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A REFINED BEHAVIOR DESCRIPTION TO THE NORMALIZED RICCI FLOW ON SOME HOMOGENEOUS SPACES

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The article is devoted to the Ricci curvature positivity on homogeneous spaces under the normalized Ricci flow. We obtained some refinements to our previous results devoted to the case of such spaces with coincided parameters showing the existence of a special value of the parameter that separates the entire interval into the disjoint union of two subintervals. One of them represents a narrow subinterval such that for values of the parameter chosen from there trajectories can both enter and leave the domain of positivity of the Ricci curvature. No trajectory leaves the domain once entering there, if the corresponding values are chosen from another subinterval.

Keywords: normalized Ricci flow, generalized Wallach space, Riemannian metrics, Ricci curvature, dynamical system.

Макала жалпыланган Уоллах мейкиндиктеринде оң Риччи ийриликтин нормалдаштырылган Риччи агымы таасири астында сакталуу суроолоруна арналган. Бүтүн интервалды кесилишпөөчү эки интервалдын биригүүсүнө ажыраткан параметрдин атайы мааниси бар экенин көрсөтүү аркылуу ушундай мейкиндиктердин параметрлери дал келген учуруна арналган мурунку натыйжаларыбызга кээ бир тактоолорду алдык. Интервалдардын бирөөсү кууш жана параметрдин мында жаткан маанилери тандалганда, траекториялар Риччи ийриликтин оң облусуна кирип да, чыгып да жатышы мүмкүн. Параметрдин экинчи интервалда жаткан маанилеринде траекториялар айтылган облуска бир кирсе, кайра чыга алышпайт.

Урунттуу сөздөр: нормалдаштырылган Риччи агымы, жалпыланган Уоллах мейкиндиги, римандык метрика, Риччи ийрилиги, динамикалык система.

Статья посвящена вопросам сохранения положительности кривизны Риччи на обобщенных пространствах Уоллаха под влиянием нормализованного потока Риччи. Мы получили некоторые уточнения к нашим предыдущим результатам, посвященным случаю таких пространств с совпадающими параметрами, показывая существование специального значения параметра, которое разделяет целый интервал на объединение двух непересекающихся интервалов. Один из них представляет собой узкий интервал такой, что

при значениях параметра, выбранных оттуда, траектории могут как входить в область положительности кривизны Риччи, так и выходить оттуда. При значениях параметра, выбранных из другого интервала, траектории никогда не выходят из упомянутой области, попав в нее однажды.

Ключевые слова: нормализованный поток Риччи, обобщённое пространство Уоллаха, риманова метрика, кривизна Риччи, динамическая система.

The paper is devoted to study the behavior of positively curved invariant Riemannian metrics on generalized Wallach spaces under the influence of the normalized Ricci flow (NRF)

$$\dot{\mathbf{g}}(t) = -2\mathrm{Ric}_{\mathbf{g}} + 2n^{-1}\mathbf{g}(t)\mathrm{S}_{\mathbf{g}} \tag{1}$$

introduced in [1], where $\mathbf{g}(t)$ is an one-parameter family of Riemannian metrics on a n-dimensional Riemannian manifold, $\mathrm{Ric}_{\mathbf{g}}$ and $\mathrm{S}_{\mathbf{g}}$ are respectively the Ricci tensor and the scalar curvature of $\mathbf{g}$. Assume that G/H is an almost effective compact homogeneous space with a compact semi-simple connected Lie group G and its closed Lie subgroup H. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras and $\mathfrak{p}$ is an orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$ with respect to an inner product $\langle\cdot,\cdot\rangle = -B(\cdot,\cdot)$ on $\mathfrak{g}$, where B is a Killing form of $\mathfrak{g}$. Then G/H is called a *generalized Wallach space* (GWS) if $\mathfrak{p}$ is decomposed into a direct sum $\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ of three $\mathrm{Ad}(H)$-invariant irreducible modules $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{p}_3$, pairwise orthogonal with respect to $\langle\cdot,\cdot\rangle$ and satisfying $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for all $i \in \{1,2,3\}$ (see details in [2] and references therein). Every GWS can be characterized (descripted) by a triple of real numbers $(a_1, a_2, a_3) \in (0,1/2]^3$, connected with dimensions $d_i = \dim(\mathfrak{p}_i)$ via $a_i = A/d_i \in (0,1/2]$, where A is some nonnegative number. Any G-invariant metric $\mathbf{g}(\cdot,\cdot)$ on GWS and its Ricci tensor $\mathrm{Ric}_{\mathbf{g}}$ admit the following decompositions:

$$\mathbf{g}(\cdot,\cdot) = x_1\langle\cdot,\cdot\rangle|_{\mathfrak{p}_1} + x_2\langle\cdot,\cdot\rangle|_{\mathfrak{p}_2} + x_3\langle\cdot,\cdot\rangle|_{\mathfrak{p}_3}, \tag{2}$$

$$\mathrm{Ric}_{\mathbf{g}}(\cdot,\cdot) = x_1\mathbf{r}_1\langle\cdot,\cdot\rangle|_{\mathfrak{p}_1} + x_2\mathbf{r}_2\langle\cdot,\cdot\rangle|_{\mathfrak{p}_2} + x_3\mathbf{r}_3\langle\cdot,\cdot\rangle|_{\mathfrak{p}_3}, \tag{3}$$

where x_1, x_2, x_3 are arbitrary positive numbers and

$$\mathbf{r}_i = \frac{1}{2x_i} + \frac{a_i}{2}\left(\frac{x_i}{x_j x_k} - \frac{x_k}{x_i x_j} - \frac{x_j}{x_k x_i}\right)$$

are components of the Ricci tensor on each $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{p}_3$, $\{i, j, k\} = \{1,2,3\}$. Using (2) and (3), on GWS NRF (1) can be reduced to an equivalent dynamical system

where x_1, x_2, x_3 are arbitrary positive numbers and $\mathbf{r}_i = \frac{1}{2x_i} + \frac{a_i}{2}\left(\frac{x_i}{x_j x_k} - \frac{x_k}{x_i x_j} - \frac{x_j}{x_k x_i}\right)$ are components of the Ricci tensor on each $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{p}_3$, $\{i, j, k\} = \{1,2,3\}$. Using (2) and (3), on GWS NRF (1) can be reduced to an equivalent dynamical system

$$\dot{x}_i = -2x_i\left(\mathbf{r}_i - \frac{a_1^{-1}\mathbf{r}_1 + a_2^{-1}\mathbf{r}_2 + a_3^{-1}\mathbf{r}_3}{a_1^{-1} + a_2^{-1} + a_3^{-1}}\right). \tag{4}$$

Denote by $\mathcal{R}_+$ the set of parameters x_1, x_2, x_3 of the metric (2) with a positive Ricci curvature: $\mathcal{R}_+ = \{(x_1, x_2, x_3) \in \mathbb{R}_+^3 \mid \mathbf{r}_1 > 0,\ \mathbf{r}_2 > 0,\ \mathbf{r}_3 > 0\}$.

Then $\mathrm{Ric}_{\mathbf{g}} > 0$ is equivalent to $(x_1, x_2, x_3) := \mathbf{x} \in \mathcal{R}_+$.

Denote by Σ a surface $\mathrm{Vol} = 1$, where $\mathrm{Vol} = x_1^{1/a_1} x_2^{1/a_2} x_3^{1/a_3}$ is the volume function of the metric (2). As proved in [3], Σ plays an important role as the surface of dominant motions of trajectories of the system (5) in the case $a_1 = a_2 = a_3 = a \in (0, 1/2)$ (slow motions can be neglected, because they happen along the rays $x_1 = x_2 = x_3 = \tau > 0$ and $x_i = \frac{1-2a}{2a}\tau$, $x_j = x_k = \tau$, $\{i, j, k\} = \{1,2,3\}$, always staying in the set $\mathcal{R}_+$). As noted in [3] using new variables $x_i = X_i \sqrt[3]{c}$ and $t = \tau\sqrt[3]{c}$ the general case $\mathrm{Vol} = c > 0$ can equivalently be reduced to the case $\mathrm{Vol} = 1$ of a system of the same form as (5) with respect to $X_i(\tau)$. Therefore, it suffices to consider the set $\Sigma \cap \mathcal{R}_+$ for studying the evolution of metrics (2) under NRF (1).

We assume $x_1 \neq x_2 \neq x_3 \neq x_1$ in (2). Such metrics are called generic. The evolution of non-generic metrics, i.e. metrics with $x_i = x_j$, $i \neq j$, are trivial (see [4]). In the sequel by metrics we will mean generic metrics. A shorter phrase “a Ricci positive metric” will mean “a metric with positive Ricci curvature”.

In [4,5] the following result was obtained for GWS with $a_1 = a_2 = a_3 = a$:

Theorem 1 (The joint result of Theorem 3 in [4] and Theorem 3 in [5]). Let G/H be a GWS with $a_1 = a_2 = a_3 = a \in (0, 1/2)$.

a) If $a \in (0, 1/6)$, then NRF (1) evolves all metrics (2) with positive Ricci curvature into metrics with mixed Ricci curvature.

b) If $a \in (1/6, 1/2)$, then NRF (1) evolves all metrics (2) into metrics with positive Ricci curvature.

Theorem 1 was proved using asymptotic ideas. The time required for metric transfers could be as long as we want. The main result of the present article is contained in the following theorem, where some refinements to our previous results are obtained.

Theorem 2. Let G/H be a GWS with parameters $a_1 = a_2 = a_3 = a \in (0, 1/2)$. Then the following assertions hold for the metrics (2) under NRF (1):

a) If $a \in [a^*, 1/2)$, then all Ricci positive metrics (2) retain Ricci positive;

b) If $a \in (1/6,\ a^*)$, then some Ricci positive metrics (2) retain Ricci positive, where $a^* \in (1/6, 1/4)$. More precisely, $a^* = \frac{\left(-1+\sqrt[3]{28+3\sqrt{87}}\right)^2}{12\sqrt[3]{28+3\sqrt{87}}} \approx 0.1739051924.$

Remark 1. The case $a = 1/6$ was studied in [4] and [6]: there exist metrics (2) both preserving $\operatorname{Ric} > 0$ and losing $\operatorname{Ric} > 0$ at $a = 1/6$. It was proved in [4, Theorem 4] that metrics (2) preserve $\operatorname{Ric} > 0$ in the case $a = 1/6$ if satisfy $x_k < x_i + x_j$, $\{i, j, k\} = \{1,2,3\}$. Metrics (2) with $x_k = x_i + x_j$ are known as Kähler metrics on GWS. Theorem 6 in [6] implies also the existence of metrics (2) that lose $\operatorname{Ric} > 0$ in the case $a = 1/6$.

Proof of Theorem 2. a) As established in [6, Theorem 7] for the general case of GWS with parameters $a_1, a_2, a_3 \in (0,1/2)$ and $a_1 + a_2 + a_3 > 1/2$, NRF (1) preserves the positivity of the Ricci curvature of (2), if $\theta \geq \max\{\theta_1, \theta_2, \theta_3\}$, where $\theta = a_1 + a_2 + a_3 - 0.5$ and $\theta_i = a_i - 0.5 + 0.5\sqrt{(1 - 2a_i)(1 + 2a_i)^{-1}}$, $i = 1,2,3$.

Since $1-2a_i>0$ for $a_i\in(0,1/2)$, the condition $\theta\ge\theta_i$ is equivalent to $4\left(a_j+a_k\right)^2\ge(1-2a_i)(1+2a_i)^{-1}$. At $a_1=a_2=a_3=a$ the latter is equivalent to $h(a):=32a^3+16a^2+2a-1\ge 0$. Note also that $a>1/6$ since $a_1+a_2+a_3>1/2$. The cubic polynomial $h(a)$ has a real root a^* between $a=1/6$ and $a=1/4$, since $h(1/6)=-2/27<0$ and $h(1/4)=1>0$. Such a root is unique by negativity of the discriminant of $h(a)$. Therefore, $\theta\ge\theta_1=\theta_2=\theta_3$ for all $a\in[a^*,1/2)$, meaning that $\mathrm{Ric}_{\mathbf{g}(t)}>0$ for all $t>0$ and every initial metric (2) with $\mathrm{Ric}_{\mathbf{g}(0)}>0$.

b) For $a\in(1/6,a^*)$ we have $\theta<\theta_1=\theta_2=\theta_3$, hence, the conclusion of [6] holds as well by existence of metrics (2) with $\mathrm{Ric}_{\mathbf{g}(t)}>0$ for sufficiently large t. Theorem 2 is proved.

Remark 2. The exact value of a^* can be found by Cardano's formula. Indeed, putting $a=b-1/6$ in $h(a)=0$ we get the equation $b^3+pb+q=0$ with $p=-1/48$ and $q=-7/216$, which admits one real root $b^*=\alpha+\beta$, where $\alpha=\sqrt[3]{-q/2+\sqrt{-D}}$, $\beta=\sqrt[3]{-q/2-\sqrt{-D}}$ and $D=-(p/3)^3-(q/2)^2$. Since $\alpha\beta=-p/3$, then $a^*=b^*-1/6=\alpha+1/(144\alpha)-1/6=(12\alpha-1)^2\alpha^{-1}/144$.

Example 1. Under NRF (1) every metric (2) with $\mathrm{Ric}>0$ loses $\mathrm{Ric}>0$ on every space $\mathrm{Sp}(3k)/\mathrm{Sp}(k)\times\mathrm{Sp}(k)\times\mathrm{Sp}(k)$, since $a_1=a_2=a_3=\frac{k}{6k+2}$ according to [2] and, hence $\frac{k}{6k+2}<1/6$ for every $k\in\mathbb{N}$.

Example 2. Another example of GWS, where every metric (2) loses $\mathrm{Ric}>0$ by Theorem 1, is the Wallach space $W_{24}=F_4/\mathrm{Spin}(8)$. Indeed, $a_1=a_2=a_3=a=1/9<1/6$ for it. In the left panel of Figure 1 trajectories of the system (4) are depicted (lines in red), that leave the domain $\Sigma\cap\mathcal{R}_+$ corresponding to the space W_{24}. By lines in blue the boundary curves $\Sigma\cap\Lambda_i$ are depicted of the domain $\Sigma\cap$

$\mathcal{R}_+$, where each Λ_i means a cone, defined by the equation $a\left(x_i^2 - x_j^2 - x_k^2\right) + x_j x_k = 0$, equivalent to $\mathbf{r}_i = 0$ in (3) for $x_1, x_2, x_3 > 0$, $\{i, j, k\} = \{1,2,3\}$.

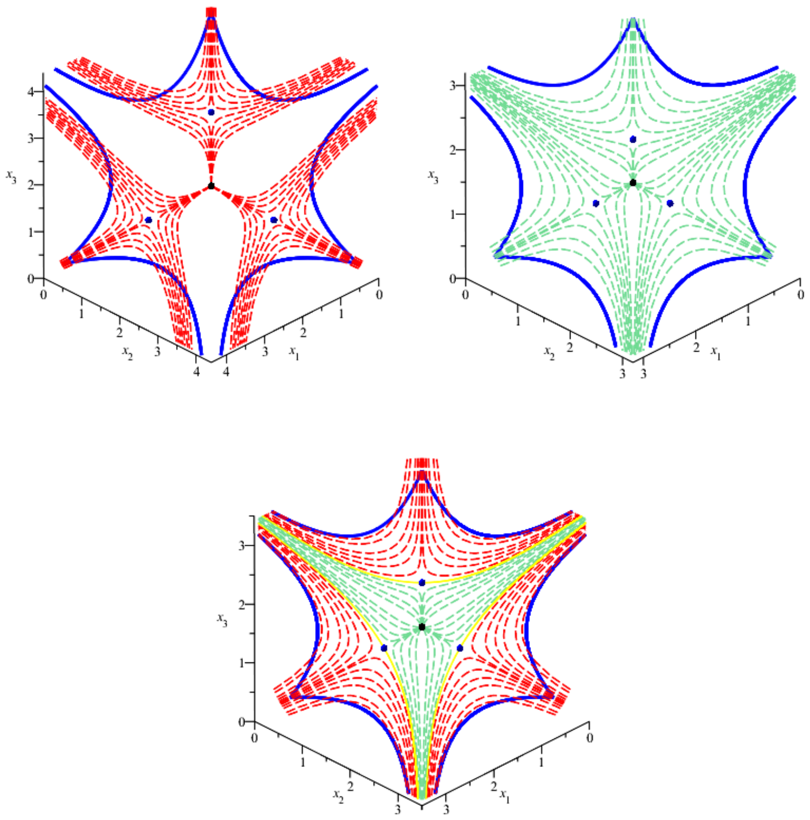

Figure 1– NRF on GWS: Trajectories of the system (4)

are leaving $\Sigma \cap \mathcal{R}_+$ at $a = 1/9$ (left panel);

are staying in $\Sigma \cap \mathcal{R}_+$ at $a = 4/24$ (middle panel)

both leaving $\Sigma \cap \mathcal{R}_+$ and staying in $\Sigma \cap \mathcal{R}_+$ at $a = 1/6$ (right panel)

Example 3. According to Theorem 2 every metric (2) with $\mathrm{Ric} > 0$ preserves $\mathrm{Ric} > 0$ on each of 15 spaces $\mathrm{SO}(3k)/\mathrm{SO}(k) \times \mathrm{SO}(k) \times \mathrm{SO}(k)$ with $k \in \{2, \ldots, 16\}$. Indeed, since $a := a_1 = a_2 = a_3 = k/(6k-4)$ due to [2], the inequalities $1/6 < a \le 1/4$ are satisfied for all $k \ge 2$. As for $a \ge a^*$ it has solutions $k \le \frac{4a^*}{6a^*-1} \approx 16.02$. In the middle panel of Figure 1 trajectories of (4) are illustrated (in green), staying in the domain $\Sigma \cap \mathcal{R}_+$ that corresponds to $\mathrm{SO}(3k)/\mathrm{SO}(k) \times \mathrm{SO}(k) \times \mathrm{SO}(k)$ at $k = 16$. Indeed, $a = \frac{k}{6k-4} = \frac{4}{23} > a^*$ for $k = 16$. Note that $4/23 \approx 0.1739130435$.

Example 4. Other examples of spaces where $\mathrm{Ric} > 0$ is preserved are $E_7/\mathrm{SO}(8) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$, $E_7/\mathrm{SO}(8)$ и $E_8/\mathrm{SO}(8) \times \mathrm{SO}(8)$ located in the rows 9, 11 and 13 in Table 1 in [2], since $a = 2/9 \approx 0.22 > a^*$, $a = 5/18 \approx 0.27 > a^*$ and $a = 4/15 \approx 0.26 > a^*$ for them respectively.

Example 5. Infinitely many homogeneous spaces $\mathrm{SU}(3k)/\mathrm{S}\big(\mathrm{U}(k) \times \mathrm{U}(k) \times \mathrm{U}(k)\big)$ are correspond to $a = 1/6$. Results are in the right panel of Figure 1. Three yellow curves mean the parameters of the Kähler metrics on Σ: $\big\{(x_1, x_2, x_3) \in \mathbb{R}_+^3 \mid x_k = x_i + x_j\big\} \cap \{(x_1, x_2, x_3) \in \mathbb{R}_+^3 \mid x_1 x_2 x_3 = 1\}$.

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ON EXISTENCE OF INFINITE FAMILIES OF HOMOGENEOUS SPACES PRESERVING/ LOSING THE POSITIVITY OF THE RICCI CURVATURE UNDER THE NORMALIZED RICCI FLOW

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In the article the questions of preserving or losing of the Ricci curvature positivity on homogeneous spaces are considered. We proved that there exist infinitely many generalized Wallach spaces on which every invariant Riemannian metric preserves the positivity of the Ricci curvature when evolved by the normalized Ricci flow. Similarly, the number of generalized Wallach spaces is infinite as well on which every invariant Riemannian metric loses the positivity of the Ricci curvature.

Keywords: normalized Ricci flow, generalized Wallach space, Riemannian metrics, Ricci curvature.

Макалада бир тектүү мейкиндиктерде оң Риччи ийриликтин сакталышы жа сакталбашы жөнүндөгү суроолор караштырылат. Метрикалардын нормалдаштырылган Риччи агымынын таасири астындагы эволюциясы учурунда ар бир инварианттуу Риман метрикасы оң Риччи ийрилигинин сактап калчу чексиз көп сандагы жалпыланган Уоллах мейкиндиктери болоорун далилдедик. Туура ушундайча, ар бир инварианттуу Риман метрикасы оң Риччи ийрилигинин жоготчу жалпыланган Уоллах мейкиндиктеринин саны да чексиз.

Урунттуу сөздөр: нормалдаштырылган Риччи агымы, бир тектүү мейкиндик, жалпыланган Уоллах мейкиндиги, Риман метрикасы, Риччи ийрилиги.

В статье рассматриваютсяя вопросы сохранения или не сохранения положительности кривизны Риччи на однородных пространствах. Мы доказали, что существует бесконечно много обобщенных пространств Уоллаха, в которых каждая инвариантная метрика Римана сохраняет положительность кривизны Риччи в процессе эволюции метрик под влиянием нормализованного потока Риччи. Аналогично, число обобщенных пространств Уоллаха, в которых каждая инвариантная метрика Римана теряет положительности кривизны Риччи, тоже бесконечно.

Ключевые слова: нормализованный поток Риччи, однородное пространство, обобщённое пространство Уоллаха, метрика Римана, кривизна Риччи.

In [1] we studied the normalized Ricci flow [2]

$$\dot{\mathbf{g}}(t) = -2\mathrm{Ric}_{\mathbf{g}} + 2n^{-1}\mathbf{g}(t)\mathrm{S}_{\mathbf{g}} \tag{1}$$

on a class of homogeneous spaces G/H called generalized Wallach spaces (GWS), see [3] for definitions. The main useful feature of GWS is that every GWS can be characterized (descripted) by a triple of real numbers $(a_1, a_2, a_3) \in (0,1/2]^3$. The second useful feature is that any G-invariant metric $\mathbf{g}(\cdot,\cdot)$ on GWS admits the form:

$$\mathbf{g}(\cdot,\cdot) = x_1\langle\cdot,\cdot\rangle|_{\mathfrak{p}_1} + x_2\langle\cdot,\cdot\rangle|_{\mathfrak{p}_2} + x_3\langle\cdot,\cdot\rangle|_{\mathfrak{p}_3}, \qquad x_1, x_2, x_3 > 0. \tag{2}$$

One of the important questions is if the NRF preserves the positivity of the Ricci curvature of Riemannian metrics on a given manifold. In general, the answer to this question turned out to be negative: the Ricci flow need not to preserve the positivity of curvatures of metrics. In 2004, Ni showed in [4] non-compact Riemannian manifolds with bounded non-negative sectional curvature on which the Ricci flow does not preserve the non-negativity of sectional curvature. The positivity of curvatures may not be preserved on compact manifolds either, as it became clear in 2007 by Böhm and Wilking proved in [5, Theorem 3.1] the following statement: "On the compact manifold $\mathrm{Sp}(3)/\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ the Ricci flow evolves certain positively curved metrics into metrics with mixed Ricci curvature". In 2015, Cheung and Wallach proved in [6, Theorem 2] that on the spaces $W_6 = \mathrm{SU}(3)/T_{\max}$, $W_{12} = \mathrm{Sp}(3)/\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ and $W_{24} = F_4/\mathrm{Spin}(8)$ the Ricci flow deforms certain positively curved metrics into metrics with mixed sectional curvature. Metrics with positive sectional curvature may lose the positivity of the Ricci curvature not only on W_{12}. The results of [5] were covered in [6, Theorem 9] by asserting such a property for the Wallach spaces W_{12} and W_{24}: "There exist homogeneous metrics of strictly positive sectional curvature on the 12 and 24 dimensional examples that deform under the Ricci flow to metrics with some negative Ricci curvature". However, the results of [6, Theorem 9] cannot be extended to the space W_6. Some of them may still lose positive sectional curvature on W_6, however, all retain positive Ricci curvature as proved in [6, Theorem 8]: "If $\mathbf{g}(0)$ is a homogeneous Riemannian structure on a 6 dimensional example with strictly positive sectional curvature, then under the Ricci flow it retains strictly positive Ricci curvature".

In 2016-2017, the mentioned results of [5,6] were generalized in [1,7] to a special class of GWS with coincided parameters with $a_1 = a_2 = a_3 = a \in (0, 1/2)$ in which W_6, and W_{12} and W_{24} are contained as the members corresponding to the values $a = 1/6$, $a = 1/8$ and $a = 1/9$. Namely, for metrics (2) of the kind $x_1 \neq x_2 \neq x_3 \neq x_1$ (called generic metrics) we proved in [1, Theorem 1] that they all lose positivity of sectional curvature on the spaces W_6, W_{12} and W_{24}; in [1, Theorem 4] a wider set of metrics was found on W_6 (the case $a = 1/6$) preserving positive the Ricci curvature, even losing the positivity of the sectional curvature; it was proved in [1, Theorem 3] and in [7, Theorem 3] that all metrics (2) with positive Ricci curvature are evolved into metrics with mixed Ricci curvature under the NRF for $a \in (0, 1/6)$, whereas the NRF evolves all metrics (2) into metrics with positive Ricci curvature in a finite time, if $a \in (1/6, 1/2)$. Note also that metrics with $x_i = x_j$, $i \neq j$ (so called exceptional metrics) admit a simple dynamics along invariant curves of the flow, and, therefore, are of no interest. Hence, in the sequel by "a metric"' we mean "a generic metric".

Recently, in 2024, new results were obtained on non-maintenance of the positivity of the Ricci curvature of some metrics on some homogeneous spaces, when evolved by NRF. Namely, González-Álvaro and Zarei obtained in [8] such a result for the series of homogeneous spaces $\mathrm{Sp}(k+1)/\mathrm{Sp}(k-1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ which contain the space W_{12} as their first member corresponding to $k = 2$. A similar result was obtained in [9] by Cavenaghi et al. for the homogeneous spaces $\mathrm{SU}(k+2m) \,/\, \mathrm{S}\big(\mathrm{U}(k) \times \mathrm{U}(m) \times \mathrm{U}(m)\big)$, where the space W_6 is contained at $k = m = 1$. Results of [8,9] related to the Ricci curvature were generalized in [10] to the general case of the spaces $\mathrm{SU}(k+l+m) \,/\, \mathrm{S}\big(\mathrm{U}(k)\times\mathrm{U}(l)\times\mathrm{U}(m)\big)$ and $\mathrm{Sp}(k+l+m) \,/\, \mathrm{Sp}(k)\times\mathrm{Sp}(l)\times\mathrm{Sp}(m)$ with arbitrary $k,l,m\in\mathbb{N}$. Due to symmetry assume $k\ge l\ge m\ge 1$. In [10] the other homogeneous spaces were also studied, given in Table 1 in [3].

In [10] the results of [1, Theorems 2-4] and [7, Theorem 3] were also extended to the general case $a_1,a_2,a_3\in(0,1/2)$. Briefly, the questions were clarified in [1,7,10]: on which spaces can the positivity of the Ricci curvature be preserved or lost and does such a property hold for all metrics or only for some of them. Another interesting question arises about the number of such spaces.

The main results of this article are described in the following two theorems for GWS with $a_1,a_2,a_3\in(0,1/2)$.

Theorem 1. There exist infinitely many GWS with $a_1+a_2+a_3\le 1/2$ such that every metric (2) on every of them loses the positivity of the Ricci curvature when evolved by NRF (1).

Theorem 2. There exist infinitely many GWS $a_1+a_2+a_3>1/2$ such that every metric (2) on every of them preserves the positivity of the Ricci curvature when evolved by NRF (1).

Proof of Theorem 1. Consider an infinite family of homogeneous spaces $\mathrm{Sp}(k+l+m) \,/\, \mathrm{Sp}(k)\times\mathrm{Sp}(l)\times\mathrm{Sp}(m)$ with $a_1=\frac{k}{2(k+l+m+1)}$, $a_2=\frac{l}{2(k+l+m+1)}$ and $a_3=\frac{m}{2(k+l+m+1)}$, where $k,l,m\in\mathbb{N}$. Choose now $k=l=m$. Then for the infinite number of homogeneous spaces $\mathrm{Sp}(3k) \,/\, \mathrm{Sp}(k)\times\mathrm{Sp}(k)\times\mathrm{Sp}(k)$ we have $a_1=a_2=a_3=\frac{k}{6k+2}<\frac{1}{6}$. According to Theorem 3 in [1] every metric (2) on every space $\mathrm{Sp}(3k) \,/\, \mathrm{Sp}(k)\times\mathrm{Sp}(k)\times\mathrm{Sp}(k)$ loses $\mathrm{Ric}>0$. Theorem 1 is proved.

Remark 1. Since $a_1+a_2+a_3-1/2=-\frac{1}{2(k+l+m+1)}<0$ the spaces $\mathrm{Sp}(k+l+m) \,/\, \mathrm{Sp}(k)\times\mathrm{Sp}(l)\times\mathrm{Sp}(m)$ satisfy Theorem 6 in [10], which provides a preliminary guarantee that some metrics lose $\mathrm{Ric}>0$ under NRF (1) on every GWS with $a_1+a_2+a_3\le 1/2$. The condition $a_1+a_2+a_3\le 1/2$ can also be satisfied by another family of homogeneous spaces $\mathrm{SU}(k+l+m)\,/\,\mathrm{S}\big(\mathrm{U}(k)\times\mathrm{U}(l)\times\mathrm{U}(m)\big)$ with $a_1=\frac{k}{2(k+l+m)}$, $a_2=\frac{l}{2(k+l+m)}$ and $a_3=\frac{m}{2(k+l+m)}$. But Theorem 3 in [1] fails for $\mathrm{SU}(3k)\,/\,\mathrm{S}\big(\mathrm{U}(k)\times\mathrm{U}(k)\times\mathrm{U}(k)\big)$ with $a_1=a_2=a_3=1/6$.

Proof of Theorem 2. Denote $\boldsymbol{a}=(a_1,a_2,a_3)$ and introduce the following sets

$$\mathcal{A}=\{\boldsymbol{a}\mid\ a_1+a_2+a_3>1/2 \text{ and NRF (1) for all (2) preserves } \mathrm{Ric}>0\},$$

$$\mathcal{B}=\{\boldsymbol{a}\ \in\mathcal{A}\mid\boldsymbol{a}\ \text{corresponds to some GWS}\}.$$

We claim that $\mathcal{A}$ is an infinite set. Indeed consider the functions $\theta_i=a_i-\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1-2a_i}{1+2a_i}}$, obtained in [10], where $i=1,2,3$. Clearly, $\theta_i(0)=\theta_i(1/2)=0$ and $\theta_i(a_i)\in(0,1)$. We claim that $\theta_i=\theta_i(a_i)<1/6$ for all $a_i\in(0,1/2)$ and every $i=1,2,3$. Indeed, $\theta_i<1/6$ is equivalent to $3\sqrt{(1-2a)(1+2a)^{-1}}<2(2-3a)$, where $a:=a_i\in(0,1/2)$. Since $1-2a>0$ and $2-3a>0$ for $a\in(0,1/2)$, raising to square and simplifying the latter irrational inequality we get an equivalent one $p(a):=-72a^3+60a^2-2a-7<0$. Possessing the negative discriminant, the cubic polynomial $p(a)$ has a unique real root. Since $p(-\infty)>0$ (where $p(-\infty)$ $:=\lim_{a\to-\infty}p(a)=+\infty$) and $p(0)<0$, that root must be negative. Therefore $p(a)<0$ for all $a\in(0,+\infty)$, in particular, for $a\in(0,1/2)$. This proves $\theta_i<1/6$.

According to Theorem 7 in [10] the condition $(a_1,a_2,a_3)\in\mathcal{A}$ is equivalent to $\theta\geq\max\{\theta_1,\theta_2,\theta_3\}$, where $\theta=a_1+a_2+a_3-1/2$. The estimate $\theta_i<1/6$ provides an opportunity to make $\theta>\theta_i$ for each $i=1,2,3$ by choosing infinitely many values of a_i from the interval $(2/9,1/2)$. Indeed,

$$\theta=a_1+a_2+a_3-\frac{1}{2}>3\cdot\frac{2}{9}-\frac{1}{2}=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}>\theta_i$$

and, hence, the conditions $a_1,a_2,a_3\in(2/9,1/2)$ are sufficient to ensure $\mathrm{Ric}>0$. Thus, $\mathcal{A}$ can be chosen equal to the infinite (uncountable) set $(2/9,1/2)^3$.

Now we claim that $\mathcal{B}$ is infinite. It is well known that every GWS is described by some triple $a_1,a_2,a_3\in(0,1/2)$, but not every triple can correspond to a certain GWS (see [3]). Therefore, we need to prove the existence of infinitely many triples

$(a_1, a_2, a_3) \in \mathcal{A}$ that correspond to actual GWSs. A suitable example for this aim is an infinite family of homogeneous spaces $\mathrm{SU}(2l)/\mathrm{U}(l)$, $l \geq 2$ (see Table 1 in [3]) with $a_1 = \frac{l+1}{4l}$, $a_2 = \frac{l-1}{4l}$, $a_3 = \frac{1}{4}$ and $\theta = a_1 + a_2 + a_3 - 1/2 = 1/4 > 1/6 > \theta_i$, $i = 1,2,3$. Hence, for $\mathcal{B}$ we can take the countable set $\mathcal{B} = \left\{\left(\frac{l+1}{4l}, \frac{l-1}{4l}, \frac{1}{4}\right) \mid l \geq 2\right\} \subset \mathcal{A}$. This means that there is an infinite number of GWSs on which any metric (2) with $\mathrm{Ric} > 0$ preserves $\mathrm{Ric} > 0$ when evolved by NRF (1). Theorem 2 is proved.

Remark 2. Unfortunately, the spaces $\mathrm{SO}(k+l+m)/\mathrm{SO}(k) \times \mathrm{SO}(l) \times \mathrm{SO}(m)$, $k \geq l \geq m \geq 1$, $l \geq 2$, cannot satisfy Theorem 2. According to [10, Theorem 8] there is only a finite number of such spaces on which NRF preserves $\mathrm{Ric} > 0$ for every initial metric (2) with $\mathrm{Ric} > 0$. In the exceptional case $l = m = 1$ we have the Stiefel spaces $\mathrm{SO}(k+2)/\mathrm{SO}(k)$ on which every initial metric is evolved by NRF into metrics with $\mathrm{Ric} > 0$ as it was proved in [11].

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PROPERTIES OF GENERALIZED DERIVATIVES OF FUNCTIONS OF REAL VARIABLE

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Local operators on non-continuous functions and the most general local operators on functions defined on topological spaces are considered: coinciding of two functions in any neighbor of a point implies a same value of the operator on such functions in the point. The notion of derivative was extended to all functions in metrical spaces. Properties of such derivatives of functions of real variable are proven in the paper.

Keywords: functional relation, local equation, generalized derivative, algorithm

Үзгүлтүк функциялар боюнча жергилик операторлор жана топологиялык мейкиндиктерде аныкталган функциялар боюнча эң жалпы жергилик оператор-лор каралат: чекиттин каалаган айланасында эки функциянын дал келиши чек-иттеги мындай функциялар боюнча оператордун бирдей маанисин билдирет. Туунду түшүнүгү метрикалык мейкиндиктердеги бардык функцияларга жайыл-тылды.

Урунттуу сөздөр: функционалдык байланыш, жергилик теңдеме, жалпыланган туунду, алгоритм

Рассмотрены локальные операторы над разрывными функциями и наиболее общие локальные операторы над функциями, заданными в топологических пространствах: совпадение двух функций в любой окрестности точки влечет за собой одинаковое значение оператора над такими функциями в этой точке. Понятие производной было распространено на все функции в метрических пространствах. В статье доказаны свойства таких производных функций вещественной переменной.

Ключевые слова: функциональное соотношение, локальное уравнение, обобщенная производная, алгоритм

1. Introduction

The notion “local” is understood in two meanings: “infinitesimal” and “on bounded domain”. We used the first meaning. Considered a general task on relations between equations with global and local operators for a same function. Obviously, a global operator involves values of the function in different points of domain. Previously, definitions of local operators involved any kind of smoothness of a function defined on a domain with locally linear structure.

Probably, the first local operator on sufficiently non-smooth functions was introduced in [1]: upper and lower right and left derivative numbers of certain continuous functions. Also, there were publications on differential operators on “almost differentiable” (for instance, convex) functions and “differentiable almost everywhere” functions.

Section 2 contains definitions on local operators in topological and metrical spaces.

Section 3 connects local operators and functional relations.

Section 4 contains proofs of properties of local operators and functional relations.

2. Local operators in topological and metrical spaces

We asked artificial intelligence about local operators.

ChatGPT responded: In mathematics, a local operator refers to a type of function or mapping that acts on objects in a way that depends only on the information within a small, localized neighborhood of each point in the domain. In other words, the effect of the operator at a given point is determined solely by values of the function (or object) within a restricted area around that point, rather than depending on global information.

Local operators are common in fields like differential equations, functional analysis, and geometry. For example: Differential operators are local because the value of a derivative at a point depends only on the behavior of the function in an arbitrarily small neighborhood around that point.

In image processing, local filters (such as blurring or sharpening filters) apply changes to each pixel based on its surrounding pixels, making them local operators.

In contrast, a non-local operator depends on the values of the object over a larger or even the entire domain.

A local operator in mathematics can be strictly defined within the context of functional analysis or differential geometry, where it typically acts on sections of a vector bundle or functions defined on a manifold.

Definition 1. Let X be a topological space, and let E and F be sheaves of sets (or vector spaces, or modules) over X. A local operator from E to F is a map $T: G(U,E)\to G(U,F)$, where $G(U,E)$ denotes the sections of E over the open set $U \subseteq X$, with the following locality condition:

For all open sets $U \subseteq X$ and for all sections $s, t \in G(U, E)$, if s and t agree on an open subset $V\subseteq U$ (i.e., $s|_V=t|_V$), then $T(s)|_V=T(t)|_V$.

In other words, the action of the operator T on a section over an open set U on-ly depends on the values of the section in small neighborhoods within U. (End of ChatGPT's text).

Simplifying Definition 1, we proposed [2], by our opinion, the most general no-tion of local operator for function defined on a domain with topological structure without any restrictions on a range. We [3] slightly improved the definition.

Definition 2. A general local operator Q of the function $u:X\to V$, where X is a topological space, at the point $x\in X$ satisfies the condition: if for some neighborhood U of the point x the restrictions $u_1 |_U = u_2 |_U$ hold then $Q(u_1, x) = Q(u_2, x)$.

Examples of local operators over non-smooth functions.

Definition 3 [1]. The upper right derivative number in a point x_0 is the upper limit

$$limsup\{(f(x)-f(x_0))/(x-x_0)|\ x\to x_0+0\} \quad (1)$$

(if it exists). Similarly, the lower right, the upper left and lower left derivative number are determined.

Such limits exist not for all functions.

We used a well-known technique of transforming the extended space of real numbers into a segment.

Definition 4. We [2] proposed to change (1) to

$liminf, limsup \{(f(x)-f(x_0))/ (x-x_0) /(| f(x)-f(x_0)| /|x-x_0|+1) \ |x\to x_0\pm 0\}.$

(2)

Such four real numbers exist for all functions of real variable. These numbers may be called "generalized derivative numbers".

It may be generalized to functions from metrical spaces to metrical spaces.

Definition 5. Let f map a Banach space B to R; $x\in B;\ v\in B\backslash\{0\}$. Upper and lower generalized derivative numbers by direction:

$$limsup, liminf \{(f(x+tv)-f(x))/ (t||v||) /(|f(x+tv)-f(x)|/ (t||v||)+1) \ |t\to +0\}. \tag{3}$$

It may be generalized further to functions from metrical spaces to metrical spaces.

Definition 6. Two numbers

$$liminf, limsup\{\rho(f(x), f(x_0))/\rho (x, x_0)/(\rho(f(x), f(x_0))/\rho (x, x_0)+1) \mid x\to x_0\}. \tag{4}$$

Remark. We introduced the general notion of functional relations [4].

We considered the functional relation for a continuous function

$$u(2x)= 2u(x),\ x\in R_+. \tag{5}$$

Theorem 1. Solutions of the equation (5) fulfil the local equation

$$Q(u,0)=a_u\in R, \tag{6}$$

$$Q(u,x) := limsup\{measure \{x'\in[x, x+h] \mid u(x')\geq 0\}/h \mid h\to +0\}; \tag{7}$$

$a_u:= sup\{ measure \{x\in[1-t, 2-t]: u(x)\geq 0\}|0 \leq t \leq 1/2\}.$

(8)

Proof. The assumption "$u(x)\neq 0$" implies contradiction: we have $u(1)$

$=2^n u(2^{-n})$ for all $n \in N$, hence $|u(1)|$ is unbounded. Hence, $u(0)=0$.

Consider the functional relation for a continuous function

$$u(2x) = -2u(x),\ x \in R_+. \quad (9)$$

Theorem 2. Solutions of the equation (9) fulfil the local equation (6) with $a_u := \sup\{\ \text{measure}\ \{x \in [1-t, 2-t] : u(x) \geq 0\} | 0 \leq t \leq 3/4\}$.

3. Properties of generalized derivative numbers

Denote in (2)

$Difsup_+(f, x_0) := \limsup\{(f(x)-f(x_0))/(x-x_0)/(|f(x)-f(x_0)|/|x-x_0|+1)\ |x \to x_0+0\}$;

$Difsup_-(f, x_0) := \limsup\{(f(x)-f(x_0))/(x-x_0)/(|f(x)-f(x_0)|/|x-x_0|+1)\ |x \to x_0-0\}$;

$Difinf_+(f, x_0) := \liminf\{(f(x)-f(x_0))/(x-x_0)/(|f(x)-f(x_0)|/|x-x_0|+1)\ |x \to x_0+0\}$;

$Difinf_-(f, x_0) := limif\{(f(x)-f(x_0))/(x-x_0)/(|f(x)-f(x_0)|/|x-x_0|+1)\ |x \to x_0-0\}$.

Theorem 3. If $k>0$ then: if $|Difsup\,(f, x_0)| = 1$ then $Difsup\,(kf, x_0) := Difsup\,(f, x_0)$;

if $|Difsup\,(f, x_0)| < 1$ then

$Difsup\,(kf, x_0) := k \cdot Difsup\,(f, x_0)/(1 + Difsup\,(f, x_0)(k-1))$.

Theorem 4. If $k<0$ then: if $|Difinf\,(f, x_0)| = 1$ then $Difsup\,(kf, x_0) := -Difinf\,(f, x_0)$;

if $|Difinf\,(f, x_0)| < 1$ then

$Difsup\,(kf, x_0) := -k \cdot Difinf\,(f, x_0)/(1 - Difsup\,(f, x_0)(k-1))$.

The similar (antisymmetrical) formulas are valid for $Difinf_+(f, x_0)$, $Difinf_-(f, x_0)$.

4. Conclusion

The work was carried out within the framework of the project of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic. We hope that the paper would draw attention to generalized derivative numbers and such definitions would involve new classes of functions in Analysis.

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ON INVERSE PROBLEM FOR A CLASS OF PSEUDOPARABOLIC EQUATIONS

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In this work based on the methods of integral equations and on the basis of a new formula for solving second-order linear differential equations with variable coefficients [19], the theorems of uniqueness and existence of solutions for inverse problems of pseudoparabolic equations are proved.

Keywords: Inverse problems, pseudo-parabolic equations, mixed problems, method of integral equations, uniqueness and existence theorems .

Интегралдык теңдемелердин методдоруна негизделген жана өзгөрүлмө коэффициенттүү экинчи тартиптеги сызыктуу дифференциалдык теңдемелерди чечүү үчүн жаңы формуланын негизинде [19], бул макалада, псевдопараболикалык теңдемелердин тескери маселелеринин чыгарылышынын жалгыздыгы жана жашашы жөнүндөгү теоремалар далилденген.

Урунттуу сөздөр: Тескери маселелер, псевдопараболикалык теңдемелер, аралаш маселелер, интегралдык теңдемелер ыкмасы, чыгарылышынын жалгыздыгы жана жашашы жөнүндөгү теоремалар.

В данной работе, основанной на методах интегральных уравнений и на основе новой формулы для решения линейных дифференциальных уравнений второго порядка с переменными коэффициентами [19], доказаны теоремы единственности и существования решения для обратных задач псевдопараболических уравнений.

Ключевые слова: Обратные задачи, псевдопараболические уравнения, смешанные задачи, метод интегральных уравнений, теоремы единственности и существования.

Many problems of the theory and applications of inverse and ill-posed Problems have been investigated in numerous works. In particular, various issues of the study of ill-posed and inverse problems are studied in [1-18]. The basics of the theory and application of ill-posed and inverse problems can be found in [1-4]. Inverse problems for parabolic type equations have been studied in many papers. In particular, inverse problems for parabolic type equations with an integral redefinition condition were studied in [5-13]. Various inverse problems for pseudoparabolic and integro-differential pseudoparabolic equations were studied in [14-18].

In the presented work, based on the methods of integral equations and on the basis of a new formula for solving second-order linear differential equations with variable coefficients [19], inverse problems for one class of pseudo-parabolic equations are investigated. The theorems of uniqueness and existence of the solution are proved.

Consider the equations of the form

$$m(x,t)u_t'(x,t)=(Au)_t'+k(x,t)(Au)+\sum_{i=1}^{n}\alpha_i(t)f_i(x,t)+$$
$$+f_0(x,t),(x,t)\in D=[a,b]\times[0,T], \tag{1}$$

with an initial condition

$$u(x,0)=\varphi(x),\ \ x\in[a,b], \tag{2}$$

and boundary conditions

$$u(a,t)=u(b,t)=0,\ \ t\in[0,T], \tag{3}$$

with additional conditions

$$u(x_i,t)=g_i(t),\ \ t\in[0,T], \tag{4}$$

where $i=1,2,...,n,\ \ a<x_1<x_2<...<x_n<b,$

$$Au=u_{xx}''+p(x,t)u_x'+q(x,t)u,\ \ (x,t)\in D, \tag{5}$$

$m(x,t), k(x,t),\ p(x,t), q(x,t),\ f_0(x,t),\ f_i(x,t)\ (i=1,2,...,n)$ - known continuous functions in the domain $D, \varphi(x)$ a known continuous function in $[a,b]$, functions $g_1(t), g_2(t),..., g_n(t)$ known continuous functions in $[0,T]$, functions $u(x,t), \alpha_1(t), \alpha_2(t),..., \alpha_n(t)$ unknown functions.

In the prezent work, based on the methods of integral equations and on the basis of a new formula for solving second-order linear differential equations with variable coefficients [18], the theorems of uniqueness and existence of a solution for the inverse problems (1)-(4) are proved.

Let's assume that the approval condition is fulfilled

$$\varphi(x_i) = g_i(0),\ (i=1,2,...,n), \tag{6}$$

and $\varphi(x) \in C^2[a,b],\ g_i(t) \in C^1[0,T],\ i=1,2,...n.$

Deciding on Au, we transform equations (1):

$$\begin{aligned} Au &= e^{-\int_0^t k(x,\tau)d\tau}\left[\varphi''(x) + p(x,0)\varphi'(x) + q(x,0)\varphi(x)\right] + \\ &+ \int_0^t e^{-\int_s^t k(x,\tau)d\tau} m(x,s)u_s'(x,s)ds - \\ &- \int_0^t e^{-\int_s^t k(x,\tau)d\tau}\left[\sum_{i=1}^n \alpha_i(s) f_i(x,s) + f_0(x,s)\right]ds, \end{aligned} \tag{7}$$

where $(x,t) \in D.$ Next, given the formula

$$\begin{aligned} \int_0^t e^{-\int_s^t k(x,\tau)d\tau} m(x,s)u_s'(x,s)ds &= m(x,t)u(x,t) - e^{-\int_0^t k(x,\tau)d\tau} m(x,0)\varphi(x) - \\ &- \int_0^t e^{-\int_s^t k(x,\tau)d\tau}\left[m_s'(x,s) + k(x,s)m(x,s)\right]u(x,s)ds, \end{aligned} \tag{8}$$

and introducing the notation

$$f(x,t)=e^{-\int_0^t k(x,\tau)d\tau}\left[\varphi''(x)+p(x,0)\varphi'(x)+q(x,0)\varphi(x)\right]-$$
$$-e^{-\int_0^t k(x,\tau)d\tau}\varphi(x)-\int_0^t e^{-\int_s^t k(x,\tau)d\tau} f_0(x,s)ds, \tag{9}$$

from (7) we have

$$u''_{xx}+p(x,t)u'_x+\left[q(x,t)-m(x,t)\right]u=$$
$$=f(x,t)-\sum_{i=1}^{n}\int_0^t e^{-\int_s^t k(x,\tau)d\tau}\alpha_i(s)f_i(x,s)ds,\ (x,t)\in D. \tag{10}$$

Assume that the following conditions are met:

a) $$q(x,t)=m(x,t)+a(x,t)\left[p(x,t)-a(x,t)\right]-l^2(x,t)+$$
$$+a'_x(x,t),(x,t)\in D, \tag{11}$$

$$l(x,t)=r(t)\exp\left\{\int_a^x\left[2a(y,t)-p(y,t)\right]dy\right\}, \tag{12}$$

where $a(x,t), a'_x(x,t), p(x,t), m(x,t), f_i(x,t)\in C(D)$,

$(i=0,1,2,...,n)$, $r(t)\neq 0$ for all $t\in[0,T], r(t)\in C^1[0,T]$;

b) $a'_t(x,t), a''_{xt}(x,t), p'_t(x,t), m'_t(x,t)\in C(D)$,

Lemma 1. Let the conditions a), b) be satisfied. Then the functions

$$u_1(x,t)=\exp\left\{-\int_a^x[a(y,t)+l(y,t)]dy\right\},(x,t)\in D, \tag{13}$$

$$u_2(x,t)=\exp\left\{-\int_a^x[a(y,t)-l(y,t)]dy\right\},(x,t)\in D, \tag{14}$$

are the solution of the following homogeneous differential equation

$$u''_{xx}+p(x,t)u'_x+[q(x,t)-m(x,t)]u=0,(x,t)\in D. \tag{15}$$

In this case $\dfrac{\partial^3 u_i(x,t)}{\partial t\partial^2 x}\in C(D),\ i=1,2.$

The proof of Lemma 1 follows from Theorem 1 of the work [18].

Remark. Let in the differential equation (15)

$$p(x,t)=p(t),q(x,t)-m(x,t)=q(t),\ p^2(t)-4q(t)>0,(x,t)\in D.$$

Then condition a) i.e. (11) is satisfied when

$$a(x,t)=\frac{1}{2}p(t),r^2(t)=\frac{1}{4}p^2(t)-q(t),(x,t)\in D.$$

Theorem 1. Let the conditions a) b) be satisfied. Then the Green functions of the boundary value problem (15), (3) exist and are determined by the formula:

$$G(x,y,t)=\frac{\left[u_1(y,t)e^{2\int_a^b l(y,t)dy}-u_2(y,t)\right][u_1(x,t)-u_2(x,t)]}{2l(y,t)u_1(y,t)u_2(y,t)\left[e^{2\int_a^b l(y,t)dy}-1\right]},(x,y,t)\in D_1\times[0,T],\quad (16)$$

$$G(x,y,t)=\frac{[u_1(y,t)-u_2(y,t)]\left[u_1(x,t)e^{2\int_a^b l(y,t)dy}-u_2(x,t)\right]}{2l(y,t)u_1(y,t)u_2(y,t)\left[e^{2\int_a^b l(y,t)dy}-1\right]},(x,y,t)\in D_2\times[0,T],\quad (17)$$

where $D_1=\{a\le x\le y\le b\}$, $D_2=\{a\le y\le x\le b\}$. Here are the functions $u_1(x,t),u_2(x,t)$ defined in Lemma 1, i.e. by formulas (13) and (14).

Proof. Since $u_1(x,t),u_2(x,t)$ are the solution of the homogeneous differential equation (15), therefore, from formulas (16) and (17) it can be seen that $G(x,t)$ satisfies equation (15). Next, from the formula (13), (14), (16), (17) let's make sure that $G(0,y,t)=0$, $G(b,y,t)=0$ for all $y\in[a,b]$ and for $t\in[0,T]$.

It is clear that

$$G(y+0,y,t)=G(y-0,y,t),\ \forall y\in[a,b],\ \forall t\in[0,T],$$

where

$$G(y+0,y,t)=\lim_{\varepsilon\to 0}G(y+\varepsilon,y,t),$$
$$G(y-0,y,t)=\lim_{\varepsilon\to 0}G(y-\varepsilon,y,t),\ \varepsilon>0.$$

From (16) and (17) we get

$$G_x^{'}(x,y,t)=\frac{\left[u_1(y,t)e^{2\int_a^b l(y,t)dy}-u_2(y,t)\right]\left[a(x,t)-l(x,t)\right]u_2(x,t)}{2l(y,t)u_1(y,t)u_2(y,t)\left[e^{2\int_a^b l(y,t)dy}-1\right]}-$$

$$-\frac{\left[u_1(y,t)e^{2\int_a^b l(y,t)dy}-u_2(y,t)\right]\left[a(x,t)+l(x,t)\right]u_1(x,t)}{2l(y,t)u_1(y,t)u_2(y,t)\left[e^{2\int_a^b l(y,t)dy}-1\right]},(x,y,t)\in D_1\times[0,T], \tag{18}$$

$$G_x^{'}(x,y,t)=\frac{\left[u_1(y,t)-u_2(y,t)\right]\left[a(x,t)-l(x,t)\right]u_2(x,t)}{2l(y,t)u_1(y,t)u_2(y,t)\left[e^{2\int_a^b l(y,t)dy}-1\right]}-$$

$$-\frac{\left[u_1(y,t)-u_2(y,t)\right]-e^{2\int_a^b l(y,t)dy}\left[a(x,t)+l(x,t)\right]u_1(x,t)}{2l(y,t)u_1(y,t)u_2(y,t)\left[e^{2\int_a^b l(y,t)dy}-1\right]},(x,y,t)\in D_2\times[0,T]. \tag{19}$$

By virtue of formulas (18) and (19), we will make sure that

$G_x^{'}(y+0,y,t)=G_x^{'}(y-0,y,t)+1,$ for all $(y,t)\in D.$

Thus the function $G(x,y,t)$ defined by formulas (16) and (17) is the Green function of the boundary value problem (15), (3). Theorem 1 has been proved.

By applying the Green's function $G(x,y,t)$ defined in Theorem 1, from (10) and (3) we have

$$u(x,t)=\sum_{j=1}^{n}\int_0^t p_j(x,t,s)\alpha_j(s)ds+F(x,t),\ (x,t)\in D, \tag{20}$$

where

$$p_j(x,t,s)=-\int_a^b G(x,y,t)e^{-\int_s^t k(\tau,y)d\tau}f_j(y,s)dy,\ \ (j=1,2,...,n), \tag{21}$$

$$F(x,t)=\int_a^b G(x,y,t)f(y,t)dy,\ \ (x,t)\in D. \tag{22}$$

Substituting $x=x_i\ (i=1,2,...,n)$ and by virtue of condition (4), from (20) we get

$$\int_0^t Q(t,s)\alpha(s)ds = \beta(t),\ t\in[0,T], \tag{23}$$

where

$$Q(t,s)=\begin{pmatrix} q_{11}(t,s) & q_{12}(t,s) & \dots & q_{1n}(t,s) \\ q_{21}(t,s) & q_{22}(t,s) & \dots & q_{2n}(t,s) \\ \dots & \dots & \dots & \dots \\ q_{n1}(t,s) & q_{n2}(t,s) & \dots & q_{nn}(t,s) \end{pmatrix}, \tag{24}$$

$$\beta(t) = (\beta_1(t), \beta_2(t),...\beta_n(t))^T,\ \alpha(t) = (\alpha_1(t), \alpha_2(t),...,\alpha_n(t))^T,$$

$$q_{ij}(t,s) = p_j(x_i,t,s),\ \ \beta_i(t) = g_i(t) - F(x_i,t), \tag{25}$$

$(i, j = 1, 2,\ \dots\ ,n)$.

It is clear that system (23) is a system of linear integral Volterra equations of the first kind.

Theorem 2. Let the conditions a) b) and $\det Q(t,t)\neq 0$ for all $t\in[0,T]$, where the matrix- function $Q(t,s)$ is determined by the formula (24), (25) and (21). Then the inverse problem (1) - (4) in space $C^{1,2}(D)\times C_n[0,T]$ has a single solution $(u(t,x),\alpha(t))$. Here the vector- function $\alpha(t)$ is defined from the following system

$$\alpha(t) = \int_0^t Q^{-1}(t,t)Q_t^{'}(t,s)\alpha(s)ds + Q^{-1}(t,t)\beta^{'}(t),\ t\in[0,T], \tag{26}$$

where $Q^{-1}(t,t)$ is the inverse matrix to the matrix $Q(t,t)$ for $t\in[0,T]$, $\beta^{'}(t)$ is a derivative of a vector function $\beta(t)$, the matrix function $Q(t,s)$ and the vector- function $\beta(t)$ determined from the formula (24), (25), (21) and (22). Unknown function $u(t,x)$ is defined from the formula (20).

Proof. In this case, it is clear that the system (23) is equivalent to the system of linear integral Volterra equations of the second kind (26). Therefore, the system (23) has a single solution

$$\alpha(t)=(\alpha_1(t),\alpha_2(t),...,\alpha_n(t))^T$$

in space $C_n[0,T]$. Next, from (20) we define $u(t,x)$. Theorem 2 has been proved.

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ONE CLASS OF LINEAR INTEGRAL EQUATIONS OF THE THIRD KIND ON THE SEMI-AXIS

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In the present article the theorem about uniqueness of Fredholm linear integral equations of the third kind on the semi axis, with method of nonnegative quadratic forms and functional analysis methods.

Key words: linear inteqral equations, third kind, the semiaxis, uniqueness of solutions.

Бул макалада терс эмес квадраттык формалар усулунун, функционалдык анализдин усулдарынын жардамы менен жарым октогу үчүнчү түрдөгү сызыктуу интегралдык тендемелеринин чечимдеринин жалгыздыгы далилденди.

Урунттуу сөздөр: н сызыктуу интегралдык тендемелер, үчүнчү түрдөгү, жарым ок, чечимдеринин жалгыздыгы.

В настоящей статье доказана теорема о единственности линейных интегральных уравнений третьего рода на полуоси с использованием метода неотрицательных квадратичных форм и методов функционального анализа.

Ключевые слова: линейные интегральные уравнения третьего рода, полуось, единственность решений.

1. Introduction

Many problems of the theory of integral equations of the first kind were studied in [1-12]. But fundamental results for Fredholm integral equations of the first kind were obtained in [11-12], where regularizing operators in the sence of M.M. Lavrent'ev were constructed. Rezults on non-classical Volterra integral equations of the first kind can be found in [1]. In [2,6], problems of regularization, uniqueness and existence of solutions for Volterra integral and operator equations of the first kind are studied. In [13], for linear Volterra integral equations of the first and the third kind with smooth kernel, the existence of multiparameter family of solutions was proved. In [4,5], on the basis of theory of Volterra integral equations of the first kind, various inverse problems were studied. In [8], uniqueness theorems were proved and reqularizing operators in the sense of Lavrent'ev were constructed for systems of linear Fredholm integral equations of the third kind. In [10], problems of uniqueness and stability of solutions for linear integral equations of the first kind with two independent variables were investigated. In [3, 9], based on a new approach, the existence and uniquenes of solutions of Fredholm integral equations and the system linear Fredholm integral equations of the third kinds were studied.

In the prezent paper, on the basis of the method of integral transformation, uniqueness theorems for the new class of linear integral equations of the third kind in the semiaxis were proved.

2. The linear integral equations of the third kind

Consider the linear Fredholm integral equations of the third kind

$$m(t) + \int_{-\infty}^{+\infty} k(t,s)u(s)ds = f(t) \qquad (1)$$

where the u(t) is the desired function on $(-\infty, +\infty)$, $m(t)$, $f(t)$ is the given function, is the continuous on $(-\infty, +\infty)$

$$K(t,s) = \begin{cases} A(t,s), & -\infty < s \leq t < +\infty \\ B(t,s), & -\infty < t \leq s < +\infty \end{cases} \qquad (2)$$

the given functions A(t,s) and B(t,s) are continuous on the domain

$$G = \{(t,s): -\infty < s \leq t < +\infty\}.$$

Let C$(-\infty, +\infty)$ denote the space of all functions continuous on $(-\infty, +\infty)$. Here C(G) denote the space of all functions continuous on G.

We introduce the notation

$$H(t,s) = A(t,s) + B(s,t), (t,s) \in G \qquad (3)$$

Assume that the following conditions are satisfied:

$(i) H(t,s), H_t'(t,s), H_s'(t,s), H_{ts}''(t,s), \in C(G), \alpha(t) = \lim_{s\to-\infty} H(t,s),$

$t \in (-\infty,+\infty)$, $\alpha(+\infty) \geq 0, \alpha(t) \in C(-\infty,+\infty), \alpha'(t) \leq 0$ for all $t \in (-\infty,+\infty)$,

$H_{ts}''(t,s) \leq 0$ for all $(t,s) \in G, \alpha'(t) \in L_1(-\infty,+\infty), \beta(s) = H_s'(+\infty,s) \geq 0$ for all $s \in (-\infty,+\infty), \beta(s) \in C(-\infty,+\infty) \cap L_1(-\infty,+\infty)$;

(ii) $\operatorname*{Sup}_{(t,s)\in G} |H(t,s)| \leq 1 < \infty$, $\operatorname*{Sup}_{t\in(-\infty,+\infty)} \int_{-\infty}^{t} |H_s'(t,s)|ds \leq 1_1 < \infty$,

$H_{ts}''(t,s) \in L_1(G)$, $\operatorname*{Sup}_{t\in(s+\infty)} |H_s'(t,s)| \leq \gamma(s) \in L_1(-\infty,+\infty)$;

(iii) At least one of the following three conditions holds:

1) $\alpha(t) < 0$ for almost $t \in (-\infty,+\infty)$;
2) $\beta(s) > 0$ for almost all $s \in (-\infty,+\infty)$
3) $H_{ts}(t,s) < 0$ for almost all $(t,s) \in G$.

Theorem. Let conditions $(i), (ii)$ and (iii) be satisfied. Then the solution of the integral equation (1) is unique in $L_1(-\infty,+\infty)$.

Proof. Let $u(t) \in L_1(-\infty,+\infty)$ be a solution of the integral equation (1). By virtue of (2), we can write of the integral equation (1) in the form

$$m(t) + \int_{-\infty}^{+\infty} A(t,s)u(s)ds + \int_{t}^{+\infty} B(t,s)u(s)ds = f(t) \qquad (4)$$

Multiplying both sides of the equation (4) by u(t) and integrating over the domain $(-\infty,+\infty)$, we obtain.

$$\int_{-\infty}^{+\infty} m(t)u(t)dt + \int_{-\infty}^{+\infty}\int_{-\infty}^{t} A(t,s)u(s)u(t)dsdt +$$
$$+\int_{-\infty}^{+\infty}\int_{t}^{+\infty} B(t,s)u(s)u(t)dsdt = \int_{-\infty}^{+\infty} f(t)u(t)dt. \qquad (5)$$

Applying Dirichlet’s formulas to (5) and taking into account (3), we have

$$\int_{-\infty}^{+\infty} m(t)u(t)dt + \int_{-\infty}^{+\infty}\int_{-\infty}^{t} H(t,s)u(s)ds\, u(t)dt = \int_{-\infty}^{+\infty} f(t)u(t)dt \qquad (6)$$

We shall introduce the notation

$$z(t,s) = \int_s^t u(v)dv, \quad (t,s) \in G \tag{7}$$

Then from (7), we obtain

$$d_s z(t,s) = -u(s)ds, z(t,s)u(t)dt = \frac{1}{2} d_t\left(z^2(t,s)\right) \tag{8}$$

Let us transform the integral on the left hand of the identity (6). Taking into account (7), (8) and integrating by parts, we have

$$\int_{-\infty}^{+\infty} m(t)u(t)dt + \int_{-\infty}^{+\infty} \int_{-\infty}^{t} H(t,s)u(s)u(t)dsdt$$
$$= \int_{-\infty}^{+\infty} \alpha(t)z(t,-\infty)u(t)dt + \int_{-\infty}^{+\infty} \int_{-\infty}^{t} H_s(t,s)x(t,s)u(t)dsdt$$

Hence, applying Dirichlet's formula, we obtain

$$\int_{-\infty}^{+\infty} m(t)u(t)dt + \int_{-\infty}^{+\infty} \int_{-\infty}^{t} H(t,s)u(s)u(t)dsdt = \frac{1}{2}\int_{-\infty}^{+\infty} \alpha(t)d_t(z^2(t,-\infty)) +$$
$$\frac{1}{2}\int_{-\infty}^{+\infty}\left[\int_s^{+\infty} H_s'(t,s)d_t(z^2(t,s))\right] ds = \frac{1}{2}\alpha(+\infty)z^2(+\infty,-\infty) -$$
$$\frac{1}{2}\int_{-\infty}^{+\infty} \alpha'(t)z^2(t,-\infty)dt + \frac{1}{2}\int_{-\infty}^{+\infty} \beta(s)z^2(+\infty,s)ds -$$
$$\frac{1}{2}\int_{-\infty}^{+\infty}\int_s^{+\infty} H_{ts}''(t,s)z^2(t,s)dtds \tag{9}$$

Taking into account (6) and applying Dirichlet's formula from (9), we have

$$\int_{-\infty}^{+\infty} m(t)u(t)dt + \frac{1}{2}\alpha(+\infty)z^2(+\infty,-\infty) - \frac{1}{2}\int_{-\infty}^{+\infty} \alpha(t)z^2(t,-\infty)dt +$$
$$\frac{1}{2}\int_{-\infty}^{+\infty} \beta(s)z^2(+\infty,s)ds - \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{t} H_{ts}''(t,s)z^2(t,s)dsdt = \int_{-\infty}^{+\infty} f(t)u(t)dt$$
$$\tag{10}$$

Suppose that f(t)=0 for $t \in (-\infty,+\infty)$. Then, taking into account conditions (i), (ii) and (iii), we see that (10) implies

$$\int_{-\infty}^{t} u(\tau)d\tau = 0, \qquad t \in (-\infty,+\infty), \int_s^{+\infty} u(\tau)d\tau = 0\,, s \in (-\infty,+\infty)$$

or

$$\int_s^t u(\tau)d\tau = 0, \quad (t,s) \in G$$

Therefore, $u(t) = 0$ for all $t \in (-\infty, +\infty)$.

The theorem is proved.

Remark 1. If $B(s,t) = 0$ for all $(t,s) \in G$, then the integral equation (1) is the linear integral equation of the first kind. In this case the assertion of the theorem is true for $H(t,s) = A(t,s), \forall(t,s) \in G$

Remark 2. If $A(t,s) = 0$ for all $(t,s) \in G$, then the integral equation (1) is the linear integral equation of the first kind. In this case the assertion of the theorem is true for $H(t,s) = B(t,s), \forall(t,s) \in G$.

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ON THE BOUNDEDNESS OF SOLUTIONS OF FOURTH-ORDER LINEAR INTEGRO-DIFFERENTIAL EQUATIONS ON INFINITE DOMAINS

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This article studies the boundedness of solutions to partial differential equations on infinite domains.

Key words: limitations, integration by parts formula Dirichle, integration.

Бул макалада чексиз областтарда каралган жекече туундулуу дифференциалдык теңдемелердин чечимдеринин чектелгендиги изилденет.

Урунттуу сөздөр: чектелгендик, бөлүктөп интегралдоо, Дирихленин формуласы, интегралдоо.

Статья посвящена исследованию ограниченности решений уравнений в частных производных, заданных на бесконечных областях.

Ключевые слова: ограниченность, интегрируя по частям, формулы Дирихле, интегрирование.

The issue of the boundedness of solutions to integral and integro-differential equations on the half-line on infinite domains is also considered.

To demonstrate the boundedness of a solution, the method of nonnegative quadratic forms is used.

Sufficient conditions for boundedness on unbounded domains of a solution with zero initial data are established in the case where the free term belongs to the space of square-summable functions for a partial differential equation of order [number of squares]. Lemmas on integral transformations are applied, and V. Volterra's methods and the method of nonnegative quadratic forms are developed.

$$m(t,x)u_{txtx}(t,x)+l(t,x)u'_{txt}(t,x)+N(t,x)u'_{xtx}(t,x)+k(t,x)u_{tx}(t,x)+a(t,x)u_x(t,x)+b(t,x)u_t(t,x)+$$
$$+c(t,x)u(t,x)=f(t,x)(t,x)\in G=\{(t,x)/0\le t<\infty,\ \ 0\le x<\infty\}$$

(1)

with conditions

$$u(0,x)=0 \quad x\in[0,+\infty] \tag{*}$$

$$u(t,0)=0 \quad t\in[0,+\infty]$$

$$u_{tx}(0,x)=0 \quad x\in[0,+\infty]$$

$$f(t,x)\in L_2(G)\cap C(G) \tag{f}$$

Where $m(t,x), l(t,x)\ u(t,x), k(t,x), a(t,x), b(t,x), c(t,x), f(t,x)-$ known continuous functions, $u(t,x)-$ unknown function by domain $G_{tx}=\{0\le s\le t, 0\le y\le x\}$

Theorem. If conditions (*), (f) a) of the function are satisfied

$$m(t,x), l(t,x), N(t,x), k(t,x), a(t,x), b(t,x), a'_t(t,x)\ b'_x(t,x), c'_t(t,x), c'_x(t,x), c''_{tx}(t,x)\in C(G);\ N(t,x)\ge 0$$
$$m(t,x)\ge 0,\ l(t,x)\ge 0\ K(t,x)\ge 0,\ l'_t(t,x)\le 0,\ K(t,x)>\beta\ge 1,\ N'_x(t,x)\le 0$$

$$N(t,x)\ge 0, k(t,x)\ge 0, m'_t(t,x)\le 0, k(t,x)>\beta\ge 1, N'_x(t,x)\le 0$$

$$l(t,x)\ge 0\ K(t,x)\ge 0,\ l'_t(t,x)\le 0,\ K(t,x)>\beta\ge 1,\ N'_x(t,x)\le 0$$
$$a(t,x)\ge 0, a'_t(t,x)\le 0, b(t,x)\ge 0, b'_x(t,x)\le 0;\ m^2(t,x)-l'_t(t,x)\ N'_x(t,x)\le 0$$

в) $c'_t(t,x)\le 0,\ \ c'_x(t,x)\le 0,\ c(t,x)\ge\alpha>0,\ \ c''_{tx}(t,x)\ge 0$

$c^2(t,x)-a'_t(t,x)b'_x(t,x)\le 0,$ at $(t,x)\in G;$ then equation (1)-(*) has a unique solution in the space of continuous and bounded functions with their derivatives $C(G)$.

Proof: Let's make the following substitution.

$$u(t,x)=\int_0^t\int_0^x \vartheta(s,y)dyds$$

(2)

Substituting (2) into (1) we have:

$$m(t,x)\vartheta_{tx}(t,x)+l(t,x)\,\vartheta_t^{'}(t,x)+N(t,x)\vartheta_x^{'}(t,x)+K(t,x)\vartheta(t,x)+$$

$$+a(t,x)\int_0^t\vartheta(s,x)ds+b(t,x)\int_0^x\vartheta(t,y)dy+C(t,x)\int_0^t\int_0^x\vartheta(s,y)dyds=f(t,x)(t,x)\in G \quad (3)$$

We multiply both parts of identity (3) by and integrate over the domain

$G_{tx}\{0\le s\le t,\ 0\le y\le x\}.$ Then

$$\int_0^t\int_0^x m(s,y)\vartheta_{sy}^{''}(s,y)\vartheta(s,y)dyds+\int_0^t\int_0^x l(s,y)\vartheta_s^{'}(s,y)\vartheta(s,y)dyds+\int_0^t\int_0^x N(s,y)\vartheta_y^{'}(s,y)\vartheta(s,y)dyds+$$

$$+\int_0^t\int_0^x K(s,y)\vartheta^2(s,y)dyds+\int_0^t\int_0^x\int_0^s a(s,y)\vartheta(\tau,y)\vartheta(s,y)d\tau\,dyds+\int_0^t\int_0^x\int_0^y b(s,y)\vartheta(s,z)\vartheta(s,y)dzdyds+$$

$$+\int_0^t\int_0^x\int_0^s\int_0^y C(s,y)\vartheta(\tau,z)\vartheta(s,y)dzd\tau dyds=\int_0^t\int_0^x f(s,y)\,\vartheta(s,y)dyds; \quad (4)$$

Further, using Lemma 1. Lemma 1. For (4) any differentiable functions K,W the ratio is valid $KWW_s^{'}=\frac{1}{2}\left(KW^2\right)_s^{'}-\frac{1}{2}K_s^{'}W^2;$

Using the integration by parts formula and the Dirichlet formula, we transform the left side of relation (4). We have

$$\int_0^t\int_0^x l(s,y)\vartheta_s^{'}(s,y)\vartheta(s,y)dyds=\frac{1}{2}\int_0^t l(s,x)(\vartheta(s,x))^2ds-\frac{1}{2}\int_0^t\int_0^x l_s^{'}(s,y)\vartheta(s,y))^2dyds$$

$$\int_0^t\int_0^x N(s,y)\vartheta_s^{'}(s,y)\vartheta(s,y)dyds=\frac{1}{2}\int_0^x N(t,y)(\vartheta(t,y))^2dy-\int_0^t\int_0^x N_y^{'}(s,y)(\vartheta(s,y))^2dyds;$$

The question of uniqueness and belonging of the solution to the space of continuous and square-summable functions for linear integro-differential equations of Volterra type on the half-line was considered in the work [1]. For linear integral equations of Volterra type of the first kind on the half-line, this question was studied in the work [3].

For functions of two independent variables, similar issues were studied in [2].

The boundedness and stability of solutions to weakly nonlinear second-order Volterra-type integro-differential equations on the half-axis were considered in [4].

$$\int_0^t\int_0^x\int_0^s a(s,y)\vartheta(\tau,y)\vartheta(s,y)d\tau dyds=\int_0^t\int_0^x a(s,y)\left[\int_0^s\vartheta(\tau,y)d\tau\right]\left[\int_0^s\vartheta(\tau,y)dz\right]_s^{'}dyds=$$

$$=\frac{1}{2}\int_0^x a(t,y)\left(\int_0^t\vartheta(\tau,y)d\tau\right)^2dy-\frac{1}{2}\int_0^t\int_0^x a_s^{'}(s,y)\left(\int_0^s\vartheta(\tau,y)d\tau\right)^2dyds;$$

$$\int_0^t\int_0^x\int_0^y b(s,y)\vartheta(s,z)\vartheta(s,y)d\tau dyds=\int_0^t\int_0^x b(s,y)\left[\int_0^y\vartheta(s,z)dz\right]\left[\int_0^y\vartheta(s,z)dz\right]_y^{'}dyds=$$

$$=\frac{1}{2}\int_0^t b(s,x)\left(\int_0^x\vartheta(s,z)dz\right)^2ds-\frac{1}{2}\int_0^t\int_0^x b_y^{'}(s,y)\left(\int_0^y\vartheta(s,z)dz\right)^2dyds;$$

Further, using Lemma 3

$$C\nu\nu_{sy}^{''}=\frac{1}{2}\left(C\nu^2\right)_{sy}^{''}-\frac{1}{2}\left(C_s^{'}\nu^2\right)_y^{'}-\frac{1}{2}\left(C_y^{'}\nu^2\right)_s^{'}+\frac{1}{2}C_{sy}^{''}\nu^2-C\nu_y^{'}\nu_s^{'};$$

$$\int_0^t\int_0^x m(s,y)\vartheta_{sy}^{''}(s,y)\vartheta(s,y)dyds=\frac{1}{2}m(t,x)(\vartheta(t,x))^2-\frac{1}{2}\int_0^t m_s^{'}(s,x)(\vartheta(s,x))^2ds-\frac{1}{2}\int_0^x m_y^{'}(t,y)\times$$

$$\times(\vartheta(t,y))^2dy+\frac{1}{2}\int_0^t\int_0^x m_{sy}^{''}(s,y)(\vartheta(s,y))^2dyds-\int_0^t\int_0^x m(s,y)\vartheta_y^{'}(s,y)\vartheta_s^{'}(s,y)dyds;$$

$$\int_0^t\int_0^x\int_0^s\int_0^y C(s,y)\vartheta(\tau,z)\vartheta(s,y)dzd\tau dyds=\int_0^t\int_0^x C(s,y)\left[\int_0^s\int_0^y\vartheta(\tau,z)dzd\tau\right]\left[\int_0^s\int_0^y\vartheta(\tau,z)dzd\tau\right]_{sy}^{''}dyds=$$

$$=\frac{1}{2}C(t,x)\left[\int_0^t\int^x\vartheta(\tau,z)dzd\tau\right]^2-\frac{1}{2}\int_0^t C_s^{'}(s,x)\left[\int_0^s\int_0^x\vartheta(\tau,z)dzd\tau\right]^2ds-\frac{1}{2}\int_0^x C_y^{'}(t,y)\left[\int_0^t\int_0^y\vartheta(\tau,z)dzd\tau\right]^2dy+$$

$$+\frac{1}{2}\int_0^t\int_0^x C_{sy}^{''}(s,y)\left[\int_0^s\int_0^y\vartheta(\tau,z)dzd\tau\right]^2dyds-\int_0^t\int_0^x C(s,y)\left[\int_0^y\vartheta(s,z)dz\right]\left[\int_0^s\vartheta(\tau,y)d\tau\right]dyds; \tag{5}$$

(4) We rewrite the equality in the following form

$$\frac{1}{2}m(t,x)\big(\vartheta(t,x)\big)^2-\frac{1}{2}\int_0^t m_s'(s,x)\big(\vartheta(s,x)\big)^2 ds-\frac{1}{2}\int_0^x m_y'(t,y)\big(\vartheta(t,x)\big)^2 dy+$$
$$+\frac{1}{2}\int_0^t\int_0^x m_{sy}''\big(\vartheta(s,y)\big)^2 dyds+\frac{1}{2}\int_0^t\int_0^x\Big\{-l_s'(s,y)\big(\vartheta(s,y)\big)^2-2m(s,y)\vartheta_y'(s,y)\vartheta_s'(s,y)-$$
$$-N_y'(s,y)\big(\vartheta(s,y)\big)^2\Big\}dyds+\frac{1}{2}\int_0^t l(s,x)\big(\vartheta(s,x)\big)^2 ds+\frac{1}{2}\int_0^x N(t,y)\big(\vartheta(t,y)\big)^2 dy+$$

$$+\int_0^t\int_0^x k(s,y)\vartheta^2(s,y)dyds+\frac{1}{2}\left\{\int_0^x a(t,y)\left(\int_0^t\vartheta(\tau,y)d\tau\right)^2 dy+\int_0^t b(s,x)\left(\int_0^x\vartheta(s,z)dz\right)^2 ds-\right.$$
$$\left.-\int_0^t C_s'(s,x)\left(\int_0^s\int_0^x\vartheta(\tau,z)dzd\tau\right)^2 ds-\int_0^x C_y'(t,y)\left(\int_0^t\int_0^y\vartheta(\tau,z)dzd\tau\right)^2 dy\right\}+\frac{1}{2}C(t,x)\left[\int_0^t\int_0^x\vartheta(\tau,z)dzd\tau\right]^2+$$
$$+\frac{1}{2}\int_0^t\int_0^x\left\{-a_s'(s,y)\left(\int_0^s\vartheta(\tau,y)d\tau\right)^2-2C(s,y)\left(\int_0^s\vartheta(\tau,y)d\tau\right)\left(\int_0^y\vartheta(s,z)dz\right)-b_y'(s,y)\left(\int_0^y\vartheta(s,z)dz\right)^2\right\}dyds+$$

$$+\int_0^t\int_0^x C_{sy}''\left(\int_0^s\int_0^y\vartheta(\tau,z)dzd\tau\right)^2 dyds=\int_0^t\int_0^x f(s,y)\vartheta(s,y)dyds\,. \qquad (6)$$

Due to conditions a)-c), the left-hand side of relation (6) is non-negative. Therefore, the following inequality follows from this

$$\int_0^t\int_0^x k(s,y)\vartheta^2(s,y)dyds+\frac{\alpha}{2}\left(\int_0^t\int_0^x\vartheta(s,y)dyds\right)^2\le\int_0^t\int_0^x f(s,y)\vartheta(s,y)dyds\,. \qquad (7)$$

Then, by virtue of substitution (2), this is equivalent to the inequality

$$\int_0^t\int_0^x k(s,y)\vartheta^2(s,y)dyds+\frac{\alpha}{2}u^2(t,x)\le\left|\int_0^t\int_0^x f(s,y)\vartheta(s,y)dyds\right|. \qquad (8)$$

We further have

$$\int_0^t\int_0^x f(s,y)\vartheta(s,y)dyds\le\frac{1}{2}\int_0^t\int_0^x f^2(s,y)dyds+\frac{1}{2}\int_0^t\int_0^x\vartheta^2(s,y)dyds$$

Taking this into account, from (8) we have

$$\int_0^t\int_0^x [2K(s,y)-1]\vartheta^2(s,y)dyds+\alpha u^2(t,x)\le\int_0^t\int_0^x f^2(s,y)dyds. \qquad (9)$$

$K(t,x)>\beta\ge 1.$

$$\int_0^t\int_0^x [2\beta-1]\vartheta^2(s,y)dyds+\alpha u^2(t,x)\le\int_0^t\int_0^x f^2(s,y)dyds.$$

From here, by virtue of condition (f), we have

$$\alpha u^2(t,x)\le\int_0^\infty\int_0^\infty f^2(s,y)dyds<\infty,$$

$$u^2(t,x)\le\frac{1}{\alpha}\int_0^\infty\int_0^\infty f^2(s,y)dyds.$$

Hence $u(t,x)$ limited at $(t,x)\in G$. Thus, the theorem is proven.

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MSC 35A25

PROOF OF THE UNIQUENESS OF THE SOLUTION OF THE CAUCHY PROBLEM FOR FIRST-ORDER QUASILINEAR DIFFERENTIAL EQUATIONS WITH THE ADDITIONAL ARGUMENT METHOD

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Кошумча аргумент ыкмасын колдонуу менен баштапкы Коши шарты коюлган квазисызыктуу дифференциалдык теңдеменин чыгарылышынын жашашын жана жалгыздыгын далилдөө.

Макаланын максаты: кошумча аргумент ыкмасын колдонуу менен баштапкы Коши шарты койулган квазисызыктуу дифференциалдык теңдемени изилдөөнүн негизинде каралган маселенин чыгарылышынын жашашын жана жалгыздыгын көрсөтүү.

Урунттуу сөздөр: кошумча аргумент, баштапкы Коши шарты, квазисызыктуу дифференциалдык теңдеме, интегралдык теңдеме, удаалаш жакындо.

Применение метода дополнительного аргумента в доказательстве о существование и единственности решение квазилинейных дифференциальных уравнений первого порядка с начальным условием Коши.

Цель статьи: Показать применение метода дополнительного аргумента при доказательстве о существование и единственности решение квазилинейных дифференциальных уравнений первого порядка с начальным условием Коши.

Ключевые слова: дополнительный аргумент, начальное условие Коши, квазилинейное дифференциальное уравнение, интегральное уравнение, последовательное приближение.

Application of the method of additional argument in the proof of the existence and uniqueness of the solution of quasilinear differential equations of the first order with the Cauchy initial condition.

Purpose of the article: Show the application of the additional argument method in proving the existence and uniqueness of the solution of first-order quasilinear differential equations with Cauchy's initial condition.

Keywords: additional argument, initial Cauchy condition, quasilinear differential equation, integral equation, successive approximation.

In the paper [5] a first-order quasilinear differential equation with an initial Cauchy-type condition of the following form is considered:

$$\frac{\partial u}{\partial t}+a_1(t,x,y,u)\frac{\partial u}{\partial x}+a_2(t,x,y,u)\frac{\partial u}{\partial y}=f(t,x,y,u) \qquad (1)$$

$$u(0,x,y)=\varphi(x,y) \qquad (2)$$

$$t\in[o,T],\ T=const,\ (x,y)\in R^2.$$

Using the method of an additional argument, problem (1)-(2) is reduced to systems of integral equations:

$$p_1(s,t,x,y)=x-\int_s^t a_1(v,p_1(v,t,x,y),p_2(v,t,x,y),w(v,t,x,y))dv,$$

$$p_2(s,t,x,y)=y-\int_s^t a_2(v,p_1(v,t,x,y),p_2(v,t,x,y),w(v,t,x,y))dv. \qquad (6)$$

$$w(s,t,x,y)=\varphi\begin{pmatrix} x-\int_0^t a_1(v,p_1(v,t,x,y),p_2(v,t,x,y),w(v,t,x,y))dv, \\ y-\int_0^t a_2(v,p_1(v,t,x,y),p_2(v,t,x,y),w(v,t,x,y))dv \end{pmatrix}+$$

$$+\int_0^s f(v,p_1(v,t,x,y),p_2(v,t,x,y),w(v,t,x,y))dv. \qquad (7)$$

The equivalence of problems (1)-(2) and (6)-(7) is proved.

In this paper, we prove the existence of a continuously differentiable, bounded solution to the system of integral equations (6)-(7) from [5], thereby proving the existence of a solution to the problem (1)-(2) [5].

To do this, we introduce the following notation and definitions:

We will denote by $C(\Omega_*)$ and $C^{\alpha_1,\alpha_2,\ldots,\alpha_n}(\Omega_*)$ –spaces of functions defined and continuous (respectively, together with their derivatives up to the order α_i by i – the argument, i=1,2,…,n on some subset Ω_* Euclidean space R^n , n=1,2,…).

$$a_{i,(j)}(\xi_1,\xi_2,\xi_3,\xi_4)=\frac{\partial a_i(\xi_1,\xi_2,\xi_3,\xi_4)}{\partial \xi_j},$$

$$f_{i,(j)}(\xi_1,\xi_2,\xi_3,\xi_4)=\frac{\partial f_i(\xi_1,\xi_2,\xi_3,\xi_4)}{\partial \xi_j}, \quad j=1,2,3,4.$$

Q[T, K]= [0, T]×R^2×[-K, K] where K is an arbitrarily fixed positive number.

$$M_f(K)=max\left\{\sup_{Q(T,K)}|f_i|,\ \sup_{Q(T,K)}|f_{i,(j)}|,\ \ j=2,3,4\right\},$$

$$M_a(K)=max\left\{\sup_{Q(T,K)}|a_i|,\ \sup_{Q(T,K)}|a_{i,(j)}|,\ \ j=2,3,4\right\}.$$

We will also use the previously introduced notation N_φ.

Let K be fixed so that K> N_φ. For any function f∊C(Q(T, K)) one can specify a number T(K) such that the inequality N_φ+T(K) $M_f(K)$≤K.

Without loss of generality, we assume that the constant K is chosen in such a way that T(K) has the maximum possible value, i.e.

$$T=\sup_{K\epsilon(N_\varphi,\infty)}T(K)=\sup_{K\epsilon(N_\varphi,\infty)}\frac{K-N_\varphi}{M_f(K)}.$$

Let us now make a change in the system of equations (6)-(7), setting:

$$q_1=x-p_1,$$

$$q_2=y-p_2\ .$$

Then we arrive at the following system of equations:

$$q_1(s,t,x,y)=\int_s^t a_1(v,x-q_1(v,t,x,y),y-q_2(v,t,x,y),w(v,t,x,y))dv, \tag{1*}$$

$$q_2(s,t,x,y)=\int_s^t a_2(v,x-q_1(v,t,x,y),y-q_2(v,t,x,y),w(v,t,x,y))dv, \tag{2*}$$

$$w(s,t,x,y)=\varphi(x-\int_0^t a_1(v,x-q_1(v,t,x,y),y-q_2(v,t,x,y),w(v,t,x,y))dv,$$

$$y-\int_0^t a_2(v,x-q_1(v,t,x,y),y-q_2(vt,x,y),w(v,t,x,y))dv)+$$

$$+\int_0^s f(v,x-q_1(v,t,x,y),y-q_2(vt,x,y),w(v,t,x,y))dv\ . \tag{3*}$$

Lemma: Let $\varphi\in C^{1,1}(R^2),\ a_1,a_2,f\in(Q(T,K))$.

Moreover, the numbers T and K are chosen in such a way that the inequality

$$N_{\varphi}+\mathrm{T}\,M_{f}(K)\leq \mathrm{K}. \quad (4^{*})$$

Let further δ –positive root of the equation:

$$\delta=(1+N_{\varphi})(M_{1}(K)+M_{1}(K))t+M_{2}(K)\frac{t}{2}=1.$$

$$T_{1}=\begin{cases}\delta-\varepsilon, if\ \delta\leq T\\ T,\quad if\ \delta\geq T\end{cases},$$

where ε – any number from the range (0, δ). Then for $0<t<T_1$ the system of equations (1^*)- (3^*) has the only solution

$$q_{i}(s,t,x,y)\in C(\Omega_{T_{1}})\ (i=1,2)\ ,\ w(s,t,x,y)\in C(\Omega_{T_{1}}).$$

Proof: We will prove the existence of a solution to the system of equations (1^*)- (3^*) method of successive approximations.

Let's take $w^{0}(s,t,x,y)=\varphi(x,y)$, and the n - th approximation is determined from the system:

$$q_{1}^{n}(s,t,x,y)=\int_{s}^{t}a_{1}(v,x-q_{1}^{n-1}(v,t,x,y),y-q_{2}^{n-1}(v,t,x,y),w^{n-1}(v,t,x,y))dv, \quad (5^{*})$$

$$q_{2}^{n}(s,t,x,y)=\int_{s}^{t}a_{2}(v,x-q_{1}^{n-1}(v,t,x,y),y-q_{2}^{n-1}(v,t,x,y),w^{n-1}(v,t,x,y))dv,$$

$$(6^{*})\ w^{n}(s,t,x,y)=\varphi(x-\int_{0}^{t}a_{1}(v,x-q_{1}^{n-1}(v,t,x,y),y-q_{2}^{n-1}(v,t,x,y),w^{n-1}(v,t,x,y))dv,$$
$$y-\int_{0}^{t}a_{2}(v,x-q_{1}^{n-1}(v,t,x,y),y-q_{2}^{n-1}(v,t,x,y),w^{n-1}(v,t,x,y))dv)+$$
$$+\int_{0}^{s}f(v,x-q_{1}^{n-1}(v,t,x,y),y-q_{2}^{n-1}(v,t,x,y),w^{n-1}(v,t,x,y))dv\,. \quad (7^{*})$$

By virtue of (4^*) from (7^*) we get that all $w^{n}(s,t,x,y)$ n=0,1,2,… limited i.e.

$$|w^{n}|\leq \mathrm{K}. \quad (8^{*})$$

From (5^*) and (6^*) by the conditions of the lemma, it follows

$$|q_{i}^{k}|\leq tM_{i}(K),\ i=1,2;\ k=1,2,\ldots \quad (9^{*})$$

Let us prove that successive approximations converge. Will have:

$$q_i^{n+1} - q_i^n = \int_s^t [a_i(v, x - q_1^n(v,t,x,y), y - q_2^n(v,t,x,y), w^n(v,t,x,y)) -$$
$$-a_i(v, x - q_1^{n-1}(v,t,x,y), y - q_2^{n-1}(v,t,x,y), w^{n-1}(v,t,x,y))]dv, \tag{10*}$$

$$w^{n+1} - w^n = \varphi(x - \int_0^t a_1(v, x - q_1^n(v,t,x,y), y - q_2^n(v,t,x,y), w^n(v,t,x,y))dv,$$
$$y - \int_0^t a_2(v, x - q_1^n(v,t,x,y), y - q_2^n(v,t,x,y), w^n(v,t,x,y))dv) -$$
$$-\varphi(x - \int_0^t a_1(v, x - q_1^{n-1}(v,t,x,y), y - q_2^{n-1}(v,t,x,y), w^{n-1}(v,t,x,y))dv,$$
$$y - \int_0^t a_2(v, x - q_1^{n-1}(v,t,x,y), y - q_2^{n-1}(v,t,x,y), w^{n-1}(v,t,x,y))dv) +$$
$$+ \int_0^s [f(v, x - q_1^n(v,t,x,y), y - q_2^n(v,t,x,y), w^n(v,t,x,y)) -$$
$$-f(v, x - q_1^{n-1}(v,t,x,y), y - q_2^{n-1}(v,t,x,y), w^{n-1}(v,t,x,y))]dv. \tag{11*}$$

Note that due to $\varphi \in C^{1,1}(R^2)$, $\varphi(x,y)$ satisfies the Lipschitz condition with constant N_φ.

We have:

$$|\ \varphi(x^1, y^1) - \varphi(x^2, y^2)| = |\ \partial_x \varphi(x^1 - \theta_1(x^1 - x^2), y^1) - \partial_y \varphi(x^1, y^1 - \theta_2(y^1 - y^2)|,$$

where $0<\theta_1<1$, $0<\theta_2<1$.

Hence it follows that

$$|\ \varphi(x^1, y^1) - \varphi(x^2, y^2)| \leq N_\varphi[|x^1 - x^2| + |y^1 - y^2|]. \tag{12*}$$

Also for functions $a_{i,(j)}(\xi_1, \xi_2, \xi_3, \xi_4)$, $f_{i,(j)}(\xi_1, \xi_2, \xi_3, \xi_4)$:

$$f(\xi_1, \xi_2^1, \xi_3^1, \xi_4^1) - f(\xi_1, \xi_2^2, \xi_3^2, \xi_4^2) = \partial_{\xi_2} f(\xi_1, \xi_2^1 - \theta_2(\xi_2^1 - \xi_2^2), \xi_3^1, \xi_4^1)(\xi_2^1 - \xi_2^2) +$$

$$+\partial_{\xi_3} f(\xi_1,\xi_2^1,\xi_3^1-\theta_3(\xi_3^1-\xi_3^2),\xi_4^1)(\xi_3^1-\xi_3^2)+\partial_{\xi_4} f(\xi_1,\xi_2^1,\xi_3^1,\xi_4^1-\theta_4(\xi_4^1-\xi_4^2))(\xi_4^1-\xi_4^2),$$

where $0<\theta_j<1$, $j=2,3,4$.

From this equality we deduce:

$$|f(\xi_1,\xi_2^1,\xi_3^1,\xi_4^1)-f(\xi_1,\xi_2^2,\xi_3^2,\xi_4^2)|\le M_f(K)[|\xi_2^1-\xi_2^2|+|\xi_3^1-\xi_3^2|+|\xi_4^1-\xi_4^2|]. \tag{13*}$$

These inequalities also hold for a_i, $i=1,2$

$$|a_i(\xi_1,\xi_2^1,\xi_3^1,\xi_4^1)-a_i(\xi_1,\xi_2^2,\xi_3^2,\xi_4^2)|\le M_i(K)[|\xi_2^1-\xi_2^2|+|\xi_3^1-\xi_3^2|+|\xi_4^1-\xi_4^2|]. \tag{14*}$$

Using (12*), (13*), (14*) get from (10*), (11*):

$$|q_i^{n+1}-q_i^n|\le tM_i\big[||q_1^n-q_1^{n-1}||_{\Omega_t}+||q_2^n-q_2^{n-1}||_{\Omega_t}+||w^n-w^{n-1}||_{\Omega_t}\big], \tag{15*}$$

$$|w^{n+1}-w^n|\le M_{fi}\Big[|\int_0^t[a_1(v,x-q_1^n,y-q_2^n,w^n)-a_1(v,x-q_1^{n-1},y-q_2^{n-1},w^{n-1})]dv|+$$

$$+|\int_0^t[a_2(v,x-q_1^n,y-q_2^n,w^n)-a_2(v,x-q_1^{n-1},y-q_2^{n-1},w^{n-1})]dv|\Big]+$$

$$+\ tM_f\big[||q_1^n-q_1^{n-1}||_{\Omega_t}+||q_2^n-q_2^{n-1}||_{\Omega_t}+||w^n-w^{n-1}||_{\Omega_t}\big].$$

And then

$$|w^{n+1}-w^n|\le t(N_\varphi(M_1+M_2)+M_f)\Big[||q_1^n-q_1^{n-1}||_{\Omega_t}+||q_2^n-q_2^{n-1}||_{\Omega_t}+$$

$$+||w^n-w^{n-1}||_{\Omega_t}\Big]. \tag{16*}$$

From (15*) end (16*) we get:

$$||q_1^n-q_1^{n-1}||_{\Omega_t}+||q_2^n-q_2^{n-1}||_{\Omega_t}+||w^n-w^{n-1}||_{\Omega_t}\le$$

$$\le t((N_\varphi+1)(M_1+M_2)+M_f)\big[||q_1^n-q_1^{n-1}||_{\Omega_t}+||q_2^n-q_2^{n-1}||_{\Omega_t}+||w^n-w^{n-1}||_{\Omega_t}\big].$$

Denote by $\overrightarrow{z_n}$ vector function $\overrightarrow{z_n}$=colon(q_1^n,q_2^n,w^n).

Let's define the norm $\overrightarrow{z_n}$ equality $||\overrightarrow{z_n}||^*_{\Omega_t}= ||q_1^n||_{\Omega_t} + ||q_2^n||_{\Omega_t} + ||w^n||_{\Omega_t}$.

Consider the series $\overrightarrow{z_0} + (\overrightarrow{z_1} - \overrightarrow{z_0}) + (\overrightarrow{z_2} - \overrightarrow{z_1}) + \cdots + (\overrightarrow{z_n} - \overrightarrow{z_{n-1}})$.

This series is majorized in the norm $||\overrightarrow{z_n}||^*_{\mathrm{T}_1}$ converging next to:

$K + 2K(1 + A_{max}(t) + A^2_{max}(t) + \cdots + A^{n-1}_{max}(t) + \cdots$ (as a series with members of a geometric progression), where $A_{max}(t)= t((N_\varphi + 1)(M_1 + M_2) + M_f)$.

Taken from here:

$$(\overrightarrow{z_0})_{\mathrm{T}_1} \leq \text{К},\ (\overrightarrow{z_1} - \overrightarrow{z_0})_{\mathrm{T}_1} \leq 2\text{К},\ (\overrightarrow{z_2} - \overrightarrow{z_1})_{\mathrm{T}_1} \leq 2\text{К}A_{max}(t),\ (\overrightarrow{z_3} - \overrightarrow{z_2})_{\mathrm{T}_1} \leq$$
$$2\text{К}A^2_{max}(t), \dots,\ (\overrightarrow{z_n} - \overrightarrow{z_{n-1}})_{\mathrm{T}_1} \leq 2\text{К}A^{n-1}_{max}(t).$$

This means that the partial sum converges to the function $z\epsilon C(\Omega_{T_1})$ according to the norm of this space $C(\Omega_{T_1})$ i.e. w^n end q_i^n also converges to $w\epsilon C(\Omega_{T_1})$ $q_i\epsilon C(\Omega_{T_1})$ respectively. Passing to the limit at n→∞ in equality (7^*) we get that the function $w(s,t,x,y)$ satisfies the system (2^*)- (3^*). Similarly for $q_i(s,t,x,y)$.

The uniqueness follows from the fact that for the difference of two possible solutions of the system of equations (1^*)- (3^*) inequality will be satisfied

$(\overrightarrow{z_1} - \overrightarrow{z_2})_{\mathrm{T}_1} \leq A_{max}(\mathrm{T}_1)(\overrightarrow{z_1} - \overrightarrow{z_2})_{\mathrm{T}_2}$, where $\mathrm{A}(\mathrm{T}_1) < 1$.

Lemma proven.

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NUMERICAL ANALYTICAL METHOD FOR SOLVING CERTAIN BOUNDARY VALUE PROBLEMS

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В статье рассмотрен метод последовательных приближений для решения дифференциальных уравнений с граничными условиями.

Ключевые слова: дифференциальное уравнение, граничное условие, интегральное уравнение, метод последовательных приближений.

Бул макалада чек ара шарттары бар дифференциалдык теңдемелерди чечүү үчүн ырааттуу жакындатуу ыкмасы талкууланат.

Урунттуу сөздөр: дифференциалдык теңдеме, чектик шарт, интегралдык теңдеме, удаалаш жакындатуу ыкмасы.

This article discusses the method of successive approximations for solving differential equations with boundary conditions.

Keywords: differential equation, boundary condition, integral equation, method of successive approximations.

Let $g\left(x(0),\, x(T)\right)$ be a function of two variables $\{x_m\}$ $x_i(0)$, $i = 1, 2, \ldots, q$, and $j = q+1, 2, .., n$ will be unambiguously resolved relative to $x_i(0) = \varphi_i(x(T)), x_j(T) = \varphi_j(x(0))$. Then

$$\frac{dx}{dt} = f(t, x) \qquad (1)$$

$$x(t) = (1-\frac{t}{T})d_{x_j}(\varphi_j(x_T)) + \frac{t}{T}d_{x_j}(\varphi_j(x_0) + $$
$$+(1-\frac{t}{T})\int_0^t f(s, x(s))ds - \frac{t}{T}\int_t^T f(s, x(s))ds \qquad (2)$$

$$i = 1, 2, \ldots, q \quad , \quad j = q+1, 2, .., n$$

where x - n is a $f(t,x)$-dimensional vector function. Let the function be defined and continuous in the domain

$$(t,x)\in[0,T]\times D, \tag{3}$$

where D is a bounded closed set of space. Let the function $t\in[0,T]$, $x,x',x''\in D$ for $M=(M_1,M_2,....M_n)M_i>0$ bounded by vector and $K=\{K_{ij}\geq 0, i,j=1,2...n\}$

satisfies the Lipschitz condition for the matrix:

$$|f(t,x)|\leq M, \left|f(t,x'')-f(t,x')\right|\leq K\left|x''-x'\right| \quad . \tag{4}$$

Let the quantities M, K, and T for region (3) satisfy the following conditions:

1) Let D be a set lying in the neighborhood of D and not a non-empty set, i.e.

$$D_j\neq\varnothing \tag{5}$$

2) the eigenvalues of matrix $Q=\frac{T}{\pi}K$ must be within unity, i.e.

$$|\lambda_j(Q)|<1, j=1,2,...n \tag{6}$$

When these conditions are met, a sequence of functions satisfying the boundary condition (2) can be constructed that uniformly approaches the exact solution of the boundary value problems (1) and (2), and the error between these functions and their limit can be estimated.

We construct the system of differential equations (1) and the sequence of continuous functions satisfying the boundary condition (2) $\{x_m\}, m=1,2...$ as follows, initially introducing the notation,

$$d_{x_1}(\varphi_i(x_T))=\begin{pmatrix}\varphi_1(x_T)\\ \cdots\cdots\cdots\\ \varphi_g(x_T)\\ x_{g+1}(0)\\ \cdots\cdots\cdots\\ x_n(0)\end{pmatrix}, d_{x_1}(\varphi_j(x_0))=\begin{pmatrix}x_1(T)\\ \cdots\cdots\\ x_g(T)\\ \varphi_{g+1}(x_0)\\ \cdots\cdots\\ \varphi_n(x_0)\end{pmatrix}$$

then (1) is equivalent to the system of equations.

$$x(t) = x(0) + \int_0^t f(s, x(s))ds$$

$$x(t) = x(T) - \int_t^T f(s, x(s))ds$$

Accordingly, the integral equation can be replaced by the equations

$$x(t) = d_{x_1}(\varphi_i(x_T)) + \int_0^t f(s, x(s))ds \tag{7}$$

and

$$x(t) = d_{x_1}(\varphi_j(x_0)) - \int_t^T f(s, x(s))ds \tag{8}$$

The solution of the system of integral equations (7) is the system of differential equations (1), which satisfies the first of the boundary conditions (2), and the solution of the system (8) is the solution (1).

Which satisfies the second of the boundary conditions (1). Therefore, we obtain the solution of the boundary value problem (1), (2) by combining the integral equations (7) and (8).

In the integral equation (7), we multiply both sides of the equality sign by $1-\frac{t}{T}$, and in (8), we multiply both sides of the equality sign by $\frac{t}{T}$ and add the results.

Then,

$$x(t) = (1-\frac{t}{T})d_{x_j}(\varphi_j(x_T)) + \frac{t}{T}d_{x_j}(\varphi_j(x_0) +$$
$$+(1-\frac{t}{T})\int_0^t f(s, x(s))ds - \frac{t}{T}\int_t^T f(s, x(s))ds$$

Simplifying, we get

$$x(t) = (1-\frac{t}{T})d_{x_j}(\varphi_j(x_T)) + \frac{t}{T}d_{x_j}(\varphi_j(x_0) + \\ +\int_0^t f(s,x(s))ds - \frac{t}{T}\int_t^T f(s,x(s))ds \tag{9}$$

The solution of this system of integral equations is not always a solution of the system (1), but fully satisfies the boundary condition (2), and the system (1) itself satisfies the selected parameters x_0 and x_T .

Approximate solution of the system of integral equations (9)

$$x_{m+1}(t,x_0(T)) = (1-\frac{t}{T})d_{x_i}(\varphi_i(x_T)) + \frac{t}{T}d_{x_j}(\varphi_j(x_T)) + \\ +\int_0^t \left[f(s,x_m(s,x_0(T)) - \frac{1}{T}\int_0^T f(s,x_m(s,x_0(T))ds \right] ds, \\ m = 1,2... \tag{10}$$

Here $x_0(T) = \{x_1(0),...x_g(0), x_{g+1}(T), x_n(T)\}$ set of view parameters.

Function sequence defined by $r_{m+1}(t) = |x_{m+1}(t,x_0(T)) - x_m(t,x_0(T))|$

satisfies the boundary condition (2) for any m .

Now let us show that the functions $x_0(T) \in D_j$ defined by formula (10) $x_m(t,x_0(T))$ lie on set D. In fact, $x_0(t,x_0(T)) = (1-\frac{t}{T})d_{x_i}(\varphi_i(x_T)) + \frac{t}{T}d_{x_j}(\varphi_j(x_0))$

then from (10)

$$|x_1(t,x_0(T)) - x_0(t,x_0(T))| \le \left| \int_0^t \left[f(s,x_0(t,x_0(T)) - \frac{1}{T}\int_0^T f(s,x_0(t,x_T)ds \right] ds \right| \le \alpha_1(t)M \tag{11}$$

i.e. $x_0(t,x_0(T)) \in D_j$ for $t \in [0,T]$, from which it follows that $x_1(t,x_0(T)) \in D_j$ from $x_m(t,x_0(T))$ it can be seen that . It can be shown that all functions defined by the sequence for (10) all $m \ge 0$ of any $t \in [0,T], x_0(T) \in D_j$ lie on the set D. This allows for finding any approximation $x_m(t,x_0(T))$ using formula (10).

Now let us show that the sequence of functions $x_m(t,x_0(T))$ is a convergent sequence. To do this $j>1$, we will use the Cauchy convergence rule

$\left|x_{m+j}(t,x_0(T))-x_m(t,x_0(T))\right|$.

According to (10) to (4) $m\geq 1, t\in[0,T]$ for

$$\left|x_{m+1}(t,x_0(T))-x_m(t,x_0(T))\right|\leq$$
$$\leq K\left[(1-\frac{t}{T})\int_0^t\left|x_m(t,x_0(T))-x_{m-1}(t,x_0(T))\right|ds+\frac{t}{T}\int_0^T\left|x_m(t,x_0(T))-x_{m-1}(t,x_0(T))\right|ds\right]$$

We obtain the inequality $r_{m+1}(t)=\left|x_{m+1}(t,x_0(T))-x_m(t,x_0(T))\right|$

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MSC 34A37

DEFERRED ARGUMENT AND SECOND-ORDER LINEAR DIFFERENTIAL WITH IMPULSE ACTION BASIC INITIAL PROBLEMS FOR EQUATIONS

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Физика, механика жана илимдин башка тармактардагы көпчүлүк маселелер, аргументи кечигүүчү жана импульстуу дифференциалдык теңдемелер үчүн фундаменталдык маселелердин теориясын изилдөөгө алып келет.

Бул макалада аргументи кечигүүчү жана импульстуу экинчи тартиптеги сызыктуу дифференциалдык теңдемелер үчүн негизги баштапкы маселе каралат. Каралып жаткан маселенин чечилиши Дирак функциясы, Коши жана Коши тибиндеги функциялардын жардамы менен этап-этабы менен аныкталат.

Урунттуу сөздөр: дифференциалдык теңдеме, импульстуу таасир, кечигүүчү аргумент, баштапкы маселе.

Многие задачи в физике, механике и других областях науки приводят к изучению теории фундаментальных задач для дифференциальных уравнений с запаздывающим аргументом и импульсом.

В статье рассматрвается основная начальная задача для линейных дифференциальных уравнений второго порядка с запаздывающим аргументом и импульсом. Решение рассматриваемой *задачи определяется пошагово с помощью функции Дирака, функции Коши и функций типа Коши.*

Ключевые слова: дифференциальное уравнение, импульсный эффект, запаздывающий аргумент, начальная задача.

Many problems in physics, mechanics, and other fields of science lead to the study of the theory of fundamental problems for differential equations with a delayed argument and momentum.

The article examines the main initial problem for second-order linear differential equations with a delayed argument and momentum. The solution to the problem under consideration is determined stepwise using the Dirac function, Cauchy, and Cauchy-type functions.

Keywords: differential equation, impulse effect, delayed argument, initial problem.

Let $E_0[-\tau,0]$ have an initial function $x=\varphi(t)$ and a delayed argument $t-\tau$ in the initial set.

Let

$$x''(t)+p_1(t)x'(t)+p_2(t)x(t)+L(x(t-\tau))=0,\ t\neq t_{ij}, \tag{1}$$

$$\Delta x(t)\big|_{t=t_{ij}}=h_{ij},\quad \Delta x'(t)\big|_{t=t_{ij}}=h'_{ij}, \tag{2}$$

$$x(t)=\varphi(t),\ t\in[-\tau,0], \tag{3}$$

be a linear function that experiences momentum at points $i=1,2,...,\ j=1,2,...,m_i$. let the differential equation be given, where $h(t_{ij})=h_{ij},\ h'(t_{ij})=h'_{ij},$ and $h(t)$ is a function defined and continuous for $t\geq t_{11}$, along with its first-order derivatives. the function $\varphi(t)$ is a continuous antiderivative defined on the antiderivative set $\tau>0$, where $\tau>0$, and $L(x(t))=\sum_{k=1}^{2}q_k(t)x^{(2-k)}(t).$ is a constant.

Suppose $x_1(t),\ x_2(t)$ functions are homogeneous

$$x''(t)+p_1(t)x'(t)+p_2(t)x(t)=0$$

is a fundamental solution system, and its $K(t,s)$ function is defined using this fundamental solution system.

$$K(t,s)=\frac{1}{w(s)}\begin{vmatrix} x_1(s) & x_2(s) \\ x_1(t) & x_2(t) \end{vmatrix}$$

The Cauchy function, and the $K_1(t,s)$ function according to the t argument of this homogeneous equation and

$$K_1(t,t)=1,\ \frac{dK_1(t,s)}{dt}\bigg|_{s=t}=0$$

satisfactory

$$K_1(t,s)=\frac{1}{w(s)}\begin{vmatrix} x_1(t) & x_2(t) \\ x'_1(s) & x'_2(s) \end{vmatrix}$$

a function of the Cauchy type in the form, where $w(t)$ the function is the Vronsky determinant of the $x_1(t),\ x_2(t)$ fundamental solution system.

The use of these functions makes it more convenient to determine the solution of the main initial problem (1) - (3) with a delayed argument using the step method.

Indeed, in the first step, in the interval $0 \le t < \tau$, the main initial problem (1) - (3) is solved using the above notation.

$$x''(t) + p_1(t)x'(t) + p_2(t)x(t) + L(\varphi(t-\tau)) = f(t),\ t \ne t_{ij},$$

$$\Delta x(t)\big|_{t=t_{ij}} = h_{ij}, \quad \Delta x'(t)\big|_{t=t_{ij}} = h'_{ij},$$

$$x(0) = \varphi(0),\ x'(0) = \varphi'(0)$$

or using the Dirac function $\delta(t)$

$$x''(t) + p_1(t)x'(t) + p_2(t)x(t) + L(\varphi(t-\tau)) = f(t) + \tag{4}$$

$$+\sum_{j=1}^{m_1}\left(K_1(t,t_{1j})h(s)\delta(t-t_{1j}) + K(t,t_{1j})h(s)\delta'(t-t_{1j})\right),$$

$$x(0) = \varphi(0),\ x'(0) = \varphi'(0) \tag{5}$$

is replaced by the Cauchy problem, and this is the solution to the Cauchy problem

$$x(t) = K_1(t,0)\varphi(0) + K(t,0)\varphi'(0) + \int_0^t K(t,s)\left(f(s) - L(\varphi(s-\tau))\right)ds + \tag{6}$$

$$+\sum_{j=1}^{m_1}\int_0^t\left(K_1(t,t_{1j})h(s)\delta(t-t_{1j}) + K(t,t_{1j})h(s)\delta'(t-t_{1j})\right)ds$$

in the form or

$$x(t) = K_1(t,0)\varphi(0) + K(t,0)\varphi'(0) + \int_0^t K(t,s)\left(f(s) - L(\varphi(s-\tau))\right)ds + \tag{7}$$

$$+\sum_{j=1}^{m_1}\left(K_1(t,t_{1j})h_{1j} + K(t,t_{1j})h'_{1j}\right)\theta(t-t_{1j})$$

can be written.

Indeed, as can be seen from (6) or (7), for $t = 0$, this solution satisfies the initial conditions:

$$x(0) = K_1(0,0)\varphi(0) + K(0,0)\varphi'(0) = 1\cdot\varphi(0) + 0\cdot\varphi'(0) = \varphi(0),$$

$$x'(0)=\left.\frac{dK_1(t,0)}{dt}\right|_{t=0}\varphi(0)+\left.\frac{dK(t,0)}{dt}\right|_{t=0}\varphi'(0)=0\cdot\varphi(0)+1\cdot\varphi'(0)=\varphi'(0).$$

Now let's show that the solution in the form (7) satisfies the given equation. For this, we take the first and second derivatives of equation (7):

$$x'(t)=\frac{dK_1(t,0)}{dt}\varphi(0)+\frac{dK(t,0)}{dt}\varphi'(0)+\int_0^t\frac{dK(t,s)}{dt}\big(f(s)-L(\varphi(s-\tau))\big)ds+$$

$$+\sum_{j=1}^{m_1}\left(K_1'(t,t_{1j})h_{1j}+K'(t,t_{1j})h_{1j}'\right)\theta(t-t_{1j}),$$

$$x''(t)=\frac{d^2K_1(t,0)}{dt^2}\varphi(0)+\frac{d^2K(t,0)}{dt^2}\varphi'(0)+f(t)-L(\varphi(t-\tau))+$$

$$+\int_0^t\frac{\partial^2K(t,s)}{\partial t^2}\big(f(s)-L(\varphi(s-\tau))\big)ds+$$

$$+\sum_{j=1}^{m_1}\left(K_1''(t,t_{1j})h_{1j})+K''(t,t_{1j})h_{1j}')\right)\theta(t-t_{1j})$$

(7) and the first and second derivatives obtained above in (4) by substituting and, we show that the function $x(t)$ satisfies equation (1) for $t\neq t_{1j}$:

$$\frac{d^2K_1(t,0)}{dt^2}\varphi(0)+\frac{d^2K(t,0)}{dt^2}\varphi'(0)+f(t)-L(\varphi(t-\tau))+$$

$$+\int_0^t\frac{\partial^2K(t,s)}{\partial t^2}\big(f(s)-L(\varphi(s-\tau))\big)ds+$$

$$+\sum_{j=1}^{m_1}\left(K_1''(t,t_{1j})h_{1j})+K''(t,t_{1j})h_{1j}')\right)\theta(t-t_{1j})+$$

$$+p_1(t)\left(\frac{dK_1(t,0)}{dt}\varphi(0)+\frac{dK(t,0)}{dt}\varphi'(0)+\int_0^t\frac{dK(t,s)}{dt}\big(f(s)-L(\varphi(s-\tau))\big)ds+\right.$$

$$+\sum_{j=1}^{m_1}\left(K_1'(t,t_{1j})h_{1j})+K'(t,t_{1j})h_{1j}')\right)\theta(t-t_{1j})+$$

$$+p_2(t)\left(K_1(t,0)\varphi(0)+K(t,0)\varphi'(0)+\int_0^t K(t,s)\big(f(s)-L(\varphi(s-\tau))\big)ds+\right.$$

$$+\sum_{j=1}^{m_1}\left(K_1(t,t_{1j})h_{1j})+K(t,t_{1j})h'_{1j})\right)\theta(t-t_{1j})+L(\varphi(t-\tau))=$$

$$=\left(\frac{d^2K_1(t,0)}{dt^2}+p_1(t)\frac{dK_1(t,0)}{dt}+p_2(t)K_1(t,0)\right)\varphi(0)+$$

$$=\left(\frac{d^2K(t,0)}{dt^2}+p_1(t)\frac{dK(t,0)}{dt}+p_2(t)K(t,0)\right)\varphi'(0)+f(t)-L(\varphi(t-\tau))+$$

$$+\int_0^t\left(\frac{\partial^2K(t,s)}{\partial t^2}+p_1(t)\frac{\partial K(t,s)}{\partial t}+p_2(t)K(t,s)\right)\left(f(s)-L(\varphi(s-\tau))\right)ds+$$

$$+\sum_{j=1}^{m_1}\left(K_1''(t,t_{1j})+p_1(t)K_1'(t,t_{1j})+p_2(t)K_1(t,t_{1j})\right)h_{1j})\theta(t-t_{1j})+$$

$$+\sum_{j=1}^{m_1}\left(K''(t,t_{1j})+p_1(t)K'(t,t_{1j})+p_2(t)K(t,t_{1j})\right)h'_{1j})\theta(t-t_{1j})+L(\varphi(t-\tau))=$$

$$=f(t)-L(\varphi(t-\tau))+L(\varphi(t-\tau))=f(t)$$

Since the functions $K_1(t,s)$ and $K(t,s)$ satisfy a homogeneous equation to equation (1) with respect to the argument t, it follows that the latter equation holds true, the function $x(t)$, defined by formula (6) or (7), is a solution to the Cauchy problem (4), (5).

Now, let's determine the second-step solution of the main initial problem (1) - (3) with a delayed argument of momentum action, in the interval $\tau\le t<2\tau$. In this case, (1) - (3) is the main initial problem.

$$x''(t)+p_1(t)x'(t)+p_2(t)x(t)+L(x_1(t-\tau))=f(t)+$$

$$+\sum_{j=1}^{m_2}\left(K_1(t,t_{2j})h(s)\delta(t-t_{2j})+K(t,t_{2j})h(s)\delta'(t-t_{2j})\right)ds,$$

$$x(\tau)=x_1(\tau),\ x'(\tau)=x_1'(\tau)$$

and using the step method, the solution to this Cauchy problem is similar to the one in the first step.

$$x(t)=K_1(t,\tau)x_1(\tau)+K(t,\tau)x_1'(\tau)+\int_{\tau}^{t}K(t,s)\left(f(s)-L(x_1(s-\tau))\right)ds$$

$$+\sum_{j=1}^{m_2}\left(K_1(t,t_{2j})h(s)\delta(t-t_{2j})+K(t,t_{2j})h(s)\delta'(t-t_{2j})\right)ds$$

is determined by the formula here, $x_1(t)$ is the solution of the primary problem (1) - (3), defined by formula (7).

Continuing the process in this way, we determine the solution of the main initial problem (1) - (3) at step n, that is, in the interval $(n-1)\tau\le t<n\tau$. In this case, (1) - (3) is the main initial problem.

$$x''(t)+p_1(t)x'(t)+p_2(t)x(t)+L(x_{n-1}(t-\tau))=f(t)+$$

$$+\sum_{j=1}^{m_n}\left(K_1(t,t_{(n-1)j})h(s)\delta(t-t_{(n-1)j})+K(t,t_{(n-1)j})h(s)\delta'(t-t_{(n-1)j})\right)ds,$$

$$x((n-1)\tau)=x_{n-1}((n-1)\tau),\ x'((n-1)\tau)=x_{n-1}'((n-1)\tau)$$

and using the step method, the solution to this Cauchy problem is similar to the solution in the previous steps.

$$x(t)=K_1(t,\tau)x_{n-1}(\tau)+K(t,\tau)x_{n-1}'(\tau)+\int_{(n-1)\tau}^{t}K(t,s)\left(f(s)-L(x_{n-1}(s-\tau))\right)ds$$

$$+\sum_{j=1}^{m_n}\left(K_1(t,t_{(n-1)j})h_{(n-1)j})+K(t,t_{(n-1)j})h_{(n-1)j}')\right)\theta(t-t_{(n-1)j})$$

is determined by the formula here, $x_{n-1}(t)$ is the solution (1)-(3) to the primary problem at the previous step.

For example, suppose it has a pulsed effect and a delayed argument.

$$x''(t)+x'(t-2)+2x(t-2)=12t,\ t\ne 1,\ t\ne 3, \tag{8}$$

$$\Delta x(t)\big|_{t=t_{ij}}=h(t)\big|_{t=1,\ t=3},\quad \Delta x'(t)\big|_{t=t_{ij}}=h'(t)\big|_{t=1,\ t=3}, \tag{9}$$

$$x(t)=t+2,\ t\in[-2,0] \tag{10}$$

solution of the main initial problem at the first and second steps

$$h(t)=\frac{1}{2}t^3-3t^2+\frac{2}{2}t-1$$

Let's consider the problem of finding by the function.

$$x''(t)=0$$

system of fundamental solutions of a homogeneous equation $x_1(t)=t,\ x_2(t)=1$

therefore, $w(t)=-1,\ K(t,s)=t-s$ and $K_1(t,s)=\begin{vmatrix} t & 1 \\ 1 & 0 \end{vmatrix}=1.$

Now let's determine the solution using the step method. In the first step, for $t\in[0,2]$, the main initial problem (8) - (10) is solved.

$$x''(t)+L(\varphi(t-2))=12t+K_1(t,1)h(t)\delta(t-1)+K(t,1)h(t)\delta'(t-1),$$

$$x(0)=2,\ x'(0)=1$$

or

$$x''(t)+L(\varphi(t-2))=12t+h(t)\delta(t-1)+(t-1)h(t)\delta'(t-1),$$

$$x(0)=2,\ x'(0)=1$$

We will replace it with the Koshi problem, here $L(\varphi(t-2))=2t+1.$

Using Cauchy and Cauchy-type functions, the solution to this Cauchy problem is

$$x(t)=K_1(t,0)\cdot 2+K(t,0)\cdot 1+12\int_0^t(t-s)sds-$$

$$-\int_0^t(t-s)(2s+1)ds+\left(h(1)+(t-1)h'(1)\right)\theta(t-1)=$$

$$=1\cdot 2+t\cdot 1+\frac{5}{3}t^3-\frac{1}{2}t^2+h(1)\theta(t-1)+h'(1)\theta(t-1)=$$

$$=2+t+\frac{5}{3}t^3-\frac{1}{2}t^2+\theta(t-1)$$

using formula (7).

In the second step, for $t\in[2,4]$, the main initial problem (8) - (10) is solved.

$$x''(t)+L(x_1(t-2))=12t+K_1(t,3)h(t)\delta(t-3)+K(t,3)h(t)\delta'(t-3),$$

$$x(2)=\frac{49}{3},\ x'(2)=19$$

We replace it with the Cauchy problem, where $x_1(t)$ is the solution of the given problem at the first step,

$$x_1(t)=2+t+\frac{5}{3}t^3-\frac{1}{2}t^2+\theta(t-1),$$

and

$$L(x_1(t-2))=\frac{10}{3}(t-2)^3+4(t-2)^2+(t-2)+5+2\theta(t-3).$$

This is the solution to the Koshi problem

$$x(t)=K_1(t,2)\cdot\frac{49}{3}+K(t,2)\cdot 19+12\int_2^t K(t,s)sds-$$

$$-\int_2^t K(t,s)L(s-2)ds+\int_0^t\left(K_1(t,3)h(s)\delta(s-3)+K(t,3)h(s)\delta'(s-3)\right)ds=$$

$$=\frac{49}{3}+19(t-2)+12\int_2^t(t-s)sds-$$

$$-(t-2)\int_2^t\left(\frac{10}{3}(s-2)^3+4(s-2)^2+(s-2)+5+2\theta(s-3)\right)ds+$$

$$+\int_0^t(s-2)\left(\frac{10}{3}(s-2)^3+4(s-2)^2+(s-2)+5+2\theta(s-3)\right)ds+$$

$$+h(3)\theta(t-3)=2t^3-5t+\frac{31}{3}-$$

$$-(t-2)\left(\frac{5}{6}(t-2)^4+\frac{4}{3}(t-2)^3+\frac{1}{2}(t-2)^2+5(t-2)+2(t-3)\theta(t-3)\right)+$$

$$+\int_2^t\left(\frac{10}{3}(s-2)^4+4(s-2)^3+(s-2)^2+5(s-2)+2(s-2)\theta(s-3)\right)ds-$$

$$-\theta(t-3)=2t^3-5t+\frac{28}{3}-\frac{5}{6}(t-2)^5-\frac{4}{3}(t-2)^4-\frac{1}{2}(t-2)^2-5(t-2)^2-$$

$$-2(t-2)(t-3)\theta(t-3)+\frac{2}{3}(t-2)^5+(t-2)^4+\frac{1}{3}(t-2)^3+\frac{5}{2}(t-2)^2+$$

$$+\left((t-2)^2-1\right)\theta(t-3)-\theta(t-3)=2t^3-5t+\frac{31}{3}-$$

$$-\frac{1}{6}(t-2)^5-\frac{1}{3}(t-2)^4-\frac{1}{6}(t-2)^3-\frac{5}{2}(t-2)^2-(t^2-6t+10)\theta(t-3)$$

that is, the solution in the second step will be

$$x(t)=2t^3-5t-\frac{1}{6}(t-2)^5-\frac{1}{3}(t-2)^4-\frac{1}{6}(t-2)^3-\frac{5}{2}(t-2)^2-$$

$$+\frac{31}{3}-(t^2-6t+10)\theta(t-3)$$

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FRACTAL RECURSIVE M-MATRICES: RECURSIVE INVARIANTS AND SPECTRAL STABILITY

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This paper introduces **Fractal Recursive *M*-Matrices (FRM-matrices)** as a new class of recursive invariant matrix systems combining recursive organization, self-similarity, arithmetic consistency, and spectral stability within a unified mathematical framework. A formal definition of FRM-matrices and a canonical recursive construction preserving local and global invariants are proposed. General conditions for invariant preservation under recursive expansion are established, and spectral stability under admissible symmetry transformations is investigated. The recursive matrix hierarchy is interpreted as a mathematical model of distributed information structures exhibiting scalable organization and structural coherence. An illustrative recursive construction of a (9×9) invariant matrix system demonstrates the applicability of the proposed approach. The obtained results establish connections between recursive matrix theory, self-similar algebraic structures, spectral analysis, and mathematical models of distributed information architectures.

Keywords: Fractal Recursive *M*-Matrices; recursive matrices; recursive invariants; self-similar structures; spectral stability; distributed information systems; matrix hierarchies; recursive decomposition.

В работе вводится новый класс рекурсивных инвариантных матричных систем – **фрактальные рекурсивные *M*-матрицы (FRM-матрицы)**, объединяющие рекурсивную организацию, самоподобие, арифметическую согласованность и спектральную устойчивость в рамках единого математического подхода. Предлагаются формальное определение FRM-матриц и каноническая рекурсивная конструкция, сохраняющая локальные и глобальные инварианты. Получены общие условия сохранения инвариантов при рекурсивном расширении и исследована спектральная устойчивость относительно допустимых симметрических преобразований. Разработана интерпретация рекурсивной матричной иерархии как математической модели распределённых информационных

структур, обладающих масштабируемостью и структурной согласованностью. В качестве иллюстрации представлена рекурсивная конструкция инвариантной матрицы порядка (9×9). Полученные результаты устанавливают взаимосвязь между теорией рекурсивных матриц, самоподобными алгебраическими структурами, спектральным анализом и математическими моделями распределённых информационных архитектур.

Ключевые слова: фрактальные рекурсивные M-матрицы; рекурсивные матрицы; рекурсивные инварианты; самоподобные структуры; спектральная устойчивость; распределённые информационные системы; матричные иерархии; рекурсивная декомпозиция.

Бул макалада рекурсивдүү уюштурууну, өзүнө окшоштукту, арифметикалык туруктуулукту жана спектралдык туруктуулукту бириктирген жаңы класстагы **фракталдык рекурсивдүү *M*-матрицалар (FRM-матрицалар)** сунушталат. FRM-матрицалардын формалдуу аныктамасы жана локалдык жана глобалдык инварианттарды сактоочу канондук рекурсивдүү конструкция берилет. Рекурсивдүү кеңейүү учурунда инварианттарды сактоонун жалпы шарттары алынып, уруксат берилген симметриялык өзгөртүүлөргө карата спектралдык туруктуулук изилденет. Рекурсивдүү матрицалык иерархия масштабдуу жана структуралык жактан туруктуу бөлүштүрүлгөн маалыматтык системалардын математикалык модели катары каралат. Мисал катары (9×9) өлчөмүндөгү инварианттуу матрицанын рекурсивдүү конструкциясы келтирилет. Алынган жыйынтыктар рекурсивдүү матрицалар теориясынын, өзүнө окшош алгебралык структуралардын, спектралдык анализдин жана бөлүштүрүлгөн маалымат архитектураларынын математикалык моделдеринин ортосундагы байланышты көрсөтөт.

Урунттуу сөздөр: фракталдык рекурсивдүү *M*-матрицалар; рекурсивдүү матрицалар; рекурсивдүү инварианттар; өзүнө окшош структуралар; спектралдык туруктуулук; бөлүштүрүлгөн маалыматтык системалар; матрицалык иерархиялар; рекурсивдүү ажыратуу.

1. Introduction

The rapid development of distributed information technologies, cloud computing infrastructures, large-scale storage architectures, and hierarchical data-processing systems has significantly increased the demand for mathematical models capable of describing recursive organization, structural consistency, and scalable information architectures.

Modern distributed systems are characterized by multilevel structures in which local subsystems interact to form globally coordinated information environments. One of the fundamental challenges in the mathematical modeling of such systems is the preservation of global structural properties under recursive expansion and local transformations. Consequently, the study of invariant mathematical structures capable of maintaining consistency under scaling operations has become an important direction of contemporary mathematical research.

Matrix methods occupy a special position in the investigation of complex hierarchical systems because they provide a natural representation of interactions between local and global structural components. In particular, recursive matrix constructions offer a convenient framework for describing scalable systems whose organization remains stable under enlargement.

Classical magic squares constitute one of the oldest and most extensively studied classes of structured matrices [1, 2]. Traditionally, investigations of magic squares have been associated with combinatorics, number theory, and recreational mathematics. However, modern studies indicate that many classes of magic matrices possess considerably richer algebraic properties than previously assumed [3]. Their invariant characteristics, recursive constructions, symmetry groups, and spectral properties provide a natural basis for the investigation of self-organizing mathematical systems.

The concept of recursive matrix structures arises naturally when higher-order matrices are generated from smaller invariant blocks according to prescribed recursive rules. Such constructions exhibit hierarchical organization, self-similarity, and structural scalability. In many cases, local invariant properties remain preserved during recursive expansion, leading to the formation of globally consistent matrix systems.

An important feature of recursive structures is the coexistence of local and global invariants. Local invariants characterize the internal properties of individual blocks, whereas global invariants describe the structural consistency of the entire system. The preservation of both classes of invariants is essential for ensuring recursive stability and coherence.

The notion of self-similarity underlying recursive constructions is closely related to ideas originating from fractal geometry [4]. Although fractal theory is traditionally associated with geometric objects, the concept of recursive self-similarity may also be interpreted in an algebraic context. From this perspective, recursively generated matrix systems preserving invariant properties may be viewed as algebraic analogues of fractal structures.

In the present paper, a new class of recursive matrix systems called Fractal Recursive M-Matrices (FRM-matrices) is introduced. The proposed matrices are generated through recursive block constructions preserving arithmetic invariants and spectral characteristics. Unlike conventional recursive matrices, FRM-matrices are characterized by the simultaneous preservation of recursive consistency, self-similarity, arithmetic coherence, and spectral stability.

Terminological Remark

Throughout the present paper, the term M-matrix is used in the sense established in the authors' earlier works on magic invariant matrices and should not be confused with the classical theory of M-matrices developed by Berman and Plemmons. In the classical literature, M-matrices constitute a special class of matrices characterized by positivity and spectral properties. In contrast, the matrices considered in the present work arise from recursive constructions of invariant magic matrix systems and are studied from the viewpoint of recursive organization, self-similarity, arithmetic invariants, and spectral stability.

Unlike classical recursive matrix constructions, the proposed FRM-framework simultaneously combines recursive scalability, invariant preservation, self-similarity, and spectral stability within a unified mathematical structure.

Scientific Novelty and Contributions

The scientific novelty of the present work consists in the development of a new class of recursive invariant matrix structures and the investigation of their arithmetic and spectral properties.

1. A formal definition of Fractal Recursive *M*-Matrices is introduced.
2. A canonical recursive construction preserving local and global invariants is proposed.
3. Sufficient conditions for arithmetic invariant preservation under recursive expansion are established.
4. Spectral stability under admissible symmetry transformations is investigated.
5. A mathematical interpretation of FRM-matrices as models of distributed information structures is developed.
6. A recursive construction of a higher-order invariant matrix system is presented and analyzed.

The obtained results establish connections between recursive matrix theory, invariant structures, self-similar systems, spectral analysis, and hierarchical information architectures.

The remainder of the paper is organized as follows. Section 2 introduces the formal definition of FRM-matrices and the canonical recursive construction. Section 3 investigates arithmetic invariants and recursive decomposition. Section 4 studies symmetry transformations and spectral stability. Section 5 develops an interpretation of FRM-matrices as models of distributed information structures. Section 6 presents a recursive construction of a 9×9 invariant matrix system. Section 7 summarizes the main results and outlines directions for future research.

2. Fractal Recursive Structures and Definition of FRM-Matrices

2.1 Recursive Matrix Structures

Recursive matrix systems arise naturally when large matrix configurations are generated from smaller structural units according to prescribed expansion rules.

Such constructions play an important role in the study of hierarchical systems, multilevel architectures, self-similar structures, and scalable information models.

Let $F_n = (f_{ij}),\ i, j = 1, \ldots, n$ be a square matrix of order n. Assume that a recursive operator

$$R:\ F_n \to F_{kn}$$

generates a higher-order matrix through recursive block expansion. The resulting matrix

$$F_{kn} = R(F_n)$$

is called a recursive expansion of F_n. The principal objective of recursive constructions is the preservation of invariant structural characteristics during the expansion process.

2.2 Local and Global Invariants

For a matrix F_n, define the row sums and the column sums by

$$R_i = \sum_{j=1}^{n} f_{ij},\quad C_j = \sum_{i=1}^{n} f_{ij}.$$

If

$$R_i = C_j = S_n$$

for all admissible indices, then S_n is called the global invariant of the matrix. For recursive matrix systems, local invariants correspond to invariant properties of individual blocks, whereas global invariants characterize the entire recursive structure. The preservation of both classes of invariants constitutes a fundamental requirement for recursive consistency.

2.3 Self-Similarity and Recursive Consistency

A recursive matrix structure is called self-similar if each block generated during recursive expansion preserves the essential structural properties of the parent matrix. Let F_{kn} be represented as a block matrix

$$F_{kn} = (B_{i\,j})_{i,j=1}^{k},$$

where every block B_{ij} has order n. The matrix system is called recursively consistent if all blocks preserve the same invariant structure and jointly determine a unique global invariant.

Definition 1 (Fractal Recursive *M*-Matrix). A square matrix F_n is called a Fractal Recursive *M*-Matrix (FRM-matrix) if the following conditions are satisfied:

1. Recursive block structure: there exists a recursive operator $R: F_n \to F_{kn}$ generating higher-order matrices through recursive expansion.
2. Invariant preservation: the matrix possesses a global invariant S_n such that $R_i = C_j = S_n$ for all admissible indices.
3. Self-similarity: each recursively generated block preserves the essential structural properties of the parent matrix.
4. Recursive consistency: local invariants jointly determine a unique global invariant.
5. Spectral stability: the principal spectral characteristics remain invariant under admissible symmetry transformations.

A sequence

$$F_n, F_{kn}, F_{k^2n}, \ldots$$

generated by repeated application of the recursive operator R is called an FRM-system.

Definition 1A (Admissible Symmetry Transformation). An admissible symmetry transformation of an FRM-matrix is any permutation-induced transformation preserving recursive block organization and global invariant structure.

2.4. Example 1. Construction of a 9×9 Fractal Recursive Magic Matrix

Base matrix $F_3 = \begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix}$.

Let the coefficient matrix be defined by $A = \begin{pmatrix} 63 & 0 & 45 \\ 18 & 36 & 54 \\ 27 & 72 & 9 \end{pmatrix}$.

Block definition $B_{ij} = F_3 + \alpha_{ij}$.

Recursive matrix

$$F_9 = \begin{pmatrix} F_3 + 63 & F_3 + 0 & F_3 + 45 \\ F_3 + 18 & F_3 + 36 & F_3 + 54 \\ F_3 + 27 & F_3 + 72 & F_3 + 9 \end{pmatrix}.$$

Expanded matrix:

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

Magic constant

$$S_9 = 3S_3 + 3 \cdot 108 = 369.$$

Thus, every row and every column of the resulting matrix M_9 has the common sum 369.

Example 1 demonstrates that the proposed recursive construction generates higher-order matrix structures preserving local and global invariants. This confirms the constructive nature and scalability of the proposed FRM-matrix model.

The obtained structure preserves recursive consistency and invariant stability. Therefore, FRM-matrices may be considered as mathematical models of recursive distributed information structures.

2.5. Discussion

The introduced concept combines recursive scalability, invariant preservation, self-similar organization, and spectral robustness within a unified mathematical framework. Unlike classical magic matrices, which are usually considered as isolated objects, FRM-matrices form recursive hierarchies of arbitrarily large order. Such structures naturally arise in recursive information systems, hierarchical architectures, distributed storage environments, and multilevel organizational networks. The subsequent sections investigate arithmetic invariants and spectral properties of FRM-matrices and establish the fundamental invariance principles governing recursive matrix systems.

3. Arithmetic Invariants and Recursive Decomposition

3.1. Recursive Invariants of FRM-Matrices

One of the fundamental properties of recursive matrix systems is the preservation of invariant characteristics under recursive expansion. Let F_n be an FRM-matrix possessing invariant S_n. Let $A=(a_{ij})$ be a coefficient matrix of order k with invariant S_A. Consider the canonical FRM-construction introduced in Section 2.

For every coefficient a_{ij}, define

$$B_{ij} = F_n + a_{ij} E_n$$

The recursively generated matrix is

$$F_{kn} = (B_{i\,j})_{i,j=1}^{k}.$$

The principal problem is to determine whether recursive expansion preserves invariant structure.

3.2. Local and Global Consistency

Each block B_{ij} inherits the invariant structure of the base matrix F_n. Consequently, local consistency is preserved at the block level. However, recursive systems also require preservation of a global invariant describing the entire matrix hierarchy. The following theorem establishes sufficient conditions for invariant preservation.

Theorem 1 (General Invariant Preservation Theorem). Let F_n be an FRM-matrix possessing invariant S_n, and let $A=(a_{ij})$ be a coefficient matrix possessing invariant S_A. Then the recursively generated matrix S_{kn} constructed by the Canonical FRM-Construction possesses invariant

$$S_{kn}=kS_n+S_A.$$

Proof. Consider a block

$$B_{ij}=F_n+a_{ij}E_n.$$

The row sum of the block equals

$$S_n+a_{ij}.$$

Summing over all blocks belonging to the same recursive row yields

$$kS_n+n\sum_{j=1}^{k}a_{ij}.$$

Since

$$\sum_{j=1}^{k}a_{ij}=S_A,$$

we obtain

$$S_{kn}=kS_n+nS_A.$$

The same argument applies to column sums. Hence every row and every column of the recursively generated matrix possesses the same invariant S_{kn}. Therefore arithmetic consistency is preserved under recursive expansion.

Corollary 1 (Invariant Matrix Hierarchy). The recursive family

$$F_n,\ F_{kn},\ F_{k^2n},\dots$$

forms an invariant hierarchy in which arithmetic consistency is preserved at every recursion level.

Corollary 1A (Recursive Invariant Propagation). If the base matrix possesses invariant S_n, then every recursion level of the FRM-system possesses a uniquely determined invariant obtained by recursive propagation.

Definition 2 (Recursive Decomposition Chain). An FRM-matrix is called recursively decomposable if it admits a decomposition chain

$$F_{k^m n} \to F_{k^{m-1} n} \to \ldots \to F_n .$$

The sequence above is called a recursive decomposition chain.

3.3. Interpretation for Distributed Information Structures

Theorem 1 provides a mathematical explanation of recursive consistency in distributed information systems. Within the FRM-framework, local block invariants correspond to subsystem consistency; global invariants correspond to system-wide coherence; and recursive expansion corresponds to scalable growth. Consequently, recursive scalability may be achieved without destroying the invariant structure of the system. Theorem 1 establishes the fundamental arithmetic principle governing FRM-matrices and forms the basis for the spectral analysis developed in the next section.

4. Symmetry Transformations and Spectral Stability

4.1. Symmetry Transformations of FRM-Matrices

An important property of recursive matrix systems consists in their behavior under admissible symmetry transformations. Let F_n be an FRM-matrix and let P be a permutation matrix representing a symmetry operation. The transformed matrix is defined by

$$F_n^* = P^{-1} F_n P .$$

Since permutation matrices are orthogonal,

$$P^{-1} = P^T ,$$

the matrices F_n and F_n^* are similar. Typical admissible symmetry transformations include vertical reflection, horizontal reflection, central inversion, matrix transposition, and permutations preserving recursive block organization.

4.2. Spectral Interpretation of Arithmetic Invariants

Let S_n denote the global invariant of an FRM-matrix. Since every row possesses the common sum S_n, the vector

$$e = (1, 1, \dots , 1)^T$$

satisfies

$$F_n e = S_n e.$$

Consequently, S_n is an eigenvalue of the matrix. Thus arithmetic invariants admit a direct spectral interpretation.

Theorem 2 (Spectral Stability Theorem). Let F_n^* be obtained from an FRM-matrix F_n through a finite sequence of admissible symmetry transformations. Then

$$sp(F_n^*) = sp(F_n),$$

where $sp(F_n)$ denotes the spectrum of the matrix.

Proof. By construction,

$$F_n^* = P^{-1} F_n P .$$

Therefore F_n^* and F_n are similar matrices. Since similar matrices possess identical characteristic polynomials

$$\det(\lambda I - F_n^*) = \det(\lambda I - F_n),$$

and hence their spectra coincide:

$$sp(F_n^*) = sp(F_n) .$$

Thus all eigenvalues together with their algebraic multiplicities remain preserved under admissible symmetry transformations.

Remark

Theorem 2 is based on the classical invariance of spectra under similarity transformations. The novelty of the present result lies not in spectral invariance itself but in its application to recursively generated FRM-systems preserving recursive organization, self-similarity, and invariant hierarchy under admissible symmetry transformations.

Corollary 2 (Spectral Radius Preservation). For every admissible symmetry transformation,

$$r(F_n^*) = r(F_n),$$

where $r(F_n)$ denotes the spectral radius.

Spectral Interpretation

The spectral radius may be interpreted as a global characteristic of recursive organization. Preservation of the spectral radius under admissible transformations indicates that large-scale structural properties remain unchanged despite local reconfiguration.

Definition 3 (Recursive Spectral Stability). An FRM-system is called recursively spectrally stable if every matrix in the hierarchy

$$F_n, F_{kn}, F_{k^2n}, \ldots$$

preserves its spectrum under admissible symmetry transformations.

4.3. Interpretation for Distributed Information Structures

Theorem 2 provides a mathematical explanation of structural robustness in recursive information systems. Within the FRM-framework, symmetry transformations correspond to structural reorganization; invariant spectra correspond to stability characteristics; and recursive consistency corresponds to preservation of system integrity. Consequently, admissible reconfiguration of local components does not affect the principal spectral properties of the global system. Together with Theorem 1, the Spectral Stability Theorem forms the theoretical foundation of the FRM-framework.

Conclusion

The present study introduces a new class of recursive invariant matrix systems, referred to as **Fractal Recursive *M*-Matrices (FRM-matrices)**, which combine recursive organization, self-similarity, arithmetic consistency, and spectral stability within a unified mathematical framework.

A formal definition of FRM-matrices has been proposed together with a canonical recursive construction preserving both local and global invariants during recursive expansion. The obtained invariant preservation theorem establishes sufficient conditions under which recursive enlargement maintains arithmetic consistency throughout the entire matrix hierarchy. In addition, the spectral stability theorem demonstrates that admissible symmetry transformations preserve the spectrum and spectral radius of FRM-matrices, providing a rigorous mathematical characterization of their structural robustness.

The recursive interpretation developed in the paper shows that FRM-matrices naturally model scalable distributed information structures, where local subsystem consistency and global system coherence are simultaneously preserved under recursive growth and structural reorganization. The illustrative construction of a recursive (9×9) invariant matrix system confirms the constructive nature and scalability of the proposed framework.

The obtained results establish new connections between recursive matrix theory, self-similar algebraic structures, spectral analysis, and mathematical models of distributed information architectures. The proposed FRM-framework may serve as a basis for further investigations of recursive invariant systems, hierarchical network models, distributed storage architectures, recursive coding theory, and scalable information-processing systems. Future research may include the study of recursive eigenstructure evolution, generalized block constructions, higher-dimensional recursive matrix systems, and algorithmic implementations for large-scale distributed applications

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FRACTAL RECURSIVE *M*-MATRICES AS MODELS OF DISTRIBUTED INFORMATION STRUCTURES

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Макалада өзүнө-өзү окшоштук, масштабдуулук жана жергиликтүү жана жалпы арифметикалык инварианттарды сактоо касиеттерине ээ болгон рекурсивдүү инварианттык матрицалардын жаңы классы – фракталдык рекурсивдүү *M*-матрицалар (FRM-матрицалар) каралат. FRM-матрицаларды рекурсивдүү түзүүнүн математикалык негиздери берилген жана рекурсивдүү кеңейүү учурунда структуралык туруктуулукту жана спектралдык инварианттуулукту сактоо шарттары изилденген. FRM-матрицалар бөлүштүрүлгөн маалыматтык түзүмдөрдүн жана көп деңгээлдүү иерархиялык архитектуралардын абстрактуу математикалык модели катары түшүндүрүлөт. Мисал катары (9×9) өлчөмүндөгү сыйкырдуу квадраттын атайын рекурсивдүү түзүлүшү көрсөтүлүп, рекурсивдүү уюшулган системаларда жалпы инварианттардын сакталуусу демонстрацияланган.

Урунттуу сөздөр: FRM-матрицалар; рекурсивдүү матрицалар; фракталдык түзүлүштөр; сыйкырдуу квадраттар; матрицалык инварианттар; спектралдык туруктуулук; бөлүштүрүлгөн маалыматтык түзүмдөр; иерархиялык системалар; өзүнө-өзү окшоштук; рекурсивдүү моделдөө.

В статье рассматриваются фрактальные рекурсивные *M*-матрицы (FRM-матрицы) как новый класс рекурсивных инвариантных матричных систем, обладающих свойствами самоподобия, масштабируемости и сохранения локальных и глобальных арифметических инвариантов. Представлены математические основы рекурсивного построения FRM-матриц, исследованы условия сохранения структурной согласованности и спектральной устойчивости при рекурсивном расширении. Предложена концептуальная интерпретация FRM-матриц как абстрактных математических моделей распределённых информационных структур и многоуровневых иерархических архитектур. В качестве иллюстрации приведена специальная рекурсивная конструкция магического квадрата порядка (9×9), демонстрирующая сохранение глобальных инвариантов при рекурсивной организации структуры.

Ключевые слова: FRM-матрицы; рекурсивные матрицы; фрактальные структуры; магические квадраты; матричные инварианты; спектральная устойчивость; распределённые информационные структуры; иерархические системы; самоподобие; рекурсивное моделирование.

The paper introduces Fractal Recursive *M*-Matrices (FRM-matrices) as a new class of recursive invariant matrix systems characterized by self-similarity, recursive scalability, and preservation of local and global arithmetic invariants. The mathematical foundations of recursive FRM-matrix construction are presented, and conditions ensuring structural consistency and spectral stability under recursive expansion are investigated. A conceptual interpretation of FRM-matrices as abstract mathematical models of distributed information structures and hierarchical architectures is proposed. As an illustrative example, a special recursive construction of a (9×9) magic matrix is presented, demonstrating the preservation of global invariants within recursively organized matrix systems.

Key words: FRM-matrices; recursive matrices; fractal structures; magic squares; matrix invariants; spectral stability; distributed information structures; hierarchical systems; self-similarity; recursive modeling.

1. Introduction

The rapid development of modern information technologies has stimulated increasing interest in mathematical models capable of describing large-scale hierarchical and recursively organized systems. Distributed information structures, cloud computing architectures, recursive databases, communication networks, and multilevel storage systems are all characterized by self-similarity, scalability, and preservation of local consistency under global expansion.

Matrix methods have long served as one of the principal mathematical tools for modeling complex systems because of their ability to represent structural relations, transformations, and invariant properties in a compact algebraic form. Classical matrix theory, combinatorial matrix constructions, and the theory of magic squares provide numerous examples of structures possessing remarkable arithmetic and symmetry characteristics. However, traditional approaches usually investigate matrices of fixed order and do not explicitly incorporate recursive hierarchical organization.

Fractal Recursive *M*-Matrices (FRM-matrices) are introduced as a new class of recursive invariant matrix systems that combine self-similar block organization with preservation of local and global arithmetic invariants. Their recursive construction naturally generates scalable matrix hierarchies while maintaining structural consistency at every recursion level. In addition to arithmetic invariance, FRM-matrices exhibit spectral stability under admissible symmetry transformations, providing a mathematical framework for studying recursively organized systems.

The purpose of the present paper is to investigate the mathematical foundations of FRM-matrices, establish their invariant properties under recursive expansion, analyze their spectral characteristics, and demonstrate their applicability as abstract mathematical models of distributed information architectures. Particular attention is devoted to recursive scalability, preservation of structural coherence, and the interpretation of recursive matrix hierarchies within information-theoretic contexts.

2. Mathematical Foundations of FRM-Matrices (Brief Review)

The theory of FRM-matrices is based on several interconnected mathematical concepts including recursive matrix constructions, block decomposition, invariant-preserving transformations, and spectral analysis. The central idea consists in generating higher-order matrices through recursive composition of lower-order invariant blocks while preserving predetermined structural properties.

Let M_0 denote an initial invariant matrix. Recursive application of a construction operator R generates a hierarchy of matrices

$$M_{k+1} = R(M_k), \quad k \geq 0,$$

where each recursion level preserves local invariants and contributes to the formation of a globally coherent structure. Such recursive organization produces self-similar matrix systems possessing hierarchical block architecture.

An essential property of the FRM-framework is the coexistence of local consistency and global invariance. Local recursive blocks retain their internal arithmetic characteristics, whereas the assembled matrix preserves global structural parameters such as common row sums, column sums, and other recursive invariants. This recursive preservation principle provides mathematical scalability independent of recursion depth.

Another important aspect of the theory concerns spectral properties. Under admissible symmetry transformations the eigenvalue structure of FRM-matrices remains invariant, implying structural robustness of the recursive hierarchy. Consequently, FRM-matrices simultaneously preserve arithmetic consistency and essential spectral characteristics during recursive expansion and structural reorganization.

These mathematical principles establish the theoretical foundation for interpreting FRM-matrices as abstract recursive systems suitable for modeling hierarchical information architectures and distributed computational structures.

3. FRM-Matrices and Distributed Information Structures

3.1. Mathematical Interpretation of Recursive Information Systems

In the present work fractal recursive *M*-matrices are introduced as recursive invariant matrix systems.

Let $F=(m_{ij})$. Consider the block-recursive structure

$$F=\begin{pmatrix} B_{11} & \dots & B_{1k} \\ \vdots & \ddots & \vdots \\ B_{k1} & \dots & B_{kk} \end{pmatrix},$$

where $B_{ij}=F_n+\alpha_{ij}$. Each block preserves local invariant properties and participates in the formation of the global recursive structure.

The row sum is defined by $S_i = \sum_{j=1}^{n} m_{ij}$. For structural consistency the condition $S_i = S$ must hold. The recursive consistency condition is given by $\sum_{j=1}^{k} \alpha_{ij} = const$.

Recursive scalability is determined by the transitions $F_n \to F_{kn}$, $F_{kn} \to F_{k^2 n}$, $F_{k^2 n} \to F_{k^3 n}$. Thus a hierarchy of self-similar recursive structures is generated.

The mathematical theory developed in the previous sections demonstrates that FRM-matrices simultaneously preserve arithmetic invariants and spectral characteristics under recursive expansion and admissible symmetry transformations. These properties naturally suggest an interpretation of FRM-matrices as abstract mathematical models of recursively organized information systems.

Modern information infrastructures are typically characterized by hierarchical organization. Large systems are constructed from multiple local subsystems that operate independently while contributing to the behavior of the global architecture. Examples include distributed databases, cloud storage infrastructures, hierarchical communication networks, recursive data-processing systems, and multilevel information repositories.

A mathematical model intended to describe such architectures must satisfy preservation of local consistency and preservation of global coherence. The FRM-framework provides a natural mathematical representation of these requirements.

Summary of Definition 1 and Theorem 1

Definition 1. Fractal Recursive *M*-Matrix An FRM matrix is a square matrix that satisfies the following conditions:

It is constructed using a recursive block operator, which allows matrices of higher order to be obtained from matrices of lower order;

This construction preserves the global invariant, meaning that all rows and columns have the same total sum;

The structure is self-similar: each recursively constructed block preserves the fundamental properties of the original matrix;

Recursive consistency is maintained: local invariants of blocks jointly define a single global invariant of the entire matrix;

Spectral stability is preserved, meaning that the fundamental spectral characteristics do not change under admissible symmetric transformations.

A sequence of matrices obtained by repeated application of a recursive operator is called an FRM system.

Theorem 1. Invariant Preservation Theorem If the original FRM matrix has an invariant S_n and the coefficient matrix used in the recursive construction has an invariant S_k, then the matrix obtained by the canonical recursive expansion will also be invariant.

Moreover, its overall invariant is expressed by the formula:

$$S = kS_n + nS_k.$$

In other words, the recursive expansion does not destroy the arithmetic consistency of the structure: after constructing the new matrix, all its rows and all its columns again have the same sum. This means that invariance is preserved at each level of recursion, and the construction itself forms a stable matrix hierarchy.

Meaning of the Result Thus, Definition 1 defines a formal model of recursive self-similar invariant matrices, and Theorem 1 shows that their key property - the general arithmetic invariant - is preserved under canonical recursive construction. This makes FRM matrices a convenient mathematical model for describing hierarchical and distributed structures.

3.2. Local and Global Information Consistency

Consider the recursive hierarchy generated by repeated application of the recursive operator *R*:

$$F_n, F_{kn}, F_{k^2n}, \dots .$$

Each matrix consists of recursively generated invariant blocks. From the information-theoretic viewpoint, every block may be interpreted as an independent local subsystem possessing its own internal structure. Theorem 1 guarantees preservation of local consistency together with the existence of a global invariant characterizing the entire recursive architecture. Consequently, local autonomy and global coherence coexist within the same mathematical framework.

3.3. Recursive Scalability

Scalability represents one of the most important requirements imposed on modern information architectures. Theorem 1 demonstrates that recursive expansion preserves invariant structure independently of the recursion level. Therefore recursive growth does not destroy organizational properties established at earlier stages. From a mathematical perspective, recursive scalability may be interpreted as invariant-preserving growth of hierarchical systems.

3.4. Structural Stability

Distributed systems frequently undergo structural modifications caused by resource redistribution, communication reconfiguration, subsystem replacement, network restructuring, and redistribution of information flows. Theorem 2 demonstrates that admissible symmetry transformations preserve the spectrum of FRM-matrices. Consequently, essential global characteristics remain unchanged despite structural reorganization. This property may be interpreted as a mathematical form of structural robustness.

3.5. Recursive Information Architectures

The recursive decomposition chain

$$F_{k^m n} \to F_{k^{m-1} n} \to \ldots \to F_n$$

provides a natural representation of hierarchical information architectures. Each level preserves its internal invariant structure while simultaneously participating in the formation of a larger system. The coexistence of local autonomy and global consistency is one of the defining characteristics of recursive information architectures.

3.6. Conceptual FRM-Model

FRM-Matrix Concept	Information Interpretation
Recursive block	Local subsystem
Recursive expansion	System scaling
Local invariant	Internal consistency
Global invariant	Global coherence
Recursive hierarchy	Multilevel architecture
Symmetry transformation	Structural reorganization
Spectrum	Stability characteristics

The table should be interpreted as a conceptual correspondence rather than a specific engineering implementation. Its purpose is to illustrate the mathematical meaning of recursive invariants and spectral stability within hierarchical systems.

3.7. Concluding Remarks

The analysis presented in this section demonstrates that FRM-matrices may be viewed as abstract mathematical models of distributed information structures characterized by recursive organization, invariant preservation, scalability, and spectral robustness. The developed interpretation establishes a bridge between recursive matrix theory and mathematical models of hierarchical information architectures.

Mathematical Scope

The proposed FRM-framework should be viewed as an abstract mathematical model intended to describe recursive invariant architectures. The present paper does not attempt to construct a direct engineering realization of distributed information systems.

4. Special Recursive Magic Construction of a 9×9 Matrix

4.1. Special Recursive Magic Construction

The matrix F_9 represents a particular realization of recursive organization principles. It should be emphasized that the present construction is not intended as a direct application of Theorem 1. Rather, it constitutes a special recursive arrangement preserving the classical magic-square property.

Proposition 1. The recursively constructed matrix F_9 is a normal magic square of order nine.

Proof. The matrix contains each integer from 1 to 81 exactly once. For a normal magic square of order *n*,

$$S_n = \frac{n(n^2+1)}{2}.$$

For $n = 9$,

$$S_9 = \frac{9(81+1)}{2} = 369.$$

Direct verification shows that every row and every column possess the common sum 369. Therefore F_9 is a normal magic square of order nine.

The presented construction illustrates how local invariant blocks generate a higher-order matrix preserving global arithmetic consistency. This phenomenon constitutes one of the fundamental principles of the FRM-framework.

4.2. Relation to the General FRM-Theory

Theorem 1 establishes invariant preservation for a broad class of recursive block constructions generated by the FRM-operator. The matrix F_9 belongs to a more specialized class of recursive magic constructions based on structured arrangements of consecutive integers. Nevertheless, the example illustrates recursive organization, hierarchical block structure, preservation of local patterns, emergence of a global invariant, and scalability of the construction process.

4.3. Recursive Consistency

The matrix F_9 demonstrates the coexistence of local and global structural levels. At the local level, each block is generated from the same invariant base matrix. At the global level, recursive assembly produces a higher-order structure characterized by the invariant

$$S_9 = 369\,.$$

4.4. Spectral Remarks

Since every row of F_9 possesses the common sum 369, the vector

$$e = (1, 1, \dots, 1)^T$$

satisfies

$$F_9 e = S_9 e.$$

Therefore 369 is an eigenvalue of the matrix. This observation establishes a direct connection between arithmetic invariance and spectral structure.

4.5. Interpretation within the FRM-Framework

Within the FRM-framework, blocks represent local invariant subsystems; recursive arrangement represents hierarchical integration; the magic constant represents a global consistency parameter; and spectral characteristics represent global stability properties. Therefore, although F_9 belongs to a special class of recursive magic constructions, it provides an illustrative example of the broader FRM-concept developed in this paper.

Discussion

The proposed FRM-framework extends classical recursive matrix constructions by integrating self-similarity, invariant preservation, and recursive scalability within a unified mathematical model. Unlike conventional matrix representations of fixed dimension, FRM-matrices generate hierarchical structures whose recursive expansion preserves both local consistency and global coherence.

From a theoretical perspective, the obtained results establish new connections between recursive matrix theory, combinatorial constructions, spectral analysis, and fractal organization principles. The coexistence of arithmetic invariants and spectral stability suggests that recursive matrix hierarchies possess a high degree of structural robustness under admissible transformations.

The interpretation of FRM-matrices as mathematical models of distributed information structures should be regarded as conceptual rather than engineering-oriented. The developed framework provides an abstract mathematical language for describing scalable recursive architectures in which independently functioning local subsystems collectively form a globally consistent information system.

The presented recursive constructions also indicate possible relationships with graph theory, network topology, recursive algorithms, and hierarchical data organization. These directions may stimulate further investigations into recursive matrix operators, invariant subspaces, recursive decomposition methods, and scalable mathematical models for distributed computational systems.

Overall, the FRM-framework provides a promising basis for future research at the intersection of recursive matrix theory, self-similar structures, and mathematical modeling of complex hierarchical architectures.

5. Conclusion and Future Research

The present paper introduces Fractal Recursive *M*-Matrices (FRM-matrices) as a new class of recursive invariant matrix systems characterized by self-similarity, recursive scalability, arithmetic consistency, and spectral stability.

A recursive mathematical framework for the construction of FRM-matrices has been proposed. The developed approach extends traditional studies of magic matrices by introducing recursive block structures preserving local and global invariants.

The first principal result of the paper consists in establishing sufficient conditions for invariant preservation under recursive expansion. It has been shown that recursively generated matrix hierarchies preserve structural consistency and admit recursive decomposition without loss of arithmetic coherence.

The second principal result is the analysis of spectral stability under admissible symmetry transformations. The obtained results demonstrate that recursive matrix systems preserve not only arithmetic invariants but also essential spectral characteristics associated with global structural stability.

A mathematical interpretation of FRM-matrices as models of distributed information structures has been developed. Within this interpretation, recursive expansion corresponds to scalable growth, local invariants characterize subsystem consistency, and global invariants describe coherence of the entire recursive architecture.

A special recursive construction of a 9×9 magic matrix has been presented as an illustrative example of recursive organization and invariant preservation. The example demonstrates how local self-similar structures can generate globally consistent matrix systems of higher order.

The obtained results establish new connections between recursive matrix theory, invariant structures, spectral analysis, self-similar systems, and mathematical models of distributed information architectures.

Future Research Directions

1. Development of an algebraic theory of FRM-matrices and recursive matrix hierarchies.
2. Classification of recursive invariant matrix systems of arbitrary order and recursion depth.
3. Investigation of eigenvalue distributions and asymptotic spectral behavior of large recursive matrices.
4. Development of recursive constructions based on arithmetic progressions, geometric progressions, and higher-order difference equations.
5. Investigation of recursive matrix operators, invariant subspaces, and transformation groups.
6. Construction of mathematical models of scalable distributed storage systems and recursive information architectures.
7. Investigation of self-similarity measures and fractal characteristics of recursive matrix structures.

Final Remark

The authors believe that the concept of Fractal Recursive *M*-Matrices provides a promising direction for further investigations in recursive matrix theory, invariant structures, and mathematical models of distributed systems. The combination of recursive scalability, arithmetic consistency, and spectral stability offers a unified framework for the study of self-similar matrix hierarchies and their applications.

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Conflict of Interest

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General topics in artificial intelligence

MSC 68T01

MATHEMATICAL HOMEWORK AND ARTIFICIAL INTELLIGENCE

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Макалада окуучулардын жасалма интеллектти дароо колдонуусуна жол бербөө үчүн үй тапшырмасын түзүү жана жасалма интеллектти мыйзамдуу колдонуу менен аткарыла тур-ган чыгармачылык үй тапшырмаларын иштеп чыгуу боюнча мугалимдерге сунуштар кам-тылган. Тексттердин каталарынын жана түшүнбөстүктөрүнүн себептери жасалма интеллект-тин тарабынан гана эмес, адамдар, анын ичинде окуучулар тарабынан да) - көп сөздөрдүн эки ачалыгы, тарыхый жактан иштелип чыккан математикалык белгилердин эки ачалыгы жана түшүнүксүздүгү авторлордон эске салат.

Урунттуу сөздөр: жасалма интеллект, математика, үй тапшырмасы, мугалим, окуучу, сунуш.

В статье содержатся рекомендации преподавателям по составлению домашних заданий таким образом, чтобы избежать немедленного использования искусственного интеллекта учащимися, а также по разработке творческих домашних заданий с законным применением искусственного интеллекта. Авторы упоминают причины ошибок и недопонимания текстов (не только искусственным интеллектом, но и людьми, в том числе студентами) – это неясность многих слов, неясность и двусмысленность исторически сложившихся математических обозначений.

Ключевые слова: искусственный интеллект, математика, домашнее задание, преподаватель, учащийся, рекомендация.

The article contains recommendations to teachers on composing homework to avoid immediate using of artificial intelligence by students, and to develop creative homework to be made with legal using of artificial intelligence. The authors mention reasons for errors and misunderstandings of texts (not only by AI, but also by people, including students) - the ambiguity of many words, the ambiguity of historically established mathematical notations.

Keywords: artificial intelligence, mathematics, homework, teacher, student, recommendation.

1. Introduction

Every new invention and discovery are a challenge to people's activities.

Let us consider from this point of view the activities of millions of teachers, pupils and students: mathematical homework, its setting, execution and checking.

The main and unresolvable problem of all times: how to ensure independent execution of homework? Nowadays: how to develop AI-resistant tasks?

We consider this matter in Section 2.

On the other hand, the legal use of AI provides unlimited opportunities for creative homework for schoolchildren and students (but increases the workload for teachers). We consider this matter in Section 3.

Remark. There are numerous attempts to identify the use of AI in works by students and schoolchildren. This sometimes leads to human dramas. We believe this to be false. Nowadays AI and its predecessors: word correctors, high-level programming languages, search engines, translators, Wolfram Mathematica etc. are built in gadgets and computers and are used permanently. Banning to use AI is almost tantamount to banning computing devices at all.

Also, by our opinion, use of detectors of AI and Anti-plagiarism, which detect not only direct borrowing but also "suspicious" phrases, contradicts the principles of jurisprudence: a prohibiting law must be published in full before it is applied.

We do not give references because there are too many texts in Internet including created by AI and to prove “novelty” is impossible.

2. Developing AI-resistant tasks

The simple advice to the teacher: after preparing a homework ask AI to solve it.

We asked gemini on this matter. We cite its fit answers.

2.1. Teachers can make homework AI-resistant by assigning tasks that require personal experiences, specific local observations, and visible thinking processes. Because AI models lack real-world experience, they fail to authentically complete assignments that center on local context, metacognition, and individual physical creation.

2.2. Specific strategies include: Ask for personal and local connections. AI tools rely on generalized, internet-wide data. You can easily circumvent this by asking stu-dents.

2.3. Connect to personal experience: Ask students to tie a concept to an event, specific personal feeling, or a real-world observation they made within their local community.

2.4. Require in-depth personalization. AI generates generic responses and cannot draw on real-world memories, local observations, or personal emotions. Local cone-ctions: ask students to connect course concepts to their immediate community, spe-cific local news, or personal experiences.

2.5. Self-reflection: Require learners to reflect on their own feelings, beliefs, or "learning journeys" over the course of a project.

2.6. Custom perspectives: Have students argue a specific point of view based on an in-class discussion they just had, which a generic AI model lacks context for.

2.7. Emphasize process over product instead of grading only the final essay or finished math equation, evaluate the steps taken to get there.

2.8. Scaffolded submissions: Break larger assignments into smaller, sequential deliverables (e.g., brainstorming notes, outlines, an annotated bibliography, and a rough draft with peer feedback).

2.9. Track your steps: Ask students to document their thought process, including false starts, obstacles, or mistakes along the way.

2.10. Revision memos: Require students to submit a brief explanation of how and why they changed their work from the rough draft to the final copy.

2.11. Change the assignment format stepping away from traditional text-based essays forces students to engage with the material creatively.

2.12. Alternative mediums: Assign work using non-traditional formats like mind-maps, flowcharts, infographics, comic strips, or podcasts.

2.13. Visual representations: Ask students to sketch a visual representation or concept map of the material.

2.14. Physical artifacts: Have students create hands-on or physical models to demonstrate their knowledge and skills.

Our remark. All these techniques require a lot of mental effort and hard work from the teacher, especially reading vast texts, explanations.

Generalizing the notion introduced in [1], we propose the notion *short time*: time insufficient for calculation or for accessing AI, but sufficient for the manifestation of other human abilities.

2.15. We propose more formal technique. Have students to make and to describe step-by-step any computation of the mathematical expression. When they submit such computation, change some signs in the expression. If they themself wrote the description, they would be able to say the answer immediately.

Example 1. Teacher: How many is $\sqrt{(60^2+80^2)}-\sqrt{(12^2+5^2)}$? Describe your computation step-by-step.

Student: shows next day:

$60^2+80^2=10000$; $12^2+5^2=169$; $\sqrt{10000}=100$; $\sqrt{169}=13$; $100-13=87$.

Teacher: changes "–" to "+". *Short time*:

2.16. Teacher writes an exam program with parameterized tasks with timer with "tandem-tasks": the first task can be solved by common methods or by using AI; the solution of the second task can be derived from the solution of the first task in *short time*.

Example 2. First task: find the positive root of the equation $X*(X+5)=546$.

Second task: find the positive root of the equation $2*Y*(2*Y+5)=546$,

or: find the positive root of the equation $Y*(2*Y+5)=546/2$.

Example 3. First task: 2*X+5*Y=25; 3*X–Y=29. Find (X+Y).

Second task: 2*X+5*Y=250; 3*X–Y=290. Find (X+Y).

Example 4. First task: Calculate $\int_0^7$ 6*X*X dX.

Second task: Calculate $\int_{-7}^7$ 6*X*X dX.

Example 5. First task: Calculate $\sum_{K=1}^{77}$ (6*K+8).

Second task: Calculate $\sum_{K=0}^{77}$ (6*K+8).

Example 6. First task: Calculate lim{√(K*K+7*K) –K : K→ ∞}.

Second task: Calculate lim{√(K*K+11*K) –K: K→ ∞}, or

Calculate lim{√(K*K–9*K) – K: K→ ∞}.

3. Creative homework with assistance of artificial intelligence

Teacher may give themes for homework involving necessary using of AI.

Remark. We gave recommendations on writing mathematical texts in general and for better understanding by AI [2], [3], [4].

gemini gave recommendations also: Avoid ambiguous symbols: For multiplication, prefer * or × over a simple dot · (which can be confused with a decimal point or a vector dot product) or implicit juxtaposition, especially in complex expressions.

The teacher may themself write an instruction on communication with AI in corresponding branch of mathematics.

Example 7. Teacher: give quest “Show five green squares and two yellow squares to wolframalpha.com. Try to understand its respond. Try to obtain five green squares and two yellow squares actually.”

AI has no space imagery.

Example 8. Teacher: Have grok.com to create an image of a chain of five links.

Example 9. Teacher: Have grok.com to create an image of a chain-ring of six links.

Example 10. Teacher: Have AI to create an image of a triangle which is within a square which is within of circle.

Example 11. Teacher: Have AI to create an image of a triangle and five points within it and arranged symmetrically.

4. Conclusion

We hope that this paper would promote teachers to work with pupils and students successfully in changing world.

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9 771694 817007