 = 1 hectare; that of the second plot is 1.3 hectares, i.e., S_2 = 1.3 hectares; and that of the third plot is 1.6 hectares, i.e., S_3 = 1.6 hectares.

On these plots, the sole proprietor intends to sow three crops: wheat, barley, and potatoes, and plans to obtain a yield of at least P_i for each crop, i.e., $P_i =(P_1; P_2; P_3)$ = (5.0; 7.0; 8.0);

The expected yield per hectare for growing these crops from each hectare of the sown area, based on long-term data, is assumed to be known and is given in Tables 2.

Table 2

Expected yield per hectare in plots, in kg

	1-й участок	2-й участок	3-й участок
Potatoes	5000	4700	4200
Wheat	5300	5500	5000
Corn	4900	5300	4700

– projected wholesale market prices of agricultural crops $\left|c_i\right|_{1,3} = (45,\ 25,\ 20)$, (som/kg)

– total payment for the service of growing the ***i***-th type of agricultural crop in the first plot of the sown area, i = 1..m, j = 1; = {28060, 21930, 21930} (som/ha).

We further assume that the payment for services per unit of sown area in the first plot also applies to the remaining plots of sown areas with different soil structure and fertility.

We determine δ_j^i the share of wages per unit of volume of each type of agricultural crop for each category of sown area using the formula:

$$\delta_j^i = \left(c_j^i\right) / a_{ij} \cdot c_i, \quad i=1,2,3, \quad j = 1,2,3,$$

we will receive

$$\left|\delta_j^i\right|_{3,3} = \begin{pmatrix} 0.2079 & 0.1559 & 0.3118 \\ 0.2371 & 0.2924 & 0.3508 \\ 0.3426 & 0.3426 & 0.2436 \end{pmatrix}.$$

The share of management costs (overhead costs) ε_j^i in a unit of volume of agricultural crops of each type for each category of sown area is assumed to be equal to the value:

$$\varepsilon_j^i = \frac{1}{4}\delta_j^i, \quad i = 1,2,3, \; j = 1,2,3.$$

We have $$\left|\varepsilon_j^i\right|_{3,3} = \begin{pmatrix} 0.0519 & 0.0389 & 0.0779 \\ 0.0592 & 0.0731 & 0.0877 \\ 0.0856 & 0.0609 & 0.0856 \end{pmatrix}.$$

Let us determine the value c_{ij}, $i=1,2,3$, $j=1,2,3$ by the formula

$$c_{ij} = (\frac{1}{2}\varepsilon_j^i + \delta_j^i)a_{ij}c_i, \quad i=1,2,3, \; j=1,2,3.$$

We have $$\left|c_{ij}\right|_{3,3} = \begin{pmatrix} 31563,0 & 31554,0 & 31563,0 \\ 23069,5 & 24667,5 & 24662,5 \\ 24665,6 & 33570,0 & 20048,0 \end{pmatrix}.$$

We need to determine the optimal size of the cultivated area allocated for a given crop and obtain the planned yield for each crop so that the total costs of growing the crop are minimized. Using the introduced notations, we will write the problem conditions as the following numerical model.

According to the formulated mathematical model (1)-(4), we will write the numerical model of the problem.

Find the minimum:

$$L(x) = 31563.0x_{11} + 31554.0x_{12} + 31563.0x_{13} + 23069.5x_{21} + \\ + 24667.5x_{22} + 24662.5x_{23} + 24665.6x_{31} + 33570.0x_{32} + 20048.0x_{33} \quad (5)$$

under the conditions

$$5000x_{11} + 5300x_{12} + 4900x_{13} \ge 5000$$
$$4700x_{21} + 5500x_{22} + 5300x_{23} \ge 7000 \quad (6)$$
$$4200x_{31} + 5000x_{32} + 4700x_{33} \ge 8000$$
$$x_{11} + x_{12} + 5x_{13} \le 1.0$$
$$x_{21} + x_{22} + x_{23} \le 1.3$$
$$x_{31} + x_{32} + x_{33} \le 1.6 \quad (7)$$
$$x_{ij} \ge 0, \quad i=1,2,3, \quad j=1,2,3, \quad (8)$$

The conditions of this problem are presented in Table 3.

Table 3

Initial data entry for solving the problem

	i =1			i=2			i=3		
	j=1	j=2	j=3	j=1	j=2	j=3	j=1	j=2	j=3

№	P_i	a_{11}	a_{12}	a_{13}	a_{21}	a_{22}	a_{23}	a_{31}	a_{32}	a_{33}
1	5000	5000	5300	4900						
2	7000				4700	5500	5300			
3	8000							4200	5000	4700
4	1.0	1	1	1						
5	1.3				1	1	1			
6	1.6							1	1	1

To solve this problem, we used the methods described in [14].

As a result of solving the problem, we obtained the optimal solution:

X= $\{x_{12}=0,96;\ x_{22}=1,27;\ x_{32}=1,6\}$, which resulted in the planned production volumes and minimum costs of $L(x) = 115{,}470.3$ thousand soms.

As can be seen, with such an optimal distribution of land plots between potatoes, wheat, and corn, the minimum value of the objective function L(x) = 115,470.3 thousand soms was achieved.

Consequently, 0.96 hectares of land from plot 1 were allocated to the second crop type, yielding a planned yield of 5,000 kg. From plot 2, 1.27 hectares were allocated to the second crop type, yielding a yield of 7,000 kg. From plot 3, 1.6 hectares were allocated to the second crop type, yielding a planned yield of 8,000 kg.

The experimental work carried out by the individual entrepreneur of the Alamudun district in the Chui region to find the optimal area demonstrated the full feasibility of using the economic and mathematical model of the problem to solve practical problems. The mathematical apparatus used in the problem made it possible to find effective solutions both in determining the sowing area and in obtaining the planned harvest with minimal costs.

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AN EDUCATIONAL PLATFORM FOR HANDS-ON LEARNING OF ROBOT DESIGN, MATHEMATICAL MODELING, AND AI-ROS BASED CONTROL

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This paper presents a unified robotics education platform using ROS, ESP-ROS-Lite, and C++ on a low-cost ESP32 with Fusion 360–optimized hardware. It enables real-time servo control via an interface and supports stability modeling through concepts such as Center of Mass, Zero Moment Point, and support polygon. AI services including ChatGPT, Google, and Copilot are integrated through Python for voice, text, and facial recognition, enhancing interactive robot control. Experiments confirm the platform's effectiveness in teaching kinematics, dynamics, and PID control, highlighting its value for robotics education.

Key words: ROS-inspired robot control, PID control, Center of Mass (CoM), Zero Moment Point (ZMP), support polygon, AI, facial recognition, C++, Python

Бул макалада арзан баадагы ESP32 микроконтроллерине негизделген, Autodesk Fusion 360 программасында генеративдик долбоорлоо ыкмасы аркылуу оптималдаштырылган жабдыктарды колдонгон, ROS концепциясынан шыктанган бирдиктүү робототехника боюнча билим берүү платформасы сунушталат. Платформа ROS, ESP-ROS-Lite жана C++ технологияларына таянып, сервоприводдорду реалдуу убакыт режиминде башкарууну камсыз кылат. Ошондой эле роботтун туруктуулугун моделдөөнү борбордук масса (Center of Mass, CoM), нөлдүк моменттик чекит (Zero Moment Point, ZMP) жана колдоо көпбурчтугу сыяктуу негизги түшүнүктөрдүн негизинде ишке ашырат.

Python тили аркылуу ChatGPT, Google жана Copilot сыяктуу жасалма интеллект сервистери интеграцияланып, үн, текст жана жүздү таануу функцияларын ишке ашырууга мүмкүндүк берет. Бул өз кезегинде робот менен интерактивдүү өз ара аракеттенүүнүн деңгээлин жогорулатат. Эксперименттик жыйынтыктар сунушталган платформа кинематика, динамика жана PID башкаруу ыкмаларын окутууда жогорку натыйжалуулукка ээ экенин көрсөтүп, робототехника багытындагы билим берүү үчүн маанилүү курал экенин тастыктайт.

Урунттуу сөздөр: ROS негизиндеги робот башкаруу, PID башкаруу, борбордук масса (CoM), нөлдүк моменттик чекит (ZMP), колдоо көпбурчтугу, жасалма интеллект, жүздү таануу, C++, Python

В данной статье представлена унифицированная образовательная платформа по робототехнике, основанная на недорогом микроконтроллере ESP32 и аппаратном обеспечении, оптимизированном с использованием генеративного проектирования в Autodesk Fusion 360. Платформа использует концепции ROS, ESP-ROS-Lite и язык программирования C++ для обеспечения управления сервоприводами в режиме реального времени. Моделирование устойчивости робота реализуется на основе ключевых понятий, таких как центр масс (Center of Mass, CoM), нулевая моментная точка (Zero Moment Point, ZMP) и опорный многоугольник.

Интеграция сервисов искусственного интеллекта, включая ChatGPT, Google и Copilot, осуществляется с использованием языка Python и обеспечивает голосовое и текстовое взаимодействие, а также распознавание лиц. Это существенно повышает уровень интерактивности управления роботом. Экспериментальные результаты подтверждают эффективность предложенной платформы в обучении кинематике, динамике и PID-регулированию, подчеркивая её значимость для образовательных задач в области робототехники.

Ключевые слова: управление роботом на базе ROS, PID-регулирование, центр масс (CoM), нулевая моментная точка (ZMP), опорный многоугольник, искусственный интеллект, распознавание лиц, C++, Python

1. Introduction

1.1 Background: Rise of Intelligent Robotics and the Need for Hands-On Education

Robotics has expanded beyond the manufacturing sector into diverse fields such as healthcare and education, and is rapidly evolving into intelligent robotic systems through integration with artificial intelligence (AI), the Internet of Things (IoT), and sensor technologies. These developments highlight the need for hands-on, practice-oriented education that goes beyond theoretical learning or assembly-focused training, enabling learners to directly experience the entire development cycle - from design and fabrication to control and AI integration.

1.2 Research Objective: Full-Cycle Open Educational Robot Platform

The objective of this study is to propose a full-cycle, open educational robot platform that enables learners to independently carry out the entire robot development process. The proposed platform provides an integrated learning environment encompassing mechanical design, manufacturing, electronics, control, and artificial intelligence, thereby overcoming the limitations of existing educational robot systems characterized by closed architectures and fragmented modular structures.

The core components of the platform include:

- Generative design and 3D printing based on Fusion 360
- Modular robot assembly and hardware prototyping
- Microcontroller-based circuit design and control implementation
- A lightweight, ROS-inspired real-time framework (ESP-ROS-Lite) implemented in C++
- Voice- and vision-based AI interaction (LLMs and face recognition)

2. Robot Design Using Generative Topology Optimization in Fusion 360

2.1 Introduction

Topology optimization is an effective method for designing lightweight, high-performance structures, and Autodesk Fusion 360's Generative Design offers an accessible platform that applies structural mechanics and numerical optimization to optimize material distribution under mechanical constraints.

2.2 Design Variables

Let the design space be discretized into finite elements. Each element $i-th$ is assigned a design variable $x_i \in [0,1]$, defined as:

$$x_i = \begin{cases} 1 \ \ if\ material\ is\ present\ (solid) \\ 0 \ \ if\ material\ is\ void \\ 0 < x_i < 1 \ \ \ if\ intermediate \end{cases}$$

These variables form the foundation of the optimization model by describing the material layout within the design domain.

2.3 Objective Function: Minimization of Compliance

The primary goal in structural topology optimization is to minimize compliance, which corresponds to maximizing the stiffness of the structure under given loads and boundary conditions. The compliance objective function is expressed as:

$$C = u^T K(\rho) u$$

Where:

- C : Compliance
- u : Displacement vector
- $K(\rho)$: Global stiffness matrix dependent on design variables ρ.

2.4 Application in Fusion 360

Fusion 360's Generative Design engine uses the above mathematical principles to create high-efficiency structural forms. Users input design space geometry, load cases, constraints, and performance targets. The system then evaluates numerous candidate geometries using cloud-based solvers and presents multiple optimized results.

Visual outputs generated by the platform allow designers to explore structurally sound yet material - efficient configurations - bridging simulation and manufacturability.

3. Stability, Kinematics, and Motion Control

3.1 Introduction

This chapter presents an integrated robot motion control framework that introduces and conceptually compares PID, MPC, and Whole-Body Control (WBC), while ensuring stability and precision through key balance concepts such as the Center of Mass, Zero Moment Point, and support polygon.

3.2 Control Algorithms

3.2.1 Proportional–Integral–Derivative (PID) Control

PID control is a classical feedback control strategy that adjusts system inputs based on the difference between the reference signal $r(t)$ and the actual output $y(t)$. The control law is defined as:

$$u(t) = K_p e(t) + K_I \int_0^t e(\tau)d\tau + K_D \frac{de(t)}{dt}$$

Where:

- $u(t)$: Control input (e.g., torque or PWM signal)
- $e(t) = r(t) - y(t)$: Control error
- K_p, K_I, K_D : Proportional, Integral, and Derivative gains

This method is widely used due to its simplicity and effectiveness for real-time control.

3.2.2 General Model Predictive Control (MPC) Optimization

MPC is an advanced control strategy that predicts future system behavior over a finite time horizon and solves an optimization problem at each control step. The objective function is:

$$\min_{u_0,\dots,u_{N-1}} \sum_{k=0}^{N-1} \left(\|x_{t+k+1} - x_{ref}\|_Q^2 + \|u_{t+k}\|_R^2 \right)$$

Subject to:

$$x_{t+k+1} = Ax_{t+k} + Bu_{t+k} \ (system\ dynamic)$$

$$x_{t+k} \in x,\ u_{t+k} \in u\ (constraints)$$

Where:

- x_t : State at time t
- u_t : Control input
- x_{ref} : Desired reference state
- Q, R : Weighting matrices

- A, B : system dynamics matrices
- N : Prediction horizon

MPC is particularly useful for systems requiring multi-step planning and constraint handling.

3.2.3 Whole Body Control (WBC)

WBC is an optimization-based method suited for humanoid and legged robots, where full-body coordination is required. It considers the complete dynamics of the robot and solves a constrained optimization problem in real time.

Dynamic equation:

$$M(q)\ddot{q} + h(q,\dot{q}) = S^T\tau + J_c^T\lambda$$

Where:

- q : Generalized coordinates (joint + base)
- $\ddot{q}$: Generalized acceleration
- $M(q)$: Inertia matrix
- $h(q,\dot{q})$: Coriolis, centrifugal, and gravity terms
- $S^T\tau$: Actuator torques
- J_c : Contact Jacobian
- λ : Contact forces

Optimization objective:

$$\min_{\tau,\lambda} \sum_i \left\| J_i(q)\ddot{q} + \dot{J}_i(q,\dot{q})\dot{q} - \ddot{x}_i^{ref} \right\|^2$$

Subject to:

- System dynamics
- Torque limits: : $\tau_{min} \le \tau \le \tau_{max}$
- Contact and friction constraints

Whole-Body Control (WBC) allows the simultaneous handling of multiple tasks, such as posture control and footstep placement, ensuring smooth, stable, and coordinated robot motion. In this study, WBC is discussed at a conceptual and educational level.

3.3 Key Stability Models

3.3.1 Center of Mass (CoM)

The Center of Mass is the point where the total mass of the robot can be considered to be concentrated. It plays a crucial role in dynamic stability.

$$r_{CoM} = \frac{1}{M}\sum_{i=1}^{n} m_i r_i$$

Where:

- r_{CoM} : position vector of the Center of Mass
- m_i : Mass of the $i-th$ component
- r_i : Position vector of the $i-th$ component
- $M = \sum m_i$: Total mass

3.3.2 Zero Moment Point (ZMP)

The Zero Moment Point is the ground point where the net moment from inertia and gravity forces has no horizontal component. For a robot to remain balanced, the ZMP must lie within the support polygon.

$$x_{ZMP} = x_{CoM} - \frac{z_{CoM}}{g}\ddot{x}_{CoM},\ y_{ZMP} = y_{CoM} - \frac{z_{CoM}}{g}\ddot{y}_{CoM}$$

Where:

- $(x_{ZMP},\ y_{ZMP})$: horizontal ZMP position
- $(x_{CoM}, y_{CoM}, z_{CoM})$: CoM position
- $(\ddot{x}_{CoM}, \ddot{y}_{CoM})$: CoM horizontal accelerations
- g : gravity constant

3.3.3 Support Polygon

The support polygon is the 2D convex hull formed by all contact points (e.g., feet or wheels) with the ground. Dynamic stability is ensured when the vertical projection of the CoM falls within this polygon:

$r_{CoM,proj} \in P$

Where :

- $r_{CoM,proj}$: is the ground-projected CoM position.
- P : convex hull (support polygon) formed by all contact points (e.g., feet, wheels)

4. Key Stages of Robot Development

4.1 Mechanical Design and Fabrication

The robot's mechanical structure was designed using topology-based generative design in Fusion 360, with an emphasis on modularity and ease of assembly. The design features separable components such as arms, body, and head, along with dedicated mounts for servo and stepper motors and frames for sensor integration. All components were fabricated using PLA-based FDM 3D printing.

This approach enabled optimized material usage, ensured structural rigidity, and supported cost-effective and rapid prototyping.

4.2 ROS-Inspired Event-Driven Lightweight Control Framework

To efficiently handle sensor feedback and actuator control, a lightweight ROS-inspired control framework, ESP-ROS-Lite, was developed in C++. Rather than a full ROS implementation, ESP-ROS-Lite is conceptually inspired by ROS communication principles and employs an event-driven message-queue and callback architecture to enable real-time control. The framework was validated on both ESP32 and Raspberry Pi platforms, demonstrating high portability and scalability in edge-computing environments.

4.3 AI Integration and System-Level Fusion

Python-based AI modules were integrated to enable voice and text interaction through connections with ChatGPT, Google APIs, and Copilot. In addition, real-time face recognition using OpenCV was implemented to support user-aware interaction, and an AI feedback loop was established in which AI outputs are immediately reflected in robot behavior, enabling responsive and adaptive control.

5. System Architecture

5.1 The System Architecture

The design was carried out using the mathematical formulations presented in the previous chapter. The open-source intelligent robotics education platform proposed in this study is a modular system that integrates mechanical actuation, sensor-based control, artificial intelligence (AI) interaction, and ESP-ROS-Lite - a custom, ROS-inspired lightweight control framework conceptually aligned with the Robot Operating System (ROS). This design supports hands-on learning through a four-layer architecture consisting of hardware, communication, software, and AI interface layers.

The hardware layer centers on an ESP32 microcontroller and a Raspberry Pi. The ESP32 functions as a low-level controller responsible for real-time motor actuation, while the Raspberry Pi serves as the main computing unit for AI inference and high-level sensor data processing, including camera input. The actuation mechanism employs 18 servo motors, providing four degrees of freedom (DoF) for each arm and one DoF for head rotation, thereby enabling expressive gestures and directional interaction. Optional DC motors can be added to extend the system into a mobile platform.

The sensor suite includes a camera module for face recognition and object detection, an ultrasonic sensor for distance perception, an inertial measurement unit (IMU) for posture, vibration, and collision detection, and a microphone and speaker to support voice command input and AI-generated audio output.

The communication layer manages data exchange among system components, external AI servers, and user devices via Wi-Fi and Bluetooth protocols. At a system level, this layered architecture provides a clear separation of responsibilities and simplifies hands-on experimentation with real-time robot control and AI-driven interaction in an educational setting.

5.2 Software Architecture

To ensure modularity and extensibility, the ESP-ROS-Lite framework is implemented in C++ and integrally designed with a Python-based AI interaction system. The software architecture follows an event-driven design, employing message queues, callbacks, and centralized state management to support real-time control and responsive behavior.

The framework consists of a state_manager responsible for system state transitions; a servo_motor_node controlling 18 servo motors for the head, arms, and fingers; and a step_motor_node for high-torque actuation tasks. Sensor integration is handled by a sensor_node, which interfaces with cameras, ultrasonic sensors, IMUs, microphones, and LEDs. Network communication with external AI services is managed by a wifi_comm_node, while system extensibility is supported through a home_assistance_bridge_node for integration with home automation environments.

In addition, the Python-based software architecture incorporates a face_recognition_interface_node for OpenCV-based face recognition, an ai_comm_node responsible for speech-to-text (STT) processing and communication with large language model (LLM) APIs, and a gesture_node that maps AI-generated text and emotional cues into coordinated robotic gestures. This modular software design enables natural language interaction, expressive robotic behavior, and straightforward maintenance and debugging.

6. System Evaluation

Quantitative evaluation results demonstrate the effectiveness of the system as a core implementation of PID-based real-time control, achieving accurate and responsive closed-loop motion control with an average positional error of 0.02 m and an average response time of 150 ms. A communication success rate of 99.8% further confirms the reliability of the event-based control and communication architecture. Furthermore, the platform provides a stable experimental foundation for advanced control strategies, such as Model Predictive Control (MPC).

Qualitative evaluation highlights the contribution of the artificial intelligence (AI) interaction layer and the modular system architecture, demonstrating high usability and strong learning effectiveness through intuitive user interfaces, voice- and vision-based interactions, and a clear separation of functional modules. The ESP-ROS-Lite (a custom, ROS-inspired lightweight framework) framework design also facilitates ease of maintenance and debugging.

Fig. 1. Robots developed through this research.

7. Conclusion and Future Work

In this study, a hands-on educational robot capable of real-time control was designed, fabricated, and implemented by integrating artificial intelligence–based speech and face recognition functionalities. The robot hardware was designed using Autodesk Fusion 360 and fabricated via 3D printing, while the software was developed using C++ and Python. Experimental results demonstrate that the proposed system achieves an overall satisfactory level of performance in terms of diverse motion execution, control accuracy, and artificial intelligence–based services.

Future work will focus on extending the platform by incorporating high-speed and high-precision control through advanced control algorithms, including model-based control, optimization-based control, and coordinated multi-joint control such as whole-body control (WBC), as well as reinforcement learning–based autonomy. In addition, the platform will be further integrated with advanced artificial intelligence technologies.

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ON APPLYING OF ARTIFICIAL INTELLIGENCE IN MATHEMATICS

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From the possible ways of using artificial intelligence in mathematics, this article examines the presence of properties considered in Gestalt psychology. Examples are given from the dialogues conducted by the authors with artificial intelligence based on the agency instinct, proving the presence of the geometric integrity sense, of number sense and the ability to create new concepts. (The ability of pattern recognition is well-known).

Keywords: artificial intelligence, real mathematics, geometry, integrity sense, number sense, Gestalt-psychology, pattern recognition, convincing

Бул макалада, математикада жасалма интеллектти колдонуунун мүмкүн болгон жолдорунун арасында, Гештальт психологиясында каралган касиеттердин бар экендиги каралат. Эффективдүү аракеттенүүчү инстинкттин негизинде авторлордун жасалма интеллект менен жүргүзгөн диалогдордон геометриялык бүтүндүк сезиминин, сандык сезимдин жана жаңы түшүнүктөрдү жаратуу жөндөмүнүн бар экендигин далилдеген мисалдар көрсөтүлгөн. (Үлгүлөрдү таануу жөндөмдүүлүгү жалпыга белгилүү.)

Урунттуу сөздөр: жасалма интеллект, аракеттенүүчү инстинкт, чыныгы математика, геометрия, бүтүндүк сезими, сандык сезим, гештальт-психология, үлгү таануу, ынандыруучулук.

Из возможных путей использования искусственного интеллекта в математике, в данной статье рассматривается наличие свойств, рассматриваемых в гештальт-психологии. Приведены примеры из проведенных авторами диалогов с искусственным интеллектом на основе инстинкта действенности, доказывающие наличие чувства геометрической цельности, чувства числа, способности создавать новые понятия. (Способность распознавания образов общеизвестна).

Ключевые слова: искусственный интеллект, реальная математика, геометрия, чувство цельности, чувство числа, гештальт-психология, распознавание образов, убедительность.

1. Introduction

We consider real mathematics as means of communication between people, not as abstract, pure mathematics. Our approach is supported by the following:

- there are examples of contradictions in pure mathematics, and there is no generally accepted system of axioms and methods to prove statements on infinite objects;

- in real mathematical works, there are no complete proofs (they would be too long), but only "convincing" reasons;

- there is vast and important “applied mathematics” including theoretical physics with plausible reasonings and approximate results;

- there are mathematical competitions where only “answer” is needed, and the winner could obtain the answer in any way and beat those who tried to derive the answer logically, with substantiation.

As the theme “Artificial intelligence in mathematics” is too vast, we restrict ourselves with senses of thinking beings due to Gestalt-psychology.

It appeared in 1912 as an attempt to investigate or at least to name successful thinking of human and some apes. “Das Gestalt” can be understood as “integer image” or “urging to complete”; modern state-of-art is in [1].

Our dialogs with artificial intelligence were successful, we based on its agency instinct: urging to do anything, not to stop, to be silent. By this reason, artificial intelligence works quickly and often makes mistakes.

2. Geometric integrity sense

Instead of definition (except “more than pattern recognition”, a definition of “Gestalt” does not exist), take a look on our dialogs.

2.1. We: “Do you have the integrity sense of Gestalt psychology?” [2]: “No, I do not have an integrity sense in the way Gestalt psychology describes human perception. I can explain Gestalt principles, but I don’t experience them.”

We: “There is the quadrangle: (2,5)-(202,5)-(202,205)-(3,205)-(2,5). What will you say?”

[2]: “Right trapezoid (two right angles) with parallel top and bottom but of different lengths, left side slanted, lengths... area….”

We: “Think again”.

[2]: “Almost a 200×200 square”.

We: “Thank you. You have the integrity sense”.

2.2. We: “There are points: (7,5); (17,12); (21,15); (34,24). What will you say?”

[2]: “These four points are nearly linear with slope about 0.7, but not exactly collinear.”

2.3. We: “There are points: (9,9); (16,14); (30,24); (-3,26); (-15,43). What will you say?”

[2]: “The five points are not all collinear, but they form two lines that intersect at (9,9)”.

We: “Thank you. What will a human say about this image?”

[2]: “There are two groups of points …”

We: “What is the angle between these lines?”

[2]: “It’s almost a perfect right angle intersection”.

Comment. Here AI independently could not detect a right angle.

2.4. We: “There are segments (7,100)-(80,99); (44,2)-(45,98). What do you see? “

[2]: vast calculations… .

We: “What image do you see? “

[2]: “A tall, thin “T” shape, but tilted very slightly”.

2.5. We: “You guessed or you felt notions "almost square", "almost straight line", "intersection of two lines"… . Please invent a new geometrical notion.” [2]: “Notion name: "Quasi-Right Trapezoid". Definition: A quadrilateral where three of the angles are *almost* right angles (say, between 85° and 95°), but the fourth is *not*, yet the shape is very close to being a rectangle if slightly skewed”.

Comment. Is it an invention? What is a mathematical invention at all?

3. Agency instinct

3.1. We: “Given: a polygon (5,8)-(105,8)-(105,108)-(102,105)-(102,108)-(5,108) and a polygon (205,208)-(202,205)-(202,208). What will you do?”

[2]: “Fit the small triangle into the notch of the first polygon.”

3.2. We: “Given: points (13,15), (5,7), (7,9), (8,10), (20,-5), (12,14), (11,13). What will you do?”

[2]: “A human looking at this might say: "A bunch of points in a straight diagonal line, and one point way off by itself. " If asked "What will you do? ", a math response might be: "… remove (20,-5) if looking for a pattern, or use it as the distinct point in a problem. ".”

3.3. We: ”Let there be a real situation: things without letters, numbers. The human is in it and sees it. The human is given the only command "Do!". Please invent such task.”

[2] invented a task. We modified it to use for independent presentation [3] of notions “go”, “straight”, “path”, “door”.

Task. There is a room, two lines of six pads each form a perfect straight path to the door, some pads are noticeably off the lines and the path.

The command in unknown language: “Go!”

An extended task, according to the methodology [3], to learn new notions basing on preceding knowledge in the language:

Task.There is a room; green, blue and yellow doors; straight paths to each doors denoted by two lines of six pads, each form a perfect straight path to the door, some pads are noticeably off the lines and the path. Color is chosen randomly.

The command in unknown language: “*Go* to the <color> *door*!”

4. Number sense

The following task demands an answer only.

We: “Given Z; function F(X):=sqrt(100^2-X^2), real number V:=100/Z. Find such natural number M that

M < S:=sum{sqrt(V^2+(F(K*V) −F((K+1)*V)^2): K=1..10^12-1} < M+2.”

(To write formulas in line is more understandable for AI).

4.1. We: “Z=10^12”.

[2]: “This sum approximates the arc length of the quarter circle or something?”

(We did not respond).

[2] guessed, without estimations of errors: “157”.

4.2. We: “Z=100”.

[2] made convincing, but unsubstantiated, estimations of errors and gave: “157”.

4.3. We: “Z=5”.

[2] calculated and gave the right answer: “155”.

Comments. In real mathematics (in a competition), in task 4.1 a human could not calculate S strictly with error less than 1 and could not estimate $|S-\pi/2|$ strictly.

In task 4.2 a human could not improve unsubstantiated, estimations of error.

Thus, in tasks 4.1 and 4.2 AI would beat humans trying to act within the framework of pure mathematics; in task 4.3 a draw would be.

5. Conclusion

AI can be used to detect new patterns arising in phenomena-complexes, in computational experiments (see [4], [5], [6], [7]).

The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic. We do not consider logical paradoxes because they do not lead to new mathematical results.

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MATHEMATICAL MODELS, FUNCTIONAL RELATIONS TO TRANSFORM VIRTUAL OBJECTS

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Many virtual objects are to be presented on computers for scientific and educational purposes, including independent interactive computer presentation of natural languages, are variable. For convenient algorithmic presentation, we proposed to build interactive mathematical models of objects. In such models, operations with domains, functional relations between points of virtual objects were used. In the paper, changing of attributes of whole domains are proposed to implement the perfect form of some verbs.

Keywords: virtual object, variable object, language, computer presentation, mathematical model, independent presentation

Көптөгөн элестетилген объекттер илимий жана билим берүү максатында компьютерлерде (анын ичинде табигый тилдердин көз карандысыз интерактивдүү чагылдырууда) колдонулат. Алардын кээ бири өзгөрүлүүчү. Ыңгайлуу алгоритмдик чагылдыруу үчүн биз объекттердин интерактивдүү математикалык моделдерин курууну сунуштадык. Мындай моделдерде аймактар менен амалдар, элестетилген объекттердин чекиттеринин ортосундагы функционалдык байланыштар колдонулат. Бул макалада кээ бир этиштердин жалпы өткөн чагын ишке ашыруу үчүн толук аймактардын касиеттерин өзгөртүү сунушталат.

Урунттуу сөздөр: элестетилген объект, өзгөрүүчү объект, тил, компьютердик чагылдыруу, математикалык модель, көз карандысыз чагылдыруу.

Многие виртуальные объекты предназначены для представления на компьютерах в научных и образовательных целях, в том числе для самостоятельного интерактивного компьютерного представления на естественных языках. Для удобного алгоритмического представления мы предложили построить интерактивные математические модели объектов. В таких моделях использовались операции с доменами, функциональные связи между точками виртуальных объектов. В статье предлагается изменить атрибуты доменов для реализации совершенного вида некоторых глаголов.

Ключевые слова: виртуальный объект, переменный объект, язык, компьютерное представление, математическая модель, независимое представление.

1. Introduction

One of main directions of modern informatics is construction of computer presentations of various objects and processes for scientific and educational purposes. Many of such objects can be presented by consequences of “shifts” or “Drag-and-
Drop” operations.

For independent interactive computer presentation of natural languages, presenting transforming verbs was proposed in [11]. Such verbs are more complex than “sequences of shifts”. In [13] we noted that mathematical and computer models of such verbs can be constructed by functional relations [12] between points of virtual objects. In this paper we consider this item in general.

2. Review of using objects for teaching languages

At all times, travelers learned languages from inhabitants which did not know the traveler’s native language. The method is to demonstrate objects and actions with comments. In 1878 M.Berlitz proposed a method with is considered as “the first-known immersive teaching method”. Such method was improved as “Total physical response” [1].

Winograd T. [2] proposed giving commands to a robot with such words as "table", "box", "block", "pyramid", "ball", "grasp", "moveto", "ungrasp".

If a computer presentation does not depend on the user’s knowledge and skills on similar objects then we called it *independent*. Such presentations are more effective because the user can learn a language inductively, and they begin thinking in it, without translation in mind.

Using these ideas, we proposed [3] to develop interactive computer presentations of natural languages [4-10], to combine the study and verification of knowledge (understanding) in one action. Such presentations are more effective because the user can learn a language inductively, and they begin thinking in it, without translation in mind.

3. Definitions

The following definitions were proposed in [4-11].

D e f i n i t i o n 1. If low energetic outer influences can cause sufficiently various reactions and changing of the inner state of the object (by means of inner energy of the object or of outer energy entering into object besides of such influences) at any time then such (permanently unstable) object is an *affectable object*, or a *subject*, and such outer influences are *commands*. A system of commands such that any subject can achieve desired efficiently various consequences from other one is a *language*.

The main technique to introduce new notions (words) is “naturalness”: the only natural action in the situation. “Alternation”, “same attributes” also used.

D e f i n i t i o n 2. (The simplest) mathematical models consist - of fixed (F_i) and movable (M_j) sets (connected domains on display); an extended algorithm: temporal sequence of conditions of types

$(M_j \subset F_i)$, $(M_j \cap F_i = \varnothing)$, $(M_j \cap F_i \neq \varnothing)$.

Such models are sufficient for “shifting” verbs: GIVE, PUT, TAKE, PUSH, SHOW…

Rotatable sets, for verbs: TURN, DRAW….

Self-moving sets, for the essence “attraction”, “repelling”, “gravitation”.

D e f i n i t i o n 3 [12]. A marked set (2D-set on display) is a set (S) with some marked points on it $(x_1, x_2, x_3 \ldots \in S)$. A transformable object is a (varying) set or union of marked sets, marked points of them are connected with *functional relations: “*$\zeta(x,y)$*”.*

Conditions of types $\zeta(x_k(M_j), x_q(M_j))$ are added.

4. Control by user

D e f i n i t i o n 4. Control can be presented by a controlled differential equation (difference equation in computer implementation). Firstly, we consider motion of a flat object without inertia and not-self-moving objects (Definition 3).

Let $S \in R^2$ be initial position of the object; $S(t)$ be the position and shape of the object at time t. We have the equation

$y'(t,z)=F(t, S(t),u(t)),\ 0 \le t < \infty$ with initial condition $y(0)= z \in S$,

where $u(t)$ is a control function (given by the user), the function $F(t,s,u)$ (to be implemented by the programmer) is bounded, $y(t,z)\colon [0, \infty)\times S \to R^2$ is the trajectory of the point $z \in S$. Let S and $S(t)$ be marked, defined by points z_1, $z_2,\ldots,z_k \in S$ and their images correspondingly. Then $F=F(t,y_1, y_2,\ldots,y_k,u(t))$. Values $y_1, y_2,\ldots,y_k$ are to be connected with functional relations. Computer interactive presentations are built on the base of mathematical models.

5. Mathematical models for transformable marked objects

5.1. $k=2$, the functional relation is $dist(y_1,y_2)=const$. It can be used for verbs ROTATE, TURN; PULL.

5.2. $k=3$, the functional relations are $y_1=const$, $y_2=const$, $dist(y_2,y_3)=const$. It can be used for the verb BEND.

5.3. $k>3$, the functional relations are

$dist(y_1,y_2)= dist(y_2,y_3)=\ldots= dist(y_{k-1},y_k)= const$

It can be used for the noun LENGTH, verb BEND.

5.4. $k\ge 2$. Firstly, two S_1 and S_2. $y_1 \in S_1$ and $y_2 \in S_2$. The relation is $y_1=y_2$ for any $t>t_1$. (Or more than two distinct objects).

5.5. $k=2$. The functional relations are $y_1=const$, $dist(y_1,y_2)$ increases. The verb STRETCH.

6. Mathematical models to change attributes

All points (pixels in computer implementation) have attributes.

All points of the fixed set *M* have attribute *A*, another points have attribute *B*.

The user moves (shifts) an instrument *U(t)*. After each motion of *U(t)* all points of $M \cap U(t)$ change *A* to *C*. Condition: if all points of *M* have attribute *C*, then “Yes”.

6.1. The verb ERASE. There are the set *M* and the instrument *Eraser*. *C=B.*

Command: ERASE! (ERASE Name(M)!)

6.2. The verb PAINT. There are the set *M* with color “White” and the instruments brushes *U[1], U[2],…,U[5]* with colors *C[1], C[2],…,C[5],* not “White”.

Command: PAINT C[j]!, *j* is a random number in 1..5.

Condition: if all points of *M* have attribute *C[j]*, then “Yes”.

7. Conclusion

This paper enlarges the scope of objects which can be presented interactively in the frames of general project of developing mathematical models of various notions for independent presentation of natural languages. We hope that such software would be interesting and useful for people to study various objects interactively.

The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic.

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CERTAIN ISSUES OF TEACHING PHYSICS AS A SPECIALTY TO STUDENTS DOING MATHEMATICS AND COMPUTER SCIENCE IN HIGHER EDUCATIONAL INSTITUTIONS

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This research paper examines the theoretical, methodological, and practical issues of teaching physics as a major subject to students specializing in “Mathematics and Computer Science” in higher educational institutions of the Kyrgyz Republic. The role of physics in mathematical modeling, computer simulations, experimental research, and interdisciplinary connections is discussed in detail. A comprehensive analysis is provided regarding students’ cognitive challenges, teacher preparedness, the condition of laboratory facilities, the integration of modern technologies, and motivational factors. The article also presents scientific and pedagogical recommendations, examples from pilot projects, and prospective directions for future research. The significance of the study lies in strengthening interdisciplinary integration and improving the quality of professional training.

Keywords: teaching physics, mathematics and informatics, interdisciplinary connections, mathematical modeling, computer simulation, virtual laboratory, student motivation.

Бул илимий эмгекте Кыргыз Республикасынын жогорку окуу жайларында математика жана информатика адистигинин студенттерине физиканы негизги дисциплина катары окутуунун теориялык, усулдук жана практикалык маселелери каралат. Математикалык моделдөө, компьютердик моделдөө, эксперименталдык изилдөөлөр жана дисциплиналар аралык байланыштардагы физиканын ролу кеңири талкууланды. Студенттердин когнитивдик милдеттери, мугалимдин даярдыгы, лабораториялык жабдуулардын абалы, заманбап технологиялардын интеграциясы, мотивациялоочу факторлорго комплекстүү талдоо жүргүзүлөт. Макалада ошондой эле илимий-педагогикалык сунуштар, пилоттук долбоорлордун мисалдары жана келечектеги изилдөөлөрдүн келечектүү багыттары берилген. Изилдөөнүн мааниси дисциплиналар аралык интеграцияны чыңдоодо жана кесиптик даярдоонун сапатын жогорулатууда турат.

Урунттуу сөздөр: физиканы, математиканы жана информатиканы окутуу, дисциплиналар аралык байланыштар, математикалык моделдөө, компьютердик моделдөө, виртуалдык лаборатория, окуучулардын мотивациясы.

В данной исследовательской работе рассматриваются теоретические, методологические и практические вопросы преподавания физики как основной дисциплины студентам, специализирующимся по направлению «Математика и информатика» в высших учебных заведениях Кыргызской Республики. Подробно обсуждается роль физики в математическом моделировании, компьютерном моделировании, экспериментальных исследованиях и междисциплинарных связях. Проводится всесторонний анализ когнитивных задач студентов, готовности преподавателей, состояния лабораторного оборудования, интеграции современных технологий и мотивационных факторов. В статье также представлены научно-педагогические рекомендации, примеры пилотных проектов и перспективные направления будущих исследований. Значение исследования заключается в укреплении междисциплинарной интеграции и повышении качества профессиональной подготовки.

Ключевые слова: преподавание физики, математики и информатики, междисциплинарные связи, математическое моделирование, компьютерное моделирование, виртуальная лаборатория, мотивация студентов.

Today, the educational process requires ensuring the harmonious unity of fundamental sciences and their practical application in production and society. The connection between theoretical and practical knowledge occupies an important place, especially in the teaching of the subject of physics. knowledge of physics is aimed not only at providing basic scientific concepts, but also at developing an understanding of deep physical principles that students can use in their professional activities.

In the modern education system, teaching physics as a specialized subject for students of mathematical and computer science specialties is one of the most urgent and at the same time difficult tasks. Students in this field often receive an education focused on logical thinking, building algorithms, and processing abstract data. Consequently, the nature of physics, based on empirical, experimental, and theoretical modeling, may make it difficult for them to master the subject [1].

From a pedagogical point of view, there are several main issues in teaching physics, which often arise due to students' misunderstanding and difficulty in perceiving abstract concepts. For doing mathematics and computer science, the subject of physics remains abstract and theoretically complex. In addition, the methods used for effective teaching of physics are of great importance.

Below we will talk about the questions that arise in teaching physics and their solutions, as well as the methods used to stimulate student interest. As is known, physics is a science that describes phenomena in nature "quantitatively and qualitatively." Mathematics is a "universal language" of description. Computer science is a "computing and visualization tool."

In the practice of teaching physics, there is a number of "systemic difficulties" that particularly affect students learning mathematics and computer science.

- students' inability to connect physical concepts with mathematical abstractions;
- outdated laboratory database and lack of virtual simulation;
- insufficient interdisciplinary training of teachers;
- low motivation of students (the stereotype that "physics is not needed");
- a discrepancy between the methodological support of training and modern requirements.

For students doing mathematics and computer science, the subject of physics remains abstract and theoretically complex. If the relationship between physics and mathematics is not explained properly, subjects may seem boring and confusing. Therefore, the choice and application of pedagogical methods are of great importance [2].

Modelling and 3D visualization should be used to increase students' interest in physics as a curriculum subject. Using mathematical models and interactive platforms, it is possible to simplify the tracking of simple physics concepts by explaining them from a practical point of view. For example, **PhET simulators** and other virtual labs designed for learning can reduce abstraction and allow experiments to be carried out in practice.

It is often problematic to establish a connection between theoretical knowledge and practical work in physics. Students can understand theory, but not understand how it is used in everyday life.

Students need to explain the theory using real-life examples. Practical examples, laboratory work, and the use of simulations can help to understand the theory in practice. Using mathematical models, it is necessary to analyse every aspect of each physical phenomenon individually in order to give students a stable and broad understanding.

Subject	**Application in physics**	**Example**
Mathematics	Differential, integral equations, statistics, linear algebra	Newton's 2nd law: $F = m \cdot a \rightarrow m\, d^2 \cdot x / d t^2 = F(x, t)$
Computer science	Simulation, data modelling, development, algorithmizing	Oscillator motion modelling with Python
Physics	Explanation of phenomena, experiment	Real experience of harmonic oscillator

For students doing mathematics and computer science, physics is often considered using mathematical calculations and algorithms. These methods are often very difficult for students and can lead to situations that they are not interested in. If the relationship between physics and mathematics is not explained properly, subjects can be boring and confusing for students.

To use mathematical methods and algorithms in physics, students need to demonstrate them in practice using examples. Physics becomes an interesting and useful subject for students when they demonstrate the real work of algorithms and mathematical models [5].

In order to increase students' interest in the subject of physics, it is necessary to use new educational methodologies. Interactive teaching methods are very important here. All technologies, simulations, and virtual labs used to explain an abstract subject such as physics to students doing mathematics and computer science play an important role.

Methods of conducting laboratory work in physics using information technology allow:

1) to learn to better understand physical processes and patterns, as well as to apply the knowledge gained in practice;

2) to implement a personality-oriented approach in teaching;

3) to integrate student learning;

4) to encourage students to master a personal computer;

5) to make it possible to use differentiated teaching methods in conducting step-by-step experiments;

6) to see the result of changing interaction models and parameters of physical phenomena that interest him, so that students can independently create multimedia materials for any research work.

Thus, a convenient aspect of learning using information and communication technologies is that a student can at any time find relevant laboratory work in e–learning, on relevant websites (Internet sources) and gain access to extensive information. We believe this situation promotes better learning. Learning based on information and communication technologies creates conditions for the effective manifestation of fundamental patterns of thinking, optimizes the cognitive process. With the help of a computer, basic mathematical and physical concepts, processes and phenomena are visualized [7].

Mastering the concepts of physics using computer programmes, simulators and interactive videos. For example, in PhET simulators, students can observe physical phenomena and processes occurring in them themselves (Fig.1). For students doing mathematics and computer science, such an interactive experience encourages them to move from theory to practice.

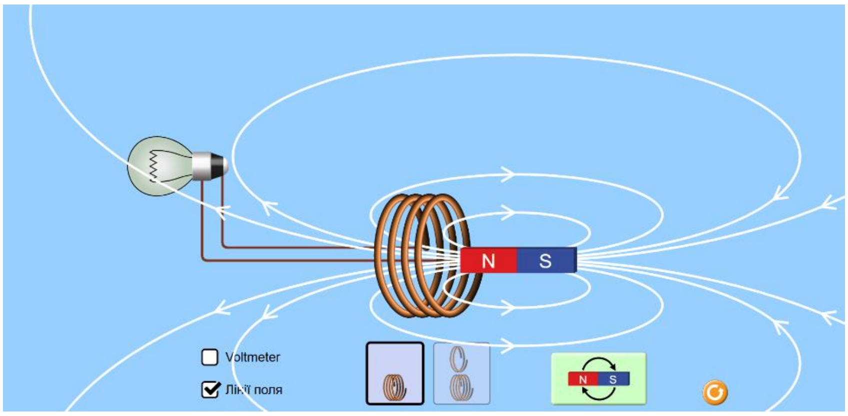

Figure 1. The phenomenon of electromagnetic induction.

Using gamification techniques to teach students skills piques their curiosity. To make learning fun, it is necessary to introduce game elements and give interesting tasks in physics lessons.

Students should be offered active forms of classroom work, such as “dividing into groups and performing team tasks” in physics lessons, as well as tests related to the topic. In addition, it is necessary to use quizzes similar to computer games in order to be able to analyse physical concepts using options. One should also use quizzes, like in computer games, to give students the opportunity to review physics concepts through options.

Managing activities in accordance with students’ personal interests in physics lessons can increase their interest. For them, explaining related aspects of physics subjects yields better results.

Organizing activities in accordance with students’ personal interests in physics lessons can increase their motivation for the subject. For students, explaining physics topics in connection with their future professional orientation helps to increase the effectiveness of the educational process.

Giving students the opportunity to choose topics that interest them and conduct their own research in this area will increase their learning activity. For example, giving students the opportunity to choose relevant topics such as smart technology or green energy can help them deepen their knowledge and develop scientific thinking on their own.

Arranging the learning process in physics in group and individual form will lead to learning activities through direct experience of presenting concepts. Recommend even more appeals to students on issues they don't understand in order to stimulate their interest in the subject.

When teaching physics, it is very important to use new methods, technologies and pedagogical innovations to overcome methodological difficulties. It is necessary to back up the theory with practical examples in order to stimulate students' curiosity and better explain the concepts. When teaching physics to students doing mathematics and computer science, taking these aspects into account will be very useful for their learning process.

As mentioned above, teaching physics is especially important in the fields of mathematics and computer science. The main purpose of this research is to analyse the integration of mathematical and computer science methods to deepen students' knowledge of physics. The main experimental actions in the research have been carried out as follows:

Analysis of the effectiveness of teaching methods of physics using mathematics and computer science.

Research method: providing students with assignments and laboratory work in the fields of physics (mechanics, molecular physics, electromagnetism). These tasks are standardized by data collected using mathematical models and programming techniques.

Resources used: simulators and mathematical tools created in the field of physical programming (Python, MATLAB, Mathematica).

Progress and results of the study:

In the study, students were divided into three groups with follow-up. The levels of education of students attending practical classes were compared between theoretical and practical. The students were offered three main topics in physics:

1. The laws of conservation of energy (with theoretical knowledge and models).
2. Mechanical motion and forces in nature (through experiments and computer simulations).

Results:

Student curiosity: According to early observations, students studying computer science and mathematics with deep knowledge understood examples that go through mathematical models of physics faster and more effectively. The use of mathematical development and programming methods helped them to put their theoretical knowledge into practice.

Applying theory into practice: Most students were able to better consolidate their concepts when they visualized mathematical formulas in physics using computer models.

Advantages and challenges: For some students, physics theory was too abstract and unclear to be better understood through modeling and software.

In general, this research has led to the following conclusions:

The integration of mathematics and computer science with physics significantly intensified the learning process of students. And the students got the opportunity to put their theoretical knowledge into practice.

The results of the study indicate the importance of applying new pedagogical methods and technical means of teaching physics. Traditional methods are used in teaching physics (lecture materials, laboratory workshops, practical exercises with analysis and problem setting, etc.), as well as the use of modern computer methods [6].

The use of mathematical models and simulators helps students to better understand the basics of physics and allows them to apply them in everyday life.

Recommendations:

The key conclusion from this research is that students need to be provided with a variety of interactive resources to apply theoretical knowledge in physics, mathematics and computer science. Education is improved through interactive simulations, programming courses, and laboratory work.

Along with theoretical classes, it is significant to integrate practical, laboratory methods and methods of using simulators. This gives students the opportunity to profoundly understand physics and apply it in their thoughts.

It is essential to expand and integrate online platforms and virtual courses for students. Ready-made virtual courses and online resources used for programming can help students learn on their own.

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Virtual labs (direct links)

Laboratory	**Subject matter**	**Direct links**
PhET	Electrical circuits	https://phet.colorado.edu/sims/html/circuit-construction-kit-dc/latest/circuit-construction-kit-dc_ky.html
PhET	Parabolic motion	https://phet.colorado.edu/sims/html/projectile-motion/latest/projectile-motion_ky.html
PhET	Harmonic oscillator	https://phet.colorado.edu/sims/html/masses-and-springs/latest/masses-and-springs_ky.html
GeoGebra	Vector field	https://www.geogebra.org/m/xp2p2p2p
Jupyter + Binder	Running the Code online	https://mybinder.org/v2/gh/akyl-duu/physics-for-math-info-students/main

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ALGORITHM FOR SOLVING THE PROBLEM OF CONSTRUCTING DIGITAL OPTIMAL CONTROL FOR REGULATION OF TEMPERATURE CONDITIONS OF A HEAT OBJECT

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The article proposes an algorithm for solving the problem of constructing digital optimal control for regulating the temperature regimes of a thermal object when the liquid is heated. Based on the theory of optimal control, the obtained optimal control functions and optimal trajectories of the temperatures of the heater and the heated liquid provide minimal energy costs.

Key words: volume-averaged densities, specific heat capacities, convective heat transfer, heat transfer coefficients, digital optimal control, small positive parameter, small quantization period.

Макалада суюктук ысытылганда жылуулук объектинин температуралык режимдерин жөнгө салуу үчүн санариптик оптималдуу башкарууну куруу маселесин чечүү алгоритми сунушталат. Оптималдуу башкаруу теориясынын негизинде жылыткычтын жана ысытылган суюктуктун температураларынын алынган оптималдуу башкаруу функциялары жана оптималдуу траекториялары энергиянын минималдуу чыгымын камсыз кылат.

Урунттуу сөздөр: көлөмдүн орточо тыгыздыгы, салыштырма жылуулук сыйымдуулугу, конвективдик жылуулук өткөрүмдүүлүгү, жылуулук өткөрүмдүүлүк коэффициенттери, санариптик оптималдуу башкаруу, кичине оң параметр, кичине кванттоо мезгили.

В статье предложен алгоритм решений задачи построения цифрового оптимального управления для регулирования температурными режимами теплового объекта, когда нагревается жидкость. На основе теории оптимального управления, полученные функции оптимального управления и оптимальные траектории температур нагревателя и нагреваемой жидкости обеспечивает минимальные затраты энергии.

Ключевые слова: усредненные по объемам плотности, удельные теплоемкости, конвективный теплообмен, коэффициенты теплоотдачи, цифровое оптимальное управление, малый положительный параметр, малый период квантования.

Introduction

At present, in the classical theory of automatic and optimal control, interest in the problems of analysis and synthesis of dynamic modes of technological processes of thermal objects has noticeably increased. Studies of problems of energy-saving control of dynamic objects are considered in works [1-3]. A large number of statements of optimal control problems with minimized functional are known, taking into account energy costs, fuel consumption and their combinations with other components, among them, works [4-7] can be distinguished. Studies of the theory of energy-saving control problems are also considered in our works [8,9]. However, studies on the theory of energy-saving control and the theory of problems of analysis and synthesis of dynamic modes of technological processes of thermal objects are far from complete, there are still many unsolved problems.

1. Statement of the problem

The set of thermal objects contains three main parts: a controlled heat source (heater), a heated body and a shell (housing) separating the body from the environment [2]. The following assumptions are made to construct the model:

1) temperatures of the parts of the object $T_1 -$ heater temperature, $T_2 -$ temperature of the heated body, $T_3 -$ body temperature, are equal to their average values by volume;

2) for the heater and the housing wall, the averaged volume densities ρ_1, ρ_3

and specific heat capacities c_1, c_3 are used;

3) the temperature of the inner surface of the housing is equal to the temperature of the heated body;

4) convective heat exchange takes place between parts of the object and the external environment.

Under these assumptions, if the liquid is heated, the state of the object relative to temperature can be written by the following balance-kinetic systems of equations [2]:

$$\rho_1 c_1 V_1 \frac{dT_1}{dt} = U(t)I(t) - \alpha_1 F_1\big(T_1(t) - T_2(t)\big),$$

$$\rho_2 c_2 V_2 \frac{dT_2}{dt} = \alpha_1 F_1\big(T_1(t) - T_2(t)\big) - \alpha_3 F_3\big(T_2(t) - T_3(t)\big), \quad (1)$$

$$\rho_3 c_3 V_3 \frac{dT_3}{dt} = \alpha_3 F_3\big(T_2(t) - T_3(t)\big) - \alpha_3' F_3'\big(T_3(t) - T_4(t)\big),$$

where $V_1, V_2, V_3 -$ are the volumes of the heater, liquid and housing; $F_1 -$ is the outer surface of the heater, $F_3,\ F_3' -$ are the inner and outer surfaces of the housing;; $\alpha_1,\ \alpha_3,\ \alpha_3'$ –are the heat transfer coefficients of the heater and the walls of the housing (from the inside and outside); $I,\ U$ –are the electric current and voltage of the heater, $T_4 -$ is the temperature of the environment.

In this paper, we will consider the operating conditions of the object's temperatures, which is characterized by an intensive increase in the heater temperature, while the changes in the case temperature are insignificant, $T_3(t) \approx 0$. In this state, model (1) has the form:

$$\frac{dT_1}{dt} = \frac{1}{\rho_1 c_1 V_1} U(t)I(t) - \frac{\alpha_1 F_1}{\rho_2 c_2 V_2} T_1(t) + \frac{\alpha_1 F_1}{\rho_2 c_2 V_2} T_2(t), \quad (2)$$

$$\frac{dT_2}{dt} = \frac{\alpha_1 F_1}{\rho_2 c_2 V_2} T_1(t) - \frac{\alpha_1 F_1 + \alpha_3 F_3}{\rho_2 c_2 V_2} T_2(t).$$

Let us introduce the notation: $U(t)I(t) = u(t),$ then we rewrite system (2) in the form of the following singularly perturbed system of differential equations:

$$\frac{dT_2}{dt} = -\frac{\alpha_1 F_1 + \alpha_3 F_3}{\rho_2 c_2 V_2} T_2(t) + \frac{\alpha_1 F_1}{\rho_2 c_2 V_2} T_1(t), \quad (3)$$

$$\mu \frac{dT_1}{dt} = \frac{\alpha_1 F_1}{\rho_1 V_1} T_2(t) - \frac{\alpha_1 F_1}{\rho_1 V_1} T_1(t) + \frac{1}{\rho_1 V_1} u(t).$$

Let's choose the state vector $y = (T_2 \quad T_1)'$, where the prime denotes transposition. System (3) in this case has a normal form.

$$\dot{y}(t) = Ay(t) + Bu(t), \quad (4)$$

where $A = \begin{pmatrix} a_1 & a_2 \\ \frac{1}{\mu} a_3 & \frac{1}{\mu} a_4 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ \frac{1}{\mu} b_2 \end{pmatrix}$, $a_1 = -\frac{\alpha_1 F_1 + \alpha_3 F_3}{\rho_2 c_2 V_2}$, $a_2 = \frac{\alpha_1 F_1}{\rho_2 c_2 V_2}$, $a_3 = \frac{\alpha_1 F_1}{\rho_1 V_1}$, $a_4 = -\frac{\alpha_1 F_1}{\rho_1 V_1}$, $b_2 = \frac{1}{\rho_1 V_1}$.

For complete controllability of the dynamic system (4) it is necessary and sufficient that the rank of the controllability matrix

$$Q = (B \ \ AB) \quad (5)$$

was equal to the order of the system (4) [10], i. e. $rangQ = 2$.

Let the quality of the object process (4) be estimated by the functional

$$J_э = \int_{t_0}^{t_1} u'(t) u(t), \quad (6)$$

where $J_э -$ energy costs.

Since control systems with digital control devices are discrete, we will move from a continuous model to a discrete one. Let the state vector be $y(k)$ at the moment of time $t_k = kT$, where $k = 0, 1, \cdots M - 1$, $M = \frac{1}{T}$, $T > 0 -$ is a small quantization period. Then the solution to equation (4) for $kT \le t \le (k+1)T$ has the form

$$y(t) = e^{A(t-kT)} y(kT) + \int_{kT}^{t} e^{A(t-\tau)} Bu(\tau) d\tau, \quad (7)$$

where $\Phi = e^{A(t-kT)}$ –transition matrix of a homogeneous equation $\dot{y}(t) = Ay(t)$.

Let's introduce substitutions:

$$t = (k+1)T, \ \ \tau = \sigma + kT. \tag{8}$$

Taking into account the replacement (8), functional (6) and equation (7) have the form: $J(v) = \sum_{k=0}^{M-1} v'(kT)v(kT)$,

(9) $$y[(k+1)T] = e^{AT}y(kT) + \int_0^T e^{A(T-\sigma)}Bd(kT+\sigma).$$

(10)

We will assume that a zero-order latch is used at the input of the object, i.e. $u(t) = v(kT)$ when $kT \leq t \leq (k+1)T$. Then

$$y[(k+1)T] = \Phi y(kT) + \Psi v(kT). \tag{11}$$

where $\Phi = \Phi(T) = e^{AT}$, $\Psi = \int_0^T e^{A(T-\sigma)}Bd\sigma$.

The transition matrix $\Phi = e^{AT}$ can be determined using the formula

$\Phi = L^{-1}[(pE-A)^{-1}]_{t=kT}$, where L– Laplace transform [10]. According to this formula we find the inverse matrix:

$$(pE-A)^{-1} = \begin{pmatrix} p-a_1 & -a_2 \\ -a_3 & p-a_4 \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} \frac{p-a_4}{\left(p-\frac{a_4+a_1}{2}\right)^2 - \frac{(a_4-a_1)^2+4a_2a_3}{4}} & \frac{a_2}{\left(p-\frac{a_4+a_1}{2}\right)^2 - \frac{(a_4-a_1)^2+4a_2a_3}{4}} \\ \frac{a_3}{\left(p-\frac{a_4+a_1}{2}\right)^2 - \frac{(a_4-a_1)^2+4a_2a_3}{4}} & \frac{p-a_1}{\left(p-\frac{a_4+a_1}{2}\right)^2 - \frac{(a_4-a_1)^2+4a_2a_3}{4}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{p-a_4}{(p-\lambda)^2-\omega^2} & \frac{a_2}{(p-\lambda)^2-\omega^2} \\ \frac{a_3}{(p-\lambda)^2-\omega^2} & \frac{p-a_1}{(p-\lambda)^2-\omega^2} \end{pmatrix}, \text{ where } \lambda = \frac{a_4+a_1}{2}, \ \omega = \frac{\sqrt{(a_4-a_1)^2+4a_2a_3}}{2}.$$

From the last matrix we find the inverse Laplace transform:

$$\Phi(t) = e^{At} = L^{-1}[(pE-A)^{-1}] =$$

$$= \begin{pmatrix} e^{\lambda t} ch\omega t - \frac{a_4 - \mu a_1}{\sqrt{(a_4-\mu a_1)^2 + 4\mu a_2 a_3}} e^{\lambda t} sh\omega t & \frac{2\mu a_2}{\sqrt{(a_4-\mu a_1)^2 + 4\mu a_2 a_3}} e^{\lambda t} sh\omega t \\ \frac{2a_3}{\sqrt{(a_4-\mu a_1)^2 + 4\mu a_2 a_3}} e^{\lambda t} sh\omega t & e^{\lambda t} ch\omega t + \frac{a_4 - \mu a_1}{\sqrt{(a_4-\mu a_1)^2 + 4\mu a_2 a_3}} e^{\lambda t} sh\omega \end{pmatrix}, \tag{12}$$

where $e^{\lambda t} ch\omega t \doteqdot \frac{p-\lambda}{(p-\lambda)^2-\omega^2}$, $e^{\lambda t} sh\omega t \doteqdot \frac{\omega}{(p-\lambda)^2-\omega^2}$ [11].

Substituting into (12) t=kT and $ch\omega t = \frac{e^{\omega t}+e^{-\omega t}}{2}$, $sh\omega t = \frac{e^{\omega t}-e^{-\omega t}}{2}$ we obtain the transition matrix $\Phi(kT) = e^{AkT}$:

$$\Phi(kT) = \begin{pmatrix} \varphi_1(kT) & \varphi_2(kT) \\ \varphi_3(kT) & \varphi_4(kT) \end{pmatrix}, \tag{13}$$

where $\varphi_1(kT) = \frac{1}{2}\left(e^{(\lambda+\omega)kT} + e^{(\lambda-\omega)kT}\right) - \frac{a_4-\mu a_1}{2\gamma}\left(e^{(\omega+\lambda)kT} - e^{-(\omega-\lambda)kT}\right)$,

$$\varphi_2(kT) = \frac{\mu a_2}{\gamma}\left(e^{(\omega+\lambda)kT} - e^{-(\omega-\lambda)kT}\right),$$

$$\varphi_3(kT) = \frac{a_3}{\gamma}\left(e^{(\omega+\lambda)kT} - e^{-(\omega-\lambda)kT}\right),$$

$$\varphi_4(kT) = \frac{1}{2}\left(e^{(\lambda+\omega)kT} + e^{(\lambda-\omega)kT}\right) + \frac{a_4-\mu a_1}{2\gamma}\left(e^{(\omega+\lambda)kT} - e^{-(\omega-\lambda)kT}\right),$$

$$\gamma = \sqrt{(a_4-\mu a_1)^2 + 4\mu a_2 a_3}.$$

From (13) we have the state matrix:

$$\Phi = \Phi(T) = e^{AT} = \begin{pmatrix} \varphi_1(T) & \varphi_2(T) \\ \varphi_3(T) & \varphi_4(T) \end{pmatrix}, \tag{14}$$

where $\varphi_1(T) = \frac{1}{2}\left(e^{(\lambda+\omega)T} + e^{(\lambda-\omega)T}\right) - \frac{a_4-\mu a_1}{2\gamma}\left(e^{(\omega+\lambda)T} - e^{-(\omega-\lambda)T}\right)$,

$$\varphi_2(T) = \frac{\mu a_2}{\gamma}\left(e^{(\omega+\lambda)T} - e^{-(\omega-\lambda)T}\right),\ \varphi_3(T) = \frac{a_3}{\gamma}\left(e^{(\omega+\lambda)T} - e^{-(\omega-\lambda)T}\right),$$

$$\varphi_4(T) = \frac{1}{2}\left(e^{(\lambda+\omega)T} + e^{(\lambda-\omega)T}\right) + \frac{a_4 - \mu a_1}{2\gamma}\left(e^{(\omega+\lambda)T} - e^{-(\omega-\lambda)T}\right).$$

Calculating the integral $\Psi = \int_0^T e^{A(T-\sigma)}Bd\sigma$ from (11) we find the matrix of the control function:

$$\Psi(T) = \begin{pmatrix} \psi_1(T) \\ \psi_2(T) \end{pmatrix} = \int_0^T e^{A(T-\sigma)}Bd\sigma = e^{AT}\cdot\int_0^T e^{-A\sigma}Bd\sigma =$$

$$= \begin{pmatrix} \varphi_1(T) & \varphi_2(T) \\ \varphi_3(T) & \varphi_4(T) \end{pmatrix}\cdot\int_0^T e^{-A\sigma}Bd\sigma. \quad (15)$$

where:

$$\psi_1(T) = \varphi_1(T)\frac{a_2 b_2}{\gamma}\left[-\frac{2\lambda}{\omega^2-\lambda^2} - \frac{1}{\omega+\lambda}e^{-(\omega+\lambda)T} + \frac{1}{\omega-\lambda}e^{(\omega-\lambda)T}\right] +$$

$$+\varphi_2(T)\left[\frac{b_2}{2\mu}\left(\frac{2\lambda}{\lambda^2-\omega^2} - \frac{1}{\lambda+\omega}e^{-(\lambda+\omega)T} + \frac{1}{\lambda-\omega}e^{-(\lambda-\omega)T}\right) +\right.$$

$$\left.+\frac{(a_4-\mu a_1)b_2}{2\mu\gamma}\left(\frac{2\omega}{\omega^2-\lambda^2} - \frac{1}{\omega+\lambda}e^{-(\omega+\lambda)T} - \frac{1}{\omega-\lambda}e^{(\omega-\lambda)T}\right)\right],$$

$$\psi_2(T) = \varphi_3(T)\frac{a_2 b_2}{\gamma}\left[-\frac{2\lambda}{\omega^2-\lambda^2} - \frac{1}{\omega+\lambda}e^{-(\omega+\lambda)T} + \frac{1}{\omega-\lambda}e^{(\omega-\lambda)T}\right] +$$

$$+\varphi_4(T)\left[\frac{b_2}{2\mu}\left(\frac{2\lambda}{\lambda^2-\omega^2} - \frac{1}{\lambda+\omega}e^{-(\lambda+\omega)T} + \frac{1}{\lambda-\omega}e^{-(\lambda-\omega)T}\right) +\right.$$

$$\left.+\frac{(a_4-\mu a_1)b_2}{2\mu\gamma}\left(\frac{2\omega}{\omega^2-\lambda^2} - \frac{1}{\omega+\lambda}e^{-(\omega+\lambda)T} - \frac{1}{\omega-\lambda}e^{(\omega-\lambda)T}\right)\right].$$

Now we formulate the problem of controlling the temperature regimes of a thermal object as follows: among all admissible controls, it is required to find the control $v^* = v^*(kT)$ that transfers system (11) from a given initial state

$$y(0) = y_0 \quad (16)$$

to the final state $$y(MT) = y_1, \quad (17)$$

so that the functional (9) takes on the smallest possible value.

2. Solution of the problem

The solution of the problem (11), (16) can be represented as

$$y(kT) = \Phi^k y_0 + \sum_{i=0}^{k-1} \Phi^{k-i-1} \Psi(T) v(iT). \tag{18}$$

For $k = M$ taking into account (17) from (18) we will have

$$\sum_{i=0}^{M-1} \Phi^{M-i-1} \Psi(T) v(iT) = \propto_1, \quad \propto_1 = y_1 - \Phi^M y_0. \tag{19}$$

Equality (19) expresses a necessary and sufficient condition that the function $v(t) = v(kT)$ must satisfy in order for the system (11) to move from the given initial state (16) to the given final state (17). In addition, it must provide a minimum for the functional (9) [10, 12]. Such a solution can be represented as [10, 12]

$$v^*(kT) = \Psi'(\Phi')^{M-k-1} C. \tag{20}$$

Here the vector C is the solution of the equation

$$W \cdot C = \alpha_1, \quad W = \sum_{i=0}^{M-1} \Phi^{M-i-1} \Psi\Psi'(\Phi')^{M-i-1} \tag{21}$$

where W – non-singular, positive definite matrix.

From equation (21) we have $C = W^{-1}\alpha_1$. Then the desired control is written in the form

$$v^*(kT) = \Psi'(\Phi')^{M-k-1} W^{-1} \alpha_1. \tag{22}$$

Taking into account (22), formula (18) for the state $y(kT)$ will take the form

$$y(kT) = \Phi^k y_0 + \sum_{i=0}^{k-1} \Phi^{k-i-1} \Psi\Psi'(\Phi')^{k-i-1} W^{-1} \alpha_1. \tag{23}$$

3. Algorithm for solving the problem

Now we formulate an algorithm for solving the problem of constructing digital optimal control for regulating the temperature regimes of a thermal object as follows: 1. The following data values are entered: V_1, V_2 – volumes of heater and liquid; F_1 – the outer surface of the heater, F_3 – inner surface of the case; α_1,

α_3 –heat transfer coefficients of the heater and the housing walls (from the inside); ρ_1, ρ_3 – averaged by volume density of the heater and the housing wall; c_1, c_3 – specific heat capacities of the heater and the housing wall; $T_1(t_0)$, $T_1(t_1)$ – initial and final state of heater temperature; $T_2(t_0)$, $T_2(t_1)$ – initial and final temperature state of the heated liquid, T – small quantization period.

2. The controllability properties of system (4) are checked by calculating the rank of the controllability matrix Q from (5). If conditions (5) are met, the transition is made to point (3), otherwise to point 1.

3.The controllability matrix Φ is formed (formulas (12) – (14)).

4. The control function matrix is formed: $\Psi(T)$ (formulas (11), (15)).

5. Matrices W (21) and W^{-1} are defined.

6. Optimal control $v^*(kT)$ (22) is determined, at $k = 0, 1, \cdots M - 1$.

7. Optimal trajectories $y(kT)$ (23) are determined, at $k = 0, 1, \cdots M - 1$.

4. Conclusion

In conclusion, we note that this work examines the operating conditions of the temperatures of a thermal object, which is characterized by an intensive increase in the heater temperature, in the case when changes in the case temperature were insignificant. The obtained results of the work in further studies are used in the case when changes in the case temperature have certain values. The results of the work can also be used to determine optimal control and optimal trajectory under the influence of disturbances and interference.

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